POSSIBILITY THEORY II
CONDITIONAL POSSIBILITY

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Dedicated to Prof. dr. Etienne E. Kerre

It is shown that the notion of conditional possibility can be consistently introduced in possibility theory, in very much the same way as conditional expectations and probabilities are defined in the measure- and integral-theoretic treatment of probability theory. I write down possibilistic integral equations which are formal counterparts of the integral equations used to define conditional expectations and probabilities, and use their solutions to define conditional possibilities. In all, three types of conditional possibilities, with special cases, are introduced and studied. I explain why, like conditional expectations, conditional possibilities are not uniquely defined, but can only be determined up to almost everywhere equality, and I assess the consequences of this nondeterminacy. I also show that this approach solves a number of consistency problems, extant in the literature.

INDEX TERMS: Possibility integral, integral equation, conditional possibility.

1 CONDITIONAL POSSIBILITY: A SURVEY

This is the second in a series of three papers on the measure- and integral-theoretic aspects of possibility theory. Here I specifically deal with conditional possibility. I shall make ample use of the results, definitions and notational conventions, given in the first paper of this series, which will be referred to as Part I.

The notion of conditional possibility for variables was, albeit under a different name, first introduced by Zadeh [1978]. It was later refined by Hisdal [1978]. Both authors drew their inspiration from the notion of conditional probability in probability theory. Nguyen [1978] published his view on the subject simultaneously. Dubois and Prade [1984] also refined Zadeh's work. More recently [Dubois *et al.*, 1994], [Dubois and Prade, 1985, 1988, 1990] they have also studied conditional possibility for events, a topic which has also received attention from Ramer [1989].

In this section, I give a brief account of the most important contributions of these authors to the field of conditional possibility. This will give the reader an idea of what has been accomplished in this domain, and of the problems and difficulties that still remain. It will also help reveal the relevance and importance of the measure- and integral-theoretic approach described in this paper. In order to make this survey as tidy as possible, I shall not be using the diversity of notations and terminology employed by the above-mentioned scholars. I prefer to reformulate their results using a uniform notation and nomenclature, already established in Part I. At the same time, I do not present a literal and explicit account of their work, but restrict myself to the main ideas.

1.1 Zadeh's Approach

Zadeh starts with two universes X_1 and X_2. He considers ξ_1 and ξ_2 as variables taking values in X_1 and X_2 respectively. (ξ_1, ξ_2) is also a variable, which assumes values in the universe $X_1 \times X_2$. Information about the values that (ξ_1, ξ_2) takes in $X_1 \times X_2$ is given by the $([0,1], \leq)$-possibility measure[1] $\Pi_{(\xi_1,\xi_2)}$ on $(X_1 \times X_2, \wp(X_1 \times X_2))$, with distribution $\pi_{(\xi_1,\xi_2)}$. For any (x_1, x_2) in $X_1 \times X_2$, $\pi_{(\xi_1,\xi_2)}(x_1, x_2)$ is the $(L, \leq)$-possibility that (ξ_1, ξ_2) assumes the value (x_1, x_2). $\Pi_{(\xi_1,\xi_2)}$ is called by Zadeh the *binary possibility distribution* of (ξ_1, ξ_2). He calls $\pi_{(\xi_1,\xi_2)}$ the *binary possibility distribution function* of (ξ_1, ξ_2). Hisdal uses in this context the adjective '*joint*' instead of 'binary'. Zadeh furthermore defines the $([0,1], \leq)$-possibility measure Π_{ξ_1} on $(X_1, \wp(X_1))$ with distribution π_{ξ_1} as

$$(\forall A_1 \in \wp(X_1))(\Pi_{\xi_1}(A_1) = \Pi_{(\xi_1,\xi_2)}(A_1 \times X_2)) \tag{1}$$

or equivalently

$$(\forall x_1 \in X_1) \left(\pi_{\xi_1}(x_1) = \sup_{x_2 \in X_2} \pi_{(\xi_1,\xi_2)}(x_1, x_2) \right); \tag{2}$$

and the $([0,1], \leq)$-possibility measure Π_{ξ_2} on $(X_2, \wp(X_2))$ with distribution π_{ξ_2} as

$$(\forall A_2 \in \wp(X_2))(\Pi_{\xi_2}(A_2) = \Pi_{(\xi_1,\xi_2)}(X_1 \times A_2)) \tag{3}$$

[1] In Zadeh's work, no mention is made of the notion of measurability. In my terminology, this means that he implicitly considers the power sets $\wp(X_1)$, $\wp(X_2)$ and $\wp(X_1 \times X_2)$ as ample fields of measurable sets on the respective universes X_1, X_2 and $X_1 \times X_2$.

or equivalently

$$(\forall x_2 \in X_2)\left(\pi_{\xi_2}(x_2) = \sup_{x_1 \in X_1} \pi_{(\xi_1,\xi_2)}(x_1, x_2)\right). \tag{4}$$

Π_{ξ_1} and Π_{ξ_2} are called by Zadeh (and Hisdal and Nguyen) the *marginal possibility distributions* of ξ_1 and ξ_2 respectively. π_{ξ_1} and π_{ξ_2} are given the name *marginal possibility distribution function.* The possibility measure Π_{ξ_k} contains information about the values which ξ_k assumes in X_k, $k = 1, 2$, (see also Part I, section 4).

Furthermore, Zadeh calls the variables ξ_1 and ξ_2 *noninteractive* iff

$$\Pi_{(\xi_1,\xi_2)} = \Pi_{\xi_1} \times_{\min} \Pi_{\xi_2} \tag{5}$$

or equivalently

$$(\forall (x_1, x_2) \in X_1 \times X_2)(\pi_{(\xi_1,\xi_2)}(x_1, x_2) = \min(\pi_{\xi_1}(x_1), \pi_{\xi_2}(x_2))) \tag{6}$$

where min is the minimum operator on the real unit interval $[0, 1]$, and $\Pi_{\xi_1} \times_{\min} \Pi_{\xi_2}$ the min-product possibility measure of Π_{ξ_1} and Π_{ξ_2} (see Part I, section 8). According to Zadeh, this noninteractivity is analogous to the notion of stochastic independence in probability theory.

Finally, Zadeh introduces for any (y_1, y_2) in $X_1 \times X_2$ the following mappings

$$\begin{aligned} \pi_{\xi_1|\xi_2}(\cdot \mid y_2)&\colon X_1 \to [0, 1]\colon x_1 \mapsto \pi_{(\xi_1,\xi_2)}(x_1, y_2) \\ \pi_{\xi_2|\xi_1}(\cdot \mid y_1)&\colon X_2 \to [0, 1]\colon x_2 \mapsto \pi_{(\xi_1,\xi_2)}(y_1, x_2), \end{aligned}$$

and calls them respectively the *conditioned possibility distribution function* of ξ_1 if $\xi_2 = y_2$ is given, and the *conditioned possibility distribution function* of ξ_2 if $\xi_1 = y_1$ is given. These conditioned possibility distribution functions are simply defined as partial mappings of the binary – or joint – possibility distribution function. This implies that the concept of a conditioned possibility distribution function plays a part in Zadeh's theory that is not completely analogous with the one played by conditional probability distribution functions (or density or frequency functions, see Part I, section 5) in probability theory [Burrill, 1972], as Zadeh himself has rightly remarked. In Hisdal's view, Zadeh does not closely follow the analogy with probability theory, because this would lead to apparent incompatibilities with his theory of *approximate reasoning* [Zadeh, 1978]. Let us, for the sake of convenience, call this difficulty '*Zadeh's problem*'. A detailed explanation of this problem, and how these incompatibilities come about, would lead us too far astray. For a thorough discussion of this problem, I refer to Hisdal's work on the subject [Hisdal, 1987].

1.2 Hisdal's Approach

Hisdal attempts to restore the analogy between possibility and probability, while at the same time securing Zadeh's theory of approximate reasoning. Exploiting the analogy with what can be done in probability theory with conditional frequency functions, she proposes to *define* the conditioned possibility distribution functions using the following equations:

$$\begin{cases} (\forall (x_1, x_2) \in X_1 \times X_2)(\pi_{(\xi_1,\xi_2)}(x_1, x_2) = \min(\pi_{\xi_1|\xi_2}(x_1 \mid x_2), \pi_{\xi_2}(x_2))) \\ (\forall (x_1, x_2) \in X_1 \times X_2)(\pi_{(\xi_1,\xi_2)}(x_1, x_2) = \min(\pi_{\xi_2|\xi_1}(x_2 \mid x_1), \pi_{\xi_1}(x_1))). \end{cases} \tag{7}$$

Remark that the operator min on $[0, 1]$ plays in these equations the same role as the product operator in probability theory; see in this respect also (5) and (6). These equations have the following solution(s), for any (x_1, x_2) in $X_1 \times X_2$:

$$\pi_{\xi_1|\xi_2}(x_1 \mid x_2) \in \begin{cases} \{\pi_{(\xi_1,\xi_2)}(x_1, x_2)\} & ; \quad \pi_{\xi_2}(x_2) > \pi_{(\xi_1,\xi_2)}(x_1, x_2) \\ \left[\pi_{(\xi_1,\xi_2)}(x_1, x_2), 1\right] & ; \quad \pi_{\xi_2}(x_2) = \pi_{(\xi_1,\xi_2)}(x_1, x_2) \end{cases} \tag{8}$$

and analogously for $\pi_{\xi_2|\xi_1}(x_2 \mid x_1)$. Hisdal uses for the mappings $\pi_{\xi_1|\xi_2}(\cdot \mid x_2)$ and $\pi_{\xi_2|\xi_1}(\cdot \mid x_1)$ the adjective '*conditional*' instead of 'conditioned', probably to make clear that the analogy between probability and possibility has been restored. From (8) we deduce that Hisdal's equations (7) do not exclude Zadeh's definition, but rather extend it. It should indeed be noted that Zadeh's choice, given by

$$(\forall(x_1, x_2) \in X_1 \times X_2)(\pi_{\xi_1|\xi_2}(x_1 \mid x_2) = \pi_{\xi_2|\xi_1}(x_2 \mid x_1) = \pi_{(\xi_1,\xi_2)}(x_1, x_2)),$$

is always a solution of Hisdal's defining equations[2]. Hisdal furthermore shows that this (happy) fact provides a solution for Zadeh's problem. On the other hand, she is confronted with another difficulty, which we shall baptise '*Hisdal's problem*'. Besides Zadeh's noninteractivity of ξ_1 and ξ_2, she also introduces the notion of *possibilistic independence*, which she claims to be a possibilistic counterpart of the stochastic independence of real stochastic variables. She calls the variable ξ_1 *possibilistically independent* of the variable ξ_2 iff

$$(\forall(x_1, x_2) \in X_1 \times X_2)(\pi_{\xi_1|\xi_2}(x_1 \mid x_2) = \pi_{\xi_1}(x_1)) \tag{9}$$

which is equivalent to

$$(\forall x_1 \in X_1)(\pi_{\xi_1|\xi_2}(x_1 \mid \cdot) = \underline{\pi_{\xi_1}(x_1)}), \tag{10}$$

where, of course, for any x_1 in X_1,

$$\pi_{\xi_1|\xi_2}(x_1 \mid \cdot)\colon X_2 \to [0, 1]\colon x_2 \mapsto \pi_{\xi_1|\xi_2}(x_1 \mid x_2)$$
$$\underline{\pi_{\xi_1}(x_1)}\colon X_2 \to [0, 1]\colon x_2 \mapsto \pi_{\xi_1}(x_1).$$

Hisdal's difficulty is now the following: when ξ_1 is possibilistically independent of ξ_2, it follows that ξ_1 and ξ_2 are noninteractive, but the reverse is not necessarily true. Whereas, still according to Hisdal, the counterpart of the reverse implication is valid in probability theory. In this way, a discrepancy between possibility and probability theory creeps in, and Hisdal is forced to distinguish between Zadeh's noninteractivity and her own notion of possibilistic independence.

It is my conviction that the distinction brought to life by Hisdal is to a great extent artificial. On the one hand, if we want to define possibilistic independence in an unambiguous way using Eq. (9), the number $\pi_{\xi_1|\xi_2}(x_1 \mid x_2)$ must be *uniquely* defined for any (x_1, x_2) in $X_1 \times X_2$. On the other hand, Hisdal defines the numbers $\pi_{\xi_1|\xi_2}(x_1 \mid x_2)$ as solutions of the Eqs. (7), which in some cases have *no unique solution*, as Eq. (8) tells us. This simply implies that Hisdal's definition of possibilistic independence makes little sense.

[2] Zadeh's choice could be considered as the most restrictive (smallest) of all possible solutions of Hisdal's equations.

1.3 Dubois and Prade's Work

Dubois and Prade [1984] define conditional possibility for variables using equations which are more general than Hisdal's, because they substitute for the operator min in (7) an arbitrary isotonic operator $\hat{\wedge}$ on $[0, 1]$ which satisfies a number of additional boundary conditions. This leads them to consider the following defining equations for the conditional possibility distribution functions

$$\begin{cases} (\forall (x_1, x_2) \in X_1 \times X_2)(\pi_{(\xi_1,\xi_2)}(x_1, x_2) = \pi_{\xi_2}(x_2) \hat{\wedge} \pi_{\xi_1|\xi_2}(x_1 \mid x_2)) \\ (\forall (x_1, x_2) \in X_1 \times X_2)(\pi_{(\xi_1,\xi_2)}(x_1, x_2) = \pi_{\xi_1}(x_1) \hat{\wedge} \pi_{\xi_2|\xi_1}(x_2 \mid x_1)). \end{cases} \tag{11}$$

The attempt they make to integrate the notion of conditional possibility into the theory of approximate reasoning is essentially similar to Hisdal's. I shall not discuss this work in more detail, but instead refer the interested reader to the original paper [Dubois and Prade, 1984].

In various publications [Dubois *et al.*, 1994], [Dubois and Prade, 1985, 1988, 1990] Dubois and Prade also consider conditional possibilities of events, or propositions. Given two events A and B, and a $([0, 1], \leq)$-possibility measure Π defined on the event space $\wp(X)$, they follow and extend Hisdal's original idea in defining the *conditional possibility* $\Pi(A \mid B)$ *of* A *given* B as a solution of the equation

$$\Pi(A \cap B) = \Pi(A \mid B) * \Pi(B), \tag{12}$$

where $*$ is in general a triangular norm (see Part I, section 2). They furthermore note that for $* = \min$ this equation not necessarily has a unique solution, and invoke what they call the *principle of minimum specificity* to justify their choice of the *maximal solution*:

$$\Pi(A \mid B) = \begin{cases} \Pi(A \cap B) & ; \quad \Pi(A \cap B) < \Pi(B) \\ 1 & ; \quad \Pi(A \cap B) = \Pi(B). \end{cases}$$

They also mention that Shafer considers the case $* =$ algebraic product, which of course yields $\Pi(A \mid B) = \Pi(A \cap B)/\Pi(B)$, whenever $\Pi(B) \neq 0$, a formula that is consistent with Dempster's rule of combination [Shafer, 1976].

Ramer [1989] uses a comparable approach. He also considers Eq. (12) with $* = \min$ as a defining equation for conditional possibilities of events, and imposes either an extra continuity condition or a principle of minimal information distance in order to obtain a unique solution.

1.4 Nguyen's Approach

Nguyen uses a different approach to define conditional possibility distribution functions of variables. I shall restrict myself here to briefly explaining his way of arriving at the numbers $\pi_{(\xi_1|\xi_2)}(x_1, x_2)$, with (x_1, x_2) in $X_1 \times X_2$. Nguyen does not use Eqs. (7) for the definition of conditional possibility, but rather looks for a $[0, 1] \times]0, 1] - \mathbb{R}^+$-mapping α satisfying, for any x_1 in X_1 and x_2 in X_2:

(i) $\pi_{(\xi_1,\xi_2)}(x_1, x_2) \cdot \alpha(\pi_{\xi_1}(x_1), \pi_{\xi_2}(x_2)) \in [0, 1]$;

(ii) $\min(\pi_{\xi_1}(x_1), \pi_{\xi_2}(x_2)) \cdot \alpha(\pi_{\xi_1}(x_1), \pi_{\xi_2}(x_2)) = \pi_{\xi_1}(x_1)$.

He leaves the question as to what should be done if $\pi_{\xi_2}(x_2) = 0$ unanswered. Nguyen uses the mapping α as a normalization factor that the joint possibility distribution function must be multiplied with in order to obtain the conditional possibility distribution function:

$$(\forall (x_1, x_2) \in X_1 \times X_2)(\pi_{(\xi_1|\xi_2)}(x_1 \mid x_2) = \pi_{(\xi_1,\xi_2)}(x_1, x_2) \cdot \alpha(\pi_{\xi_1}(x_1), \pi_{\xi_2}(x_2))).$$

He proves that the only solution α for which, for any b in $]0,1]$, the partial mapping $\alpha(\cdot, b)$ is continuous, is given by

$$(\forall (a,b) \in [0,1]\times]0,1]) \left(\alpha(a,b) = \begin{cases} 1 & ; \quad a \leq b \\ \dfrac{a}{b} & ; \quad a > b \end{cases} \right).$$

Condition (ii) ensures that whenever ξ_1 and ξ_2 are noninteractive, we have for any (x_1, x_2) in $X_1 \times X_2$ that $\pi_{\xi_1|\xi_2}(x_1 \mid x_2) = \pi_{\xi_1}(x_1)$, which makes Hisdal's possibilistic independence and Zadeh's noninteractivity coincide. Nguyen finds for any (x_1, x_2) in $X_1 \times X_2$ for which $\pi_{\xi_2}(x_2) > 0$:

$$\pi_{\xi_1|\xi_2}(x_1 \mid x_2) = \begin{cases} \pi_{(\xi_1,\xi_2)}(x_1, x_2) & ; \quad \pi_{\xi_1}(x_1) \leq \pi_{\xi_2}(x_2) \\ \pi_{(\xi_1,\xi_2)}(x_1, x_2)\dfrac{\pi_{\xi_1}(x_1)}{\pi_{\xi_2}(x_2)} & ; \quad \pi_{\xi_1}(x_1) > \pi_{\xi_2}(x_2). \end{cases} \tag{13}$$

Again, what happens if $\pi_{\xi_2}(x_2) = 0$ is not made clear. Nguyen also shows that his conditional possibilities satisfy the following expression:

$$(\forall x_1 \in X_1) \left(\pi_{\xi_1}(x_1) = \sup_{x_2 \in X_2} \min(\pi_{\xi_1|\xi_2}(x_1 \mid x_2), \pi_{\xi_2}(x_2)) \right),$$

which, by the way, can also be obtained by taking the supremum on both sides of one of the Eqs. (7), also taking into account (2).

It seems to me that Nguyen's approach, although mathematically correct, has a few shortcomings as far as its justification is concerned. First of all, it is somewhat artificial and lacking in simplicity. In my opinion, the *correction factor* $\alpha(\pi_{\xi_1}(x_1), \pi_{\xi_2}(x_2))$ that $\pi_{(\xi_1,\xi_2)}(x_1, x_2)$ must be *multiplied* with in order to obtain $\pi_{\xi_1|\xi_2}(x_1 \mid x_2)$ – why introduce a *second* (product) operator *besides* min, that is unrelated to it?–, is to some extent drawn in. Nguyen defends his approach by arguing that an analogous 'correction factor' also exists in probability theory. Indeed, to give an example, the joint density function of two continuous stochastic variables Y and Z is divided by the marginal density function of Z in order to obtain the conditional density function of Y w.r.t. Z (for a more rigorous formulation, see, for instance, [Burrill, 1972] Example 15-3B, Eq. (15)). But, the division is in this probabilistic case the *inverse operation* of the product operator that plays a very important part in probability theory. Therefore, if we *must* introduce a correction factor in the possibilistic case, would it not be more straightforward and consistent to use the 'inverse operation' of the operator min that also in Nguyen's approach takes the place of the product operator? Should we not, in that case, on the basis of Part I, Example 2.3 and of a *consistent* analogy with probability theory, expect the following expression for $\pi_{\xi_1|\xi_2}(x_1 \mid x_2)$:

$$\begin{aligned} \pi_{\xi_1|\xi_2}(x_1 \mid x_2) &= \pi_{(\xi_1,\xi_2)}(x_1, x_2) \triangle_{\min} \pi_{\xi_2}(x_2) \\ &= \begin{cases} \pi_{(\xi_1,\xi_2)}(x_1, x_2) & ; \quad \pi_{\xi_2}(x_2) > \pi_{(\xi_1,\xi_2)}(x_1, x_2) \\ 1 & ; \quad \pi_{\xi_2}(x_2) = \pi_{(\xi_1,\xi_2)}(x_1, x_2), \end{cases} \end{aligned} \tag{14}$$

which is in perfect agreement with (8)?

1.5 A Solution for Hisdal's Problem

It should also be noted that Nguyen obtains *unique* values for his conditional possibilities, and at the same time solves Hisdal's problem. Since I have already indicated above that I am not entirely satisfied with Nguyen's approach, let me briefly indicate here how I propose to solve Hisdal's problem. The approach I want to defend here – and will work out in more detail in the rest of this paper, and in Part III of this series – is based upon two observations.

For one thing, it should not really be a problem that the Eqs. (7) do not uniquely determine the conditional possibility distribution functions. In probability theory, conditional probability distribution functions are not uniquely determined either, but rather are *unique in the sense of almost everywhere equality*[3] [Burrill, 1972]. We may therefore expect something analogous for conditional possibility distribution functions. Indeed, if for any (x_1, x_2) in $X_1 \times X_2$ we represent by $\pi^1_{\xi_1|\xi_2}(x_1 \mid x_2)$ and $\pi^2_{\xi_1|\xi_2}(x_1 \mid x_2)$ two arbitrary solutions of the first equation of (7), then

$$(\forall (x_1, x_2) \in X_1 \times X_2)(\min(\pi^1_{\xi_1|\xi_2}(x_1 \mid x_2), \pi_{\xi_2}(x_2)) = \min(\pi^2_{\xi_1|\xi_2}(x_1 \mid x_2), \pi_{\xi_2}(x_2)))$$

or equivalently, using the notations introduced in Part I, section 6),

$$(\forall x_1 \in X_1)\left(\pi^1_{\xi_1|\xi_2}(x_1 \mid \cdot) \overset{(\Pi_{\xi_2},\min)}{=} \pi^2_{\xi_1|\xi_2}(x_1 \mid \cdot)\right).$$

In this sense, we see that the solutions of (7) are only *unique in the sense of* $(\Pi_{\xi_2}, \min)$-*equivalence*. We conclude that the analogy between probability and possibility theory is preserved, also as far as the nonuniqueness of the solutions of (7) is concerned.

Furthermore, it is important to note that *it is not at all necessary that conditional possibilities should be uniquely determined* in order to introduce a form of possibilistic independence and at the same time solve Hisdal's problem. We need only change Hisdal's original definition of possibilistic independence in such a way that it takes this nonuniqueness into account. That ξ_1 and ξ_2 are noninteractive is indeed equivalent to

$$(\forall (x_1, x_2) \in X_1 \times X_2)(\pi_{(\xi_1,\xi_2)}(x_1, x_2) = \min(\pi_{\xi_1}(x_1), \pi_{\xi_2}(x_2)))$$

and, taking into account (7), also with

$$(\forall (x_1, x_2) \in X_1 \times X_2)(\min(\pi_{\xi_1|\xi_2}(x_1 \mid x_2), \pi_{\xi_2}(x_2)) = \min(\pi_{\xi_1}(x_1), \pi_{\xi_2}(x_2)))$$

or equivalently, with the notation $\underline{\lambda}$ for the constant $X - \{\lambda\}$-mapping, introduced in Part I, section 2,

$$(\forall x_1 \in X_1)(\pi_{\xi_1|\xi_2}(x_1 \mid \cdot) \overset{(\Pi_{\xi_2},\min)}{=} \underline{\pi_{\xi_1}(x_1)}). \tag{15}$$

A completely analogous result can be derived in probability theory (see, for instance, [Burrill, 1972] Theorem 15-3C for probability distribution functions), where the $(\Pi_{\xi_2}, \min)$-equivalence is replaced by the analogous almost everywhere equality, and therefore *not* by the ordinary pointwise equality of mappings. If we start from (15) instead of (9) for the definition of possibilistic independence, it turns out that Zadeh's noninteractivity and this new form of possibilistic independence *are indeed equivalent notions.* At the same time, the analogy with probability theory is followed more closely than was the case with Hisdal's definition.

[3] It is easily verified that something similar holds for density and frequency functions.

Hisdal's problem is therefore an artificial one, which could arise only because a probabilistic starting point was chosen for extension by analogy that was not entirely correct. This exposes the sore spot of some of the existing measure-theoretic accounts of conditional possibility: they are inspired by the analogy with conditional probability, but within possibility theory itself do not have the necessary measure- and integral-theoretic apparatus to fully exploit this analogy. It is my aim in this paper and its sequel to develop a theory of conditional possibility and conditional independence using the analogy with the theory of probability. In doing so, I shall use the measure- and integral-theoretic techniques and notions developed in Part I. In this way, the theory of conditional possibility and possibilistic independence, existing in the literature, is embedded in a broad measure- and integral-theoretic context, and at the same time corrected in certain places, as indicated above. This will serve as further evidence for the claim that the use of possibility integrals allows a uniform and consistent construction of a theory of possibility, and the harmonious unification of many results extant in the literature.

1.6 Conditional Probability

The course of reasoning followed in this paper is rather abstract, and draws its inspiration from the classical introduction of conditional probability in a measure- and integral-theoretic context. Let me very succinctly summarize this procedure. In Kolmogorov's probability theory, the starting point of the abstract measure- and integral-theoretic approach is a particular *integral equation* in a real stochastic variable. Solutions of this equation are given the name '*conditional expectation*'. *Conditional probabilities* are then defined as special cases of conditional expectations. For both notions, a number of properties can be proven that justify the use of this terminology. I also want to point out that there exist *different types* of conditional expectations and probabilities, and that each type is defined using a different integral equation. For a thorough account of the measure- and integral-theoretic approach to conditional expectation and probability, I refer for instance to [Burrill, 1972].

1.7 Sugeno's Approach

As far as I know, Sugeno is the only one who has used a purely measure- and integral-theoretic approach to the introduction of conditionality in what could be very generally called the theory of fuzzy sets. In his doctoral dissertation [Sugeno, 1974], he introduces the notion of a *conditional fuzzy measure*, using a certain type of integral equation. Let me give a brief account of his course of reasoning. For a more detailed account of this approach, and explicit definitions of the notions mentioned below, I refer to [Sugeno, 1974].

Let $(X_1, \mathcal{S}_1, v_{\xi_1})$ and $(X_2, \mathcal{S}_2, v_{\xi_2})$ be fuzzy measure spaces, where the fuzzy measure v_{ξ_k} contains information about the values a variable ξ_k takes in the universe X_k, $k = 1, 2$. Any solution of the integral equation

$$(\forall A_1 \in \mathcal{S}_1) \left(\ooalign{$\int$\cr$-$} _{X_2} f \circ v_{\xi_2} = v_{\xi_1}(A_1) \right) \tag{16}$$

in the $\mathcal{S}_2$-measurable $([0,1], \leq)$-fuzzy set f in X_2, is given the notation $\rho(A_1 \mid \cdot)$. This means that

$$(\forall A_1 \in \mathcal{S}_1) \left(\ooalign{$\int$\cr$-$} _{X_2} \rho(A_1 \mid \cdot) \circ v_{\xi_2} = v_{\xi_1}(A_1) \right). \tag{17}$$

For any x_2 in X_2 the $\mathcal{S}_1 - [0,1]$-mapping $\rho(\cdot \mid x_2)$ is called by Sugeno the *conditional fuzzy measure of X_2 to X_1*. He furthermore shows that these conditional fuzzy measures satisfy analogous properties – in the sense of almost everywhere equality – as his fuzzy measures, which justifies his terminology.

1.8 My Approach to Conditional Possibility

For the introduction of conditional possibility in this paper I shall not draw on Sugeno's work about conditional fuzzy measures, which, to my knowledge, up to now has not been worked out in full detail. My only source of inspiration will be the existing and fully developed measure-theoretic account of conditional probability. In what follows, I introduce *three related types of conditional possibility.* For each type, I start with a particular integral equation in a fuzzy variable. The fuzzy variables which solve these integral equations are given the name '*conditional possibility*', mainly because they satisfy a number of properties which cannot but remind us of possibility measures. The distinction that is made in probability theory between conditional expectation and conditional probability can in principle also be maintained here as a distinction between generalized (conditional) possibility of fuzzy events and ordinary (conditional) possibility of ordinary events, where the latter can be viewed as a special case of the former (see also Part I, section 3). Since I have chosen for a uniform approach and nomenclature for both ordinary and fuzzy events, I shall in both cases simply use the term 'conditional possibility'.

1.9 An Overview of This Paper

Starting with a first type of integral equation, the course of reasoning in section 2 leads to the introduction and study of the conditional possibility of a fuzzy variable (fuzzy event) – or in particular of a measurable set (event) – given an ample field of sets. In section 3 a second type of integral equation allows the definition of the conditional possibility of a fuzzy variable, and in particular of a measurable set, given that a second fuzzy variable assumes a particular value. From this second brand of conditional possibility, I deduce a number of special cases: the conditional possibility that a fuzzy variable assumes a particular value given that a second fuzzy variable assumes a certain value; the conditional possibility of a measurable set given a second measurable set, etc. The two above-mentioned main types of conditional possibility of fuzzy variables and measurable sets are, from a formal point of view, fairly analogous to two well-known types of conditional probability of real stochastic variables and measurable sets.

In section 4 I extend the treatment of the conditional possibility of fuzzy variables and measurable sets towards that of possibilistic variables in general. This allows me to incorporate the existing results in the field of conditional possibility into a measure- and integral-theoretic treatment.

In order to allow the reader to follow the analogy between this approach and the probabilistic one, I have given in what follows, where possible and suitable, explicit references to the relevant probabilistic literature.

Let me conclude this general overview with a number of general notational conventions, valid in the rest of this paper, unless explicitly stated to the contrary. By P we shall denote a triangular seminorm on $(L, \leq)$, such that $(L, \leq, P)$ is a complete lattice with t-seminorm. In addition, P is everywhere assumed to be *weakly left-invertible* (see Part I, section 2).

2 CONDITIONAL POSSIBILITY OF FUZZY EVENTS AND EVENTS: TYPE 1

In this section, let us denote by $(X, \mathcal{R}, \Pi)$ a $(L, \leq)$-possibility space. The possibility distribution of Π is denoted by π. $\mathcal{D}$ is an ample field on X that is coarser than $\mathcal{R}$, or, in other words, $\mathcal{D} \subseteq \mathcal{R}$.

In the first theorem, we discuss the existence and uniqueness of solutions of the integral equation (18), which will further on lead to the introduction of a first type of conditional possibility (probabilistic counterpart: [Burrill, 1972] section 15-1, Eq. (4)).

Theorem 2.1. *The restriction $\Pi|\mathcal{D}$ of Π to $\mathcal{D}$ is a $(L, \leq)$-possibility measure on $(X, \mathcal{D})$. Furthermore, let h be a $(L, \leq)$-fuzzy variable in $(X, \mathcal{R})$. Then there exists a $(L, \leq)$-fuzzy variable g in $(X, \mathcal{D})$ satisfying*

$$(\forall D \in \mathcal{D}) \left((P) \!\!\!\!\!\!\!\!\;\;\; \int\!\!\!\!\!\!- _D g \mathrm{d}(\Pi|\mathcal{D}) = (P) \!\!\!\!\!\!\!\!\;\;\; \int\!\!\!\!\!\!- _D h \mathrm{d}\Pi \right). \tag{18}$$

Any solution g of this integral equation is unique in the sense of $(\Pi|\mathcal{D}, P)$-equivalence.

Proof. It is readily verified that $\Pi|\mathcal{D}$ is a $(L, \leq)$-possibility measure on $(X, \mathcal{D})$. We shall therefore show that there exists a $X - L$-mapping g that is $\mathcal{D}$-measurable and satisfies (18). Consider the $\mathcal{R} - L$-mapping Ψ defined as

$$(\forall E \in \mathcal{R}) \left(\Psi(E) = (P) \!\!\!\!\!\!\!\!\;\;\; \int\!\!\!\!\!\!- _E h \mathrm{d}\Pi \right).$$

Then, taking into account Part I, Proposition 7.1, Ψ is a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$. The restriction $\Phi = \Psi|\mathcal{D}$ of Ψ to the ample field $\mathcal{D}$ is of course a $(L, \leq)$-possibility measure on $(X, \mathcal{D})$. The integral equation (18) can now also be written as

$$(\forall D \in \mathcal{D}) \left((P) \!\!\!\!\!\!\!\!\;\;\; \int\!\!\!\!\!\!- _D g \mathrm{d}(\Pi|\mathcal{D}) = \Phi(D) \right).$$

Furthermore, we have for any D in $\mathcal{D}$, taking into account Part I, Eq. (5) and the $\mathcal{R}$-measurability of h, that

$$\Phi(D) = \sup_{x \in D} P(h(x), \pi(x)) \leq \sup_{x \in D} \pi(x) = \Pi(D) = (\Pi|\mathcal{D})(D).$$

Since P is assumed to be weakly invertible, the existence of the solution g and its uniqueness in the sense of $(\Pi|\mathcal{D}, P)$-equivalence is implied by Part I, Theorem 7.2. □

This theorem paves the way for the following definition.

Definition 2.2. Let h be a $(L, \leq)$-fuzzy variable in $(X, \mathcal{R})$. Any member of the equivalence class of solutions of the integral equation (18) in the $(L, \leq)$-fuzzy variable g in $(X, \mathcal{D})$ is denoted by $\Pi(h \mid \mathcal{D})$. This means that

$$(\forall D \in \mathcal{D}) \left((P) \!\!\!\!\!\!\!\!\;\;\; \int\!\!\!\!\!\!- _D \Pi(h \mid \mathcal{D}) \mathrm{d}(\Pi|\mathcal{D}) = (P) \!\!\!\!\!\!\!\!\;\;\; \int\!\!\!\!\!\!- _D h \mathrm{d}\Pi \right).$$

$\Pi(h \mid \mathcal{D})$ is called the conditional $(L, \leq, P)$-possibility of h when $\mathcal{D}$ is given. For any A in $\mathcal{R}$, $\Pi(\chi_A \mid \mathcal{D})$ is also written as $\Pi(A \mid \mathcal{D})$ and is called the conditional $(L, \leq, P)$-possibility of A when $\mathcal{D}$ is given. Taking into account Part I, Eqs. (8) and (9), this means that

$$(\forall D \in \mathcal{D}) \left((P) \fint_D \Pi(A \mid \mathcal{D}) \mathrm{d}(\Pi|\mathcal{D}) = \Pi(D \cap A) \right).$$

If, for whatever reason, we do not want to mention the structure $(L, \leq, P)$ explicitly, we simply use the name 'conditional possibility' instead of 'conditional $(L, \leq, P)$-possibility'.

In the proposition below, we derive a formula that will allow us to calculate such conditional possibilities explicitly. The left-residuals, discussed briefly in Part I, section 2, and more in detail in [De Cooman and Kerre, 1994], play the same role here as the division in probability theory.

Proposition 2.3. *Let h be a $(L, \leq)$-fuzzy variable in $(X, \mathcal{R})$ and let A be an element of $\mathcal{R}$. Then*

(i) $\Pi(h \mid \mathcal{D}) \overset{(\Pi|\mathcal{D},P)}{=} (P) \fint_{[\,\cdot\,]_\mathcal{D}} h \mathrm{d}\Pi \lhd_P \Pi([\,\cdot\,]_\mathcal{D})$;

(ii) $\Pi(A \mid \mathcal{D}) \overset{(\Pi|\mathcal{D},P)}{=} \Pi([\,\cdot\,]_\mathcal{D} \cap A) \lhd_P \Pi([\,\cdot\,]_\mathcal{D})$.

The mappings on the right hand sides of (i) and (ii) are also the greatest members, w.r.t. the relation $\sqsubseteq$ on $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$, of the respective equivalence classes of the equivalence relation $\overset{(\Pi|\mathcal{D},P)}{=}$ they belong to. In other words, they are the greatest solutions of the corresponding integral equations.

Proof. Statement (ii) is an immediate consequence of (i), taking into account Part I, Eqs. (8) and (9). It therefore suffices to prove (i). Let g be any $(L, \leq)$-fuzzy variable in $(X, \mathcal{D})$. It is easily verified that the integral equation (18) is equivalent to

$$(\forall x \in X) \left(P(g(x), \Pi([\,x\,]_\mathcal{D})) = \sup_{y \in [\,x\,]_\mathcal{D}} P(h(y), \pi(y)) \right). \tag{19}$$

Since P is assumed to be weakly invertible, and furthermore for any x in X

$$\sup_{y \in [\,x\,]_\mathcal{D}} P(h(y), \pi(y)) \leq \sup_{y \in [\,x\,]_\mathcal{D}} \pi(y) = \Pi([\,x\,]_\mathcal{D}),$$

the proof is complete if we take into account Part I, Propositions 2.1 and 2.2. □

In Theorem 2.4 we see that conditional possibilities behave in a certain sense as ordinary normal possibility measures (probabilistic counterpart: [Burrill, 1972] Theorem 15-3A). I want to stress here that the 'equalities' which appear in this theorem are $(\Pi|\mathcal{D}, P)$-equivalences of fuzzy variables in $(X, \mathcal{D})$. Indeed, the conditional possibilities introduced in Definition 2.2 are fuzzy variables. Since these are only defined up to $(\Pi|\mathcal{D}, P)$-equivalence, it is absolutely normal that the equality of two conditional possibilities is expressed by means of $(\Pi|\mathcal{D}, P)$-equality of fuzzy variables instead of the normal pointwise equality of mappings.

Theorem 2.4. *(i)* $\Pi(\emptyset \mid \mathcal{D}) \overset{(\Pi|\mathcal{D},P)}{=} \underline{0_L}$ *and* $\Pi(X \mid \mathcal{D}) \overset{(\Pi|\mathcal{D},P)}{=} \underline{1_L}$.

(ii) Let $\{h_j \mid j \in J\}$ *be a family of elements of* $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$. *Then*

$$\Pi(\sup_{j\in J} h_j \mid \mathcal{D}) \stackrel{(\Pi|\mathcal{D},P)}{=} \sup_{j\in J} \Pi(h_j \mid \mathcal{D}).$$

(iii) Let $\{A_j \mid j \in J\}$ *be a family of elements of* $\mathcal{R}$. *Then*

$$\Pi(\bigcup_{j\in J} A_j \mid \mathcal{D}) \stackrel{(\Pi|\mathcal{D},P)}{=} \sup_{j\in J} \Pi(A_j \mid \mathcal{D}).$$

Proof. Statement (i) immediately follows from Part I, Corollary 6.5. Statement (iii) follows from (ii). Let us therefore prove (ii). To this end, let D be any element of $\mathcal{D}$. We then have by definition that, taking into account Part I, Eq. (11),

$$\begin{aligned}
(P)\!\!\!\!\!\!\!\!\;\;\;\; \text{-}\!\!\!\!\!\!\int_D \Pi(\sup_{j\in J} h_j \mid \mathcal{D})\mathrm{d}(\Pi|\mathcal{D}) &= (P)\text{-}\!\!\!\!\!\!\int_D \sup_{j\in J} h_j \mathrm{d}\Pi \\
&= \sup_{j\in J} (P)\text{-}\!\!\!\!\!\!\int_D h_j \mathrm{d}\Pi \\
&= \sup_{j\in J} (P)\text{-}\!\!\!\!\!\!\int_D \Pi(h_j \mid \mathcal{D})\mathrm{d}(\Pi|\mathcal{D}) \\
&= (P)\text{-}\!\!\!\!\!\!\int_D \sup_{j\in J} \Pi(h_j \mid \mathcal{D})\mathrm{d}(\Pi|\mathcal{D}).
\end{aligned}$$

Part I, Proposition 6.4(iii) now implies (ii). □

3 CONDITIONAL POSSIBILITY OF FUZZY EVENTS AND EVENTS: TYPE 2

In this section, we take a closer look at a second and more familiar type of conditional possibility for (fuzzy) events. We denote by $(X, \mathcal{R}, \Pi)$ a $(L, \leq)$-possibility space. π is the possibility distribution of Π.

3.1 Definition and Important Properties

In Theorem 3.1 we discuss the existence and uniqueness of solutions of the integral equation (20), which leads in Definition 3.2 to the introduction of a second type of conditional possibility (probabilistic counterpart: [Burrill, 1972] subsection 15-1.2, Eq. (9)). For the notations used here, I refer to Part I, section 5.

Theorem 3.1. *Let h and g be $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$. Then there exists a $L - L$-mapping f satisfying*

$$(\forall B \in \wp(L)) \left((P)\text{-}\!\!\!\!\!\!\int_B f \mathrm{d}\Gamma_g = (P)\text{-}\!\!\!\!\!\!\int_{g^{-1}(B)} h \mathrm{d}\Pi \right). \tag{20}$$

Any solution f of this integral equation is unique in the sense of (Γ_g, P)-equivalence.

Proof. Define the $\wp(L) - L$-mapping Φ as

$$(\forall B \in \wp(L)) \left(\Phi(B) = (P) \int_{g^{-1}(B)} h \mathrm{d}\Pi \right).$$

This allows us to rewrite Eq. (20) as

$$(\forall B \in \wp(L)) \left((P) \int_{B} f \mathrm{d}\Gamma_g = \Phi(B) \right).$$

It is furthermore easily proven that Φ is a $(L, \leq)$-possibility measure on $(L, \wp(L))$, using the properties of the inverse image of a mapping and a course of reasoning that is analogous to the one followed in the proof of Part I, Proposition 7.1. Also, we have for any B in $\wp(L)$ that, taking into account Part I, Eq. (5) and the $\mathcal{R}$-measurability of h,

$$\Phi(B) = \sup_{x \in g^{-1}(B)} P(h(x), \pi(x)) \leq \sup_{x \in g^{-1}(B)} \pi(x) = \Pi(g^{-1}(B)) = \Gamma_g(B).$$

Since the t-seminorm P on $(L, \leq)$ is assumed to be weakly left-invertible, the Radon-Nikodym-like Theorem 7.2 in Part I completes the proof. $\square$

Definition 3.2. Let h and g be $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$. Any member of the equivalence class of solutions of the integral equation (20) in the $L - L$-mapping f is denoted by $\Pi(h \mid g = \cdot)$. This means that

$$(\forall B \in \wp(L)) \left((P) \int_{B} \Pi(h \mid g = \cdot) \mathrm{d}\Gamma_g = (P) \int_{g^{-1}(B)} h \mathrm{d}\Pi \right).$$

For any λ in L we call $\Pi(h \mid g = \lambda)$ the conditional $(L, \leq, P)$-possibility of h, given that g takes the value λ. For any A in $\mathcal{R}$ the $L - L$-mapping $\Pi(\chi_A \mid g = \cdot)$ is also written as $\Pi(A \mid g = \cdot)$. For any λ in L we call $\Pi(A \mid g = \lambda)$ the conditional $(L, \leq, P)$-possibility of A given that g takes the value λ. Taking into account Part I, Eqs. (8) and (9), this means that

$$(\forall B \in \wp(L)) \left((P) \int_{B} \Pi(A \mid g = \cdot) \mathrm{d}\Gamma_g = \Pi(A \cap g^{-1}(B)) \right).$$

If, for whatever reason, we do not want to mention the structure $(L, \leq, P)$ explicitly, we simply use the name 'conditional possibility' instead of 'conditional $(L, \leq, P)$-possibility'.

As before, we can find explicit formulas for the calculation of these conditional possibilities, using the notion of a left-residual. The proof of this result is completely analogous to the proof of Proposition 2.3, and is therefore omitted.

Proposition 3.3. *Let h and g be elements of $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$ and let A be an element of $\mathcal{R}$. Then*

(i) $\Pi(h \mid g = \cdot) \stackrel{(\Gamma_g, P)}{=} (P) \int_{g^{-1}(\{\cdot\})} h \mathrm{d}\Pi \lhd_P \gamma_g(\cdot)$;

(ii) $\Pi(A \mid g = \cdot) \stackrel{(\Gamma_g, P)}{=} \Pi(A \cap g^{-1}(\{\cdot\})) \lhd_P \gamma_g(\cdot)$.

The mappings on the right hand sides of (i) and (ii) are also the greatest members, w.r.t. the relation $\sqsubseteq$ on $\mathcal{F}_{(L,\leq)}(L)$, of the respective equivalence classes of the equivalence relation $\stackrel{(\Gamma_g,P)}{=}$ they belong to. In other words, they are the greatest solutions of the corresponding integral equations.

In Theorem 3.4, we see that the conditional possibilities of this type also behave in a certain sense as ordinary normal possibility measures (probabilistic counterpart: [Burrill, 1972] Theorem 15-3B). Let me emphasize that the 'equalities' which appear in this theorem, are (Γ_g, P)-equivalences of $L - L$-mappings. They are of course in general less stringent than the pointwise equalities of these mappings.

Theorem 3.4. *Let g be a $(L,\leq)$-fuzzy variable in $(X,\mathcal{R})$.*

(i) $\Pi(\emptyset \mid g = \cdot) \stackrel{(\Gamma_g,P)}{=} \underline{0_L}$ *and* $\Pi(X \mid g = \cdot) \stackrel{(\Gamma_g,P)}{=} \underline{1_L}$.

(ii) Let $\{h_j \mid j \in J\}$ be a family of elements of $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$. Then

$$\Pi(\sup_{j\in J} h_j \mid g = \cdot) \stackrel{(\Gamma_g,P)}{=} \sup_{j\in J} \Pi(h_j \mid g = \cdot).$$

(iii) Let $\{A_j \mid j \in J\}$ be a family of elements of $\mathcal{R}$. Then

$$\Pi(\bigcup_{j\in J} A_j \mid g = \cdot) \stackrel{(\Gamma_g,P)}{=} \sup_{j\in J} \Pi(A_j \mid g = \cdot).$$

Proof. Statement (i) immediately follows from Part I, Corollary 6.5. It is easily seen that (iii) follows from (ii). Let us therefore prove (ii). Let B be an arbitrary element of $\wp(L)$. Then, by definition, taking into account Part I, Eq. (11),

$$\begin{aligned}
(P)\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\;\int_B \Pi(\sup_{j\in J} h_j \mid g = \cdot)\mathrm{d}\Gamma_g &= (P)\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\;\int_{g^{-1}(B)} \sup_{j\in J} h_j \mathrm{d}\Pi, \\
&= \sup_{j\in J}\, (P)\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\;\int_{g^{-1}(B)} h_j \mathrm{d}\Pi \\
&= \sup_{j\in J}\, (P)\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\;\int_B \Pi(h_j \mid g = \cdot)\mathrm{d}\Gamma_g \\
&= (P)\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\;\int_B \sup_{j\in J} \Pi(h_j \mid g = \cdot)\mathrm{d}\Gamma_g.
\end{aligned}$$

It now follows from Part I, Proposition 6.4(iii) that (ii) holds. □

It appears from Theorem 3.5 that there exists a natural relationship between the two types of conditional possibilities that we have up to now defined (probabilistic counterpart: [Burrill, 1972] Theorem 15-1K). In order to uncover this relationship, let us first introduce a new notion. Let g be an arbitrary $X - L$-mapping. We denote by $\tau(g)$ the smallest ample field on X w.r.t. which g is still measurable, i.e., $\tau(g) = \tau(\{\, g^{-1}(B) \mid B \in \wp(L)\,\})$. It is easily proven that for any A in $\wp(X)$, $A \in \tau(g) \Leftrightarrow g^{-1}(g(A)) = A$.

Theorem 3.5. *Let h and g be elements of $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$. Consider the $L - L$-mapping α defined by $\alpha(\lambda) = \Pi(h \mid g = \lambda)$, $\lambda \in L$. Then $\Pi(h \mid g = g(\cdot)) = \alpha \circ g \stackrel{(\Pi|\tau(g),P)}{=} \Pi(h \mid \tau(g))$.*

Proof. Since g is $\mathcal{R}$-measurable, we know by definition that $\tau(g) \subseteq \mathcal{R}$. Let D be an arbitrary element of $\tau(g)$. Then $D = g^{-1}(g(D))$. Since $g(D) \in \wp(L)$, we may write on the one hand that, by definition,

$$(P)\!\!\!\!\!\!\!\;\;\int_{g(D)} \Pi(h \mid g = \cdot)\mathrm{d}\Gamma_g = (P)\!\!\!\!\!\!\!\;\;\int_D h\mathrm{d}\Pi = (P)\!\!\!\!\!\!\!\;\;\int_D \Pi(h \mid \tau(g))\mathrm{d}(\Pi|\tau(g)).$$

On the other hand, taking into account Part I, Eq. (5) and the $\tau(g)$-measurability of $\alpha \circ g$,

$$\begin{aligned}(P)\!\!\!\!\!\!\!\;\;\int_D (\alpha \circ g)\mathrm{d}(\Pi|\tau(g)) &= \sup_{y \in D} P((\alpha \circ g)(y), \Pi([\,y\,]_{\tau(g)}))\\ &= \sup_{y \in D} P((\alpha \circ g)(y), \sup_{x \in [\,y\,]_{\tau(g)}} \pi(x))\\ &= \sup_{y \in D} \sup_{x \in [\,y\,]_{\tau(g)}} P((\alpha \circ g)(y), \pi(x))\end{aligned}$$

and once again using the $\tau(g)$-measurability of $\alpha \circ g$, and Part I, Eq. (1), since $D \in \tau(g)$,

$$\begin{aligned}&= \sup_{x \in \bigcup_{y \in D} [\,y\,]_{\tau(g)}} P((\alpha \circ g)(x), \pi(x))\\ &= \sup_{x \in D} P((\alpha \circ g)(x), \pi(x)).\end{aligned}$$

Again, since $\alpha \circ g$ is $\tau(g)$-measurable and therefore also $\mathcal{R}$-measurable, it follows from Part I, Eq. (5) that

$$= (P)\!\!\!\!\!\!\!\;\;\int_D (\alpha \circ g)\mathrm{d}\Pi.$$

The right hand side of this equality can be further transformed by invoking the integral transport formula (Part I, Theorem 4.4), with the following correspondences: $X_1 \to X$, $X_2 \to L$, $f \to g$, $\mathcal{R}_1 \to \mathcal{R}$, ${\mathcal{R}_1}^{(f)} \to \mathcal{R}^{(g)} = \wp(L)$, $\Pi_1 \to \Pi$, ${\Pi_1}^{(f)} \to \Pi^{(g)} = \Gamma_g$, $h \to \alpha$, $E \to g(D)$, $f^{-1}(E) \to g^{-1}(g(D)) = D$. This leads to

$$(P)\!\!\!\!\!\!\!\;\;\int_D (\alpha \circ g)\mathrm{d}(\Pi|\tau(g)) = (P)\!\!\!\!\!\!\!\;\;\int_{g(D)} \alpha\mathrm{d}\Pi^{(g)} = (P)\!\!\!\!\!\!\!\;\;\int_{g(D)} \Pi(h \mid g = \cdot)\mathrm{d}\Gamma_g.$$

We conclude that for any D in $\tau(g)$

$$(P)\!\!\!\!\!\!\!\;\;\int_D (\alpha \circ g)\mathrm{d}(\Pi|\tau(g)) = (P)\!\!\!\!\!\!\!\;\;\int_D \Pi(h \mid \tau(g))\mathrm{d}(\Pi|\tau(g)).$$

which completes the proof, taking into account Part I, Proposition 6.4(iii). □

3.2 Some Special Cases

The conditional possibilities introduced in this section have a number of important special cases, which we shall presently investigate. The first case is singled out in Definition 3.6 (probabilistic counterpart: [Burrill, 1972] subsection 15-3.2, Eq. (13)). In Corollary 3.7 we show that these special cases also behave to a certain extent as ordinary normal possibility measures.

Definition 3.6. Let h and g be $(L,\leq)$-fuzzy variables $(X,\mathcal{R})$. We introduce the following mappings, for any λ in L and any B in $\wp(L)$:

$$\gamma_{h|g}(\lambda \mid \cdot)\colon L \to L\colon \mu \mapsto \Pi(h^{-1}(\{\lambda\}) \mid g = \mu)$$
$$\Gamma_{h|g}(B \mid \cdot)\colon L \to L\colon \mu \mapsto \Pi(h^{-1}(B) \mid g = \mu).$$

For any μ in L,

(i) $\gamma_{h|g}(\lambda \mid \mu) = \Pi(h^{-1}(\{\lambda\}) \mid g = \mu)$ is called the conditional $(L,\leq,P)$-possibility that h takes the value λ given that g takes the value μ;

(ii) $\Gamma_{h|g}(B \mid \mu) = \Pi(h^{-1}(B) \mid g = \mu)$ is called the conditional $(L,\leq,P)$-possibility that h takes a value in B given that g takes the value μ.

If, for whatever reason, we do not want to mention the structure $(L,\leq,P)$ explicitly, we simply use the name 'conditional possibility' instead of 'conditional $(L,\leq,P)$-possibility'.

Corollary 3.7. *Let h and g be $(L,\leq)$-fuzzy variables in $(X,\mathcal{R})$.*

(i) $\Gamma_{h|g}(\emptyset \mid \cdot) \overset{(\Gamma_g,P)}{=} \underline{0_L}$ *and* $\Gamma_{h|g}(L \mid \cdot) \overset{(\Gamma_g,P)}{=} \underline{1_L}$.

(ii) For any family $\{B_j \mid j \in J\}$ of elements of $\wp(L)$

$$\Gamma_{h|g}(\bigcup_{j\in J} B_j \mid \cdot) \overset{(\Gamma_g,P)}{=} \sup_{j\in J} \Gamma_{h|g}(B_j \mid \cdot).$$

(iii) For any B in $\wp(L)$

$$\Gamma_{h|g}(B \mid \cdot) \overset{(\Gamma_g,P)}{=} \sup_{\lambda\in B} \gamma_{h|g}(\lambda \mid \cdot).$$

Proof. Statements (i) and (ii) of this corollary are special cases of Theorem 3.4(i) and (iii), taking into account the properties of the inverse image of a mapping. Statement (iii) can be easily derived from (ii). □

We should not lose sight of the following: the mappings $\Pi(h^{-1}(A) \mid g = \cdot)$, for A in $\wp(L)$, underlying this new type of conditional possibility, are only determined up to (Γ_g, P)-equivalence. The same must therefore hold for the conditional possibilities introduced in Definition 3.6. In the next theorem, we show that these can also be considered as solutions of a particular integral equation, which is of course a special case of (20): for any h and g in $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$ and for any A in $\wp(L)$

$$(\forall B \in \wp(L)) \left((P)\!\!\!\!\!\!\int_B f \mathrm{d}\Gamma_g = \Gamma_{(h,g)}(A \times B) \right), \tag{21}$$

where, of course, the $L-L$-mapping f is the unknown. We might just as well have defined these conditional possibilities as arbitrary members of the equivalence class of solutions of this integral equation. Both approaches are clearly equivalent.

Let me also mention that the notations for these new notions are intended to remind the reader of the possibility distributions and distribution functions of fuzzy variables, introduced in Part I. Theorem 3.8 also tells us that there exists an important relationship between all of these notions (probabilistic counterpart: [Burrill, 1972] Example 15-3B, Theorems 15-3C and 15-3D). This observation may serve as a justification for the notations used here.

Theorem 3.8. *(i) For any A and B in $\wp(L)$:*

$$(P)\!\!\int_B \Gamma_{h|g}(A \mid \cdot)\mathrm{d}\Gamma_g = \Gamma_{(h,g)}(A \times B).$$

(ii) For any λ and μ in L:

$$P(\gamma_{h|g}(\lambda \mid \mu), \gamma_g(\mu)) = \gamma_{(h,g)}(\lambda, \mu).$$

Proof. Statement (ii) is an immediate consequence of (i), with $A = \{\lambda\}$ and $B = \{\mu\}$. Let us therefore prove (i). Consider arbitrary A and B in $\wp(L)$. Then, by definition,

$$\begin{aligned}(P)\!\!\int_B \Gamma_{h|g}(A \mid \cdot)\mathrm{d}\Gamma_g &= (P)\!\!\int_B \Pi(h^{-1}(A) \mid g = \cdot)\mathrm{d}\Gamma_g \\ &= \Pi(h^{-1}(A) \cap g^{-1}(B)) \\ &= \Pi((h,g)^{-1}(A \times B)) \\ &= \Gamma_{(h,g)}(A \times B). \quad \square\end{aligned}$$

Let us now look at a second special case. Consider arbitrary events D and E in $\mathcal{R}$. The characteristic $X-L$-mappings χ_D and χ_E are of course $(L,\leq)$-fuzzy variables in $(X,\mathcal{R})$. It follows from Theorem 3.8(ii) that for any λ and μ in L, the element $\gamma_{\chi_D|\chi_E}(\lambda \mid \mu)$ of L satisfies

$$P(\gamma_{\chi_D|\chi_E}(\lambda \mid \mu), \gamma_{\chi_E}(\mu)) = \gamma_{(\chi_D,\chi_E)}(\lambda, \mu).$$

If we choose $\lambda = \mu = 1_L$, this may, taking into account $\gamma_{\chi_E}(1_L) = \Pi(\chi_E^{-1}(\{1_L\})) = \Pi(E)$ and $\gamma_{(\chi_D,\chi_E)}(1_L, 1_L) = \Pi((\chi_D, \chi_E)^{-1}(\{(1_L, 1_L)\})) = \Pi(D \cap E)$, also be written as

$$P(\gamma_{\chi_D|\chi_E}(1_L \mid 1_L), \Pi(E)) = \Pi(D \cap E). \tag{22}$$

We now deduce the following definition from Definition 3.6 (probabilistic counterpart: [Burrill, 1972] chapter 15, Eq. (1)).

Definition 3.9. Let D and E be arbitrary elements of $\mathcal{R}$. We know that, by definition,

$$\gamma_{\chi_D|\chi_E}(1_L \mid 1_L) = \Pi(\chi_D^{-1}(\{1_L\}) \mid \chi_E = 1_L) = \Pi(D \mid \chi_E = 1_L).$$

We therefore call $\gamma_{\chi_D|\chi_E}(1_L \mid 1_L)$ the conditional $(L,\leq,P)$-possibility of D given E. $\gamma_{\chi_D|\chi_E}(1_L \mid 1_L)$ is also written as $\Pi(D \mid E)$, and (22) can therefore be rewritten as

$$P(\Pi(D \mid E), \Pi(E)) = \Pi(D \cap E).$$

If, for whatever reason, we do not want to mention the structure $(L,\leq,P)$ explicitly, we simply use the name 'conditional possibility' instead of 'conditional $(L,\leq,P)$-possibility'.

Again, it should be noted that $\Pi(D \mid E)$ has been defined using the $L-L$-mapping $\Pi(D \mid \chi_E = \cdot)$, which is only determined up to (Γ_{χ_E}, P)-equivalence. Another, completely equivalent, approach would consist in defining $\Pi(D \mid E)$ as an arbitrary member of the set of solutions of the equation

$$P(\nu, \Pi(E)) = \Pi(D \cap E)$$

in the element ν of L. Note the correspondence between this equation and Eq. (12) used by Dubois and Prade, Shafer, and Ramer to define conditional possibility for events.

3.3 A Possibilistic Counterpart for Bayes' Theorem

We can use the two types of conditional possibilities defined thus far to derive a possibilistic counterpart for Bayes' theorem. Let us consider a partition $\mathcal{A}$ of the universe X that is $\mathcal{R}$-measurable, i.e., $\mathcal{A} \subset \mathcal{R}$. Then clearly $\tau(\mathcal{A}) \subseteq \mathcal{R}$, so that, for any A in $\mathcal{R}$, we may consider the conditional possibility $\Pi(A \mid \tau(\mathcal{A}))$ of A given $\tau(\mathcal{A})$. Remark that the elements of the partition $\mathcal{A}$ are the atoms of the ample field $\tau(\mathcal{A})$. By Definition 2.2, we find that for any B in $\tau(\mathcal{A})$, since $\Pi(A \mid \tau(\mathcal{A}))$ is a $\tau(\mathcal{A})$-measurable $X - L$-mapping:

$$\begin{aligned}
\Pi(A \cap B) &= (P)\int_B \Pi(A \mid \tau(\mathcal{A}))\mathrm{d}(\Pi|\tau(\mathcal{A})) \\
&= \sup_{x \in B} P(\Pi(A \mid \tau(\mathcal{A}))(x), \Pi([\, x\,]_{\tau(\mathcal{A})})) \\
&= \sup_{D \in \mathcal{A}, D \subseteq B} \sup_{x \in D} P(\Pi(A \mid \tau(\mathcal{A}))(x), \Pi(D)) \\
&= \sup_{D \in \mathcal{A}, D \subseteq B} P(\kappa(A, D), \Pi(D)),
\end{aligned}$$

where we have denoted the constant value of $\Pi(A \mid \tau(\mathcal{A}))$ on D by $\kappa(A, D)$. As a special case, we find for $B = D \in \mathcal{A}$ that $\Pi(A \cap D) = P(\kappa(A, D), \Pi(D))$ and Definition 3.9 tells us that on the other hand $\Pi(A \cap D) = P(\Pi(A \mid D), \Pi(D))$. Therefore,

$$\Pi(A \cap B) = \sup_{D \in \mathcal{A}, D \subseteq B} P(\Pi(A \mid D), \Pi(D)).$$

and in particular for $B = X$,

$$\Pi(A) = \sup_{D \in \mathcal{A}} P(\Pi(A \mid D), \Pi(D)),$$

the possibilistic equivalent of the *total probability rule*. On the other hand, again by Definition 3.9, we find for any E in $\mathcal{A}$ that $\Pi(A \cap E) = P(\Pi(E \mid A), \Pi(A)) = P(\Pi(A \mid E), \Pi(E))$. The greatest value of $\Pi(E \mid A)$ is therefore given by

$$\Pi(E \mid A) = P(\Pi(A \mid E), \Pi(E)) \lhd_P \sup_{D \in \mathcal{A}} P(\Pi(A \mid D), \Pi(D))$$

which is clearly a possibilistic counterpart for Bayes' theorem, where sup assumes the role of addition, P the role of multiplication, and $\lhd_P$ the role of division.

3.4 An Interesting Example

In this paper, I give an abstract treatment of conditional possibility which is based upon the analogy with probability theory. The following example provides a justification for this approach from an interpretational point of view. It shows that in the case of *classical possibility* this abstract treatment leads to results which have a natural intuitive interpretation.

Classical possibility, in a sense, has been the starting point and source of inspiration for the introduction of more general forms of possibility. It has a straightforward interpretation. Indeed, consider an experiment E, the outcome o of which can take values in a universe X. The elements of the ample field $\mathcal{R}$ on X are the measurable sets (or events) associated with the universe. Consider an event A, different from $\emptyset$. If we know for certain that the outcome of the experiment must belong to A – i.e., that the event A occurs –, we can represent this information by the $\mathcal{R} - \{0,1\}$-mapping Π_A, defined by

$$(\forall B \in \mathcal{R}) \left(\Pi_A(B) = \begin{cases} 1 & ; \quad A \cap B \neq \emptyset \\ 0 & ; \quad A \cap B = \emptyset \end{cases} \right).$$

For any B in $\mathcal{R}$, $\Pi_A(B)$ is the possibility of the occurrence of the event B, based upon the information that A occurs with certainty: if $\Pi_A(B) = 1$ the occurrence of B is possible, and if $\Pi_A(B) = 0$ the occurrence of B is impossible. Of course, Π_A is a $(\{0,1\}, \leq)$-possibility measure on $(X, \mathcal{R})$. The complete Boolean chain $(\{0,1\}, \leq)$ is completely determined by $0 < 1$. On this structure, there exists only one triangular (semi)norm: the meet or Boolean multiplication $\wedge$. The structure $(\{0,1\}, \leq, \wedge)$ is a complete chain with t-norm [De Cooman and Kerre, 1994].

Let us now consider two elements B and C of $\mathcal{R}$ and look for solutions $\Pi_A(B \mid C)$ of the particular equation for conditional possibility

$$\Pi_A(B \cap C) = \Pi_A(B \mid C) \wedge \Pi_A(C), \tag{23}$$

which is a special case of (22). This equation is equivalent with

$$\begin{cases} 1 & ; \quad A \cap B \cap C \neq \emptyset \\ 0 & ; \quad A \cap B \cap C = \emptyset \end{cases} = \Pi_A(B \mid C) \wedge \begin{cases} 1 & ; \quad A \cap C \neq \emptyset \\ 0 & ; \quad A \cap C = \emptyset, \end{cases}$$

whence

$$\Pi_A(B \mid C) = \begin{cases} 1 & ; \quad A \cap B \cap C \neq \emptyset \text{ and } A \cap C \neq \emptyset \\ 0 & ; \quad A \cap B \cap C = \emptyset \text{ and } A \cap C \neq \emptyset \\ 0 \text{ or } 1 & ; \quad A \cap C = \emptyset. \end{cases}$$

How can this result be interpreted? Whenever the occurrence of C is impossible – $A \cap C = \emptyset$, or equivalently $\Pi_A(C) = 0$ – Eq. (23) imposes no restrictions on the values $\Pi_A(B \mid C)$ can assume in $\{0,1\}$. When, however, the occurrence of C is possible – $A \cap C \neq \emptyset$, or equivalently $\Pi_A(C) = 1$ – we must distinguish between two possible cases:

- $A \cap B \cap C = \emptyset$; in this case the events B and C cannot occur together, and we must have that $\Pi_A(B \mid C) = 0$;
- $A \cap B \cap C \neq \emptyset$; in this case B and C can occur together, and we must have that $\Pi_A(B \mid C) = 1$.

The interpretation of these results is straightforward. If $A \cap C \neq \emptyset$, $\Pi_A(B \mid C)$ may be considered as the *possibility that B occurs, given that C occurs* – keeping in mind the information that the occurrence of the event A is certain. When indeed in this case $A \cap B \cap C = \emptyset$ and we know that C occurs, it immediately follows that the outcome o of the experiment E *cannot* belong to B. This is in agreement with $\Pi_A(B \mid C) = 0$. When on the other hand $A \cap B \cap C \neq \emptyset$ and we know that C occurs, it is still perfectly possible that the outcome o of the experiment belongs to B. This corresponds with $\Pi_A(B \mid C) = 1$.

Finally, if $A \cap C = \emptyset$, the event C cannot occur, and it therefore strictly speaking makes little sense to talk about the possibility that B occurs given that C occurs. This is reflected in the fact that in this case Eq. (23) imposes no restriction whatsoever on the values of $\Pi_A(B \mid C)$. By the way, a similar interpretational difficulty occurs in the case of conditional probability.

4 CONDITIONAL POSSIBILITY OF POSSIBILISTIC VARIABLES

In this section we study a third type of conditional possibility, which is, in fact, a generalization of the second. In this generalization, possibilistic variables play the part of the fuzzy variables that appear, for instance, in Definition 3.6. It is precisely this new type of conditional possibility which can serve as a formalization of the conditional possibilities introduced by Zadeh and Hisdal. Many results in this paragraph are related to the results of Zadeh [1978], Hisdal [1978], Dubois and Prade [1984], briefly summarized in section 1. How these results are arrived at, is however very different in both approaches. To give an example, Zadeh and others use a more or less intuitive notion of a variable, whereas I prefer a formal definition: in the view presented here, a variable is a measurable mapping from a basic space to a sample space (see also the discussion in Part I, subsection 4.2). Furthermore, I define conditional possibilities for these variables starting from a particular type of integral equation, and closely follow the treatment in the previous sections. The approach of Zadeh and others could be considered as a first, more intuitive attempt to introduce conditional possibility. The measure- and integral-theoretic results, developed in the first paper of this series, allows me to give a more systematic and formal treatment, which solves some of the problems of the more intuitive approach.

4.1 Introductory Remarks and Definitions

In this section, Ω is a universe and $\mathcal{R}_\Omega$ an ample field of subsets of Ω. We consider $(\Omega, \mathcal{R}_\Omega)$ as a *basic space*, provided with a $(L, \leq)$-possibility measure Π_Ω with distribution π_Ω.

X_1 and X_2 are two universes, considered as *sample spaces*. $\mathcal{R}_1$ is an ample field on X_1 and $\mathcal{R}_2$ an ample field on X_2. We also consider the product ample field $\mathcal{R}_1 \times \mathcal{R}_2$ of $\mathcal{R}_1$ and $\mathcal{R}_2$, which is an ample field on the Cartesian product $X_1 \times X_2$ of the universes X_1 and X_2 (see Part I, section 2).

Furthermore, we consider a $\Omega - X_1$-mapping f_1 and a $\Omega - X_2$-mapping f_2. f_1 is assumed to be $\mathcal{R}_\Omega - \mathcal{R}_1$-measurable and is therefore a possibilistic variable in $(X_1, \mathcal{R}_1)$. Likewise f_2 is assumed to be $\mathcal{R}_\Omega - \mathcal{R}_2$-measurable, and is therefore a possibilistic variable in $(X_2, \mathcal{R}_2)$.

Finally, we consider the following well-known projection operators

$$\begin{aligned}\mathrm{proj}_1 &: X_1 \times X_2 \to X_1 : (x_1, x_2) \mapsto x_1\\ \mathrm{proj}_2 &: X_1 \times X_2 \to X_2 : (x_1, x_2) \mapsto x_2.\end{aligned}$$

We start this discussion with the following observation: besides X_1 and X_2, the universe $X_1 \times X_2$ can also be considered as a sample space, and besides the mappings f_1 and f_2 we may consider the $\Omega - X_1 \times X_2$-mapping (f_1, f_2). The following proposition tells us that this mapping is a possibilistic variable in $(X_1 \times X_2, \mathcal{R}_1 \times \mathcal{R}_2)$.

Proposition 4.1. *(f_1, f_2) is $\mathcal{R}_\Omega - \mathcal{R}_1 \times \mathcal{R}_2$-measurable if and only if f_1 is $\mathcal{R}_\Omega - \mathcal{R}_1$-measurable and f_2 is $\mathcal{R}_\Omega - \mathcal{R}_2$-measurable.*

Proof. Assume on the one hand that f_1 is $\mathcal{R}_\Omega - \mathcal{R}_1$-measurable and that f_2 is $\mathcal{R}_\Omega - \mathcal{R}_2$-measurable. We must show that $(f_1, f_2)^{-1}(\mathcal{R}_1 \times \mathcal{R}_2) = \{\, (f_1, f_2)^{-1}(B) \mid B \in \mathcal{R}_1 \times \mathcal{R}_2 \,\} \subseteq \mathcal{R}_\Omega$. If we write $\mathcal{A} = \{\, B_1 \times B_2 \mid B_1 \in \mathcal{R}_1 \text{ and } B_2 \in \mathcal{R}_2 \,\}$, then we have by definition that

$$\begin{aligned}(f_1, f_2)^{-1}(\mathcal{A}) &= \{\, (f_1, f_2)^{-1}(B_1 \times B_2) \mid B_1 \in \mathcal{R}_1 \text{ and } B_2 \in \mathcal{R}_2 \,\}\\ &= \{\, f_1^{-1}(B_1) \cap f_2^{-1}(B_2) \mid B_1 \in \mathcal{R}_1 \text{ and } B_2 \in \mathcal{R}_2 \,\}\end{aligned}$$

and, since by assumption $f_1^{-1}(\mathcal{R}_1) \subseteq \mathcal{R}_\Omega$ and $f_2^{-1}(\mathcal{R}_2) \subseteq \mathcal{R}_\Omega$, and since furthermore the ample field $\mathcal{R}_\Omega$ is closed under intersections, it follows that $(f_1, f_2)^{-1}(\mathcal{A}) \subseteq \mathcal{R}_\Omega$. Taking into account the properties of the closure operator τ (see Part I, section 2), we deduce that $\tau((f_1, f_2)^{-1}(\mathcal{A})) \subseteq \tau(\mathcal{R}_\Omega) = \mathcal{R}_\Omega$, and since the inverse image preserves complements, and arbitrary unions and intersections, this can be rewritten as $(f_1, f_2)^{-1}(\tau(\mathcal{A})) \subseteq \mathcal{R}_\Omega$. By definition, $\tau(\mathcal{A}) = \mathcal{R}_1 \times \mathcal{R}_2$, whence $(f_1, f_2)^{-1}(\mathcal{R}_1 \times \mathcal{R}_2) \subseteq \mathcal{R}_\Omega$, and therefore (f_1, f_2) is $\mathcal{R}_\Omega - \mathcal{R}_1 \times \mathcal{R}_2$-measurable.

Conversely, assume that (f_1, f_2) is $\mathcal{R}_\Omega - \mathcal{R}_1 \times \mathcal{R}_2$-measurable. We will show that f_1 is $\mathcal{R}_\Omega - \mathcal{R}_1$-measurable. The proof of the $\mathcal{R}_\Omega - \mathcal{R}_2$-measurability of f_2 is completely analogous. Consider an arbitrary element B_1 of $\mathcal{R}_1$. Since $B_1 \times X_2$ belongs to $\mathcal{R}_1 \times \mathcal{R}_2$, we have, by assumption, that

$$\mathcal{R}_\Omega \ni (f_1, f_2)^{-1}(B_1 \times X_2) = f_1^{-1}(B_1) \cap f_2^{-1}(X_2) = f_1^{-1}(B_1) \cap \Omega = f_1^{-1}(B_1). \quad \square$$

With the possibilistic variables f_1, f_2 and (f_1, f_2) we can associate possibility distributions and possibility distribution functions. They are obtained by transforming the possibility measure Π_Ω using these variables (see Part I, subsection 4.1). From the assumptions made above and Proposition 4.1 we easily deduce that $\mathcal{R}_1 \subseteq \mathcal{R}_\Omega{}^{(f_1)}$, $\mathcal{R}_2 \subseteq \mathcal{R}_\Omega{}^{(f_2)}$ and $\mathcal{R}_1 \times \mathcal{R}_2 \subseteq \mathcal{R}_\Omega{}^{((f_1,f_2))}$. This allows the introduction of the following possibility measures: $\Pi_{f_1} = \Pi_\Omega{}^{(f_1)}|\mathcal{R}_1$, $\Pi_{f_2} = \Pi_\Omega{}^{(f_2)}|\mathcal{R}_2$ and $\Pi_{(f_1,f_2)} = \Pi_\Omega{}^{((f_1,f_2))}|\mathcal{R}_1 \times \mathcal{R}_2$. Π_{f_1} is the transformed $(L, \leq)$-possibility measure on $(X_1, \mathcal{R}_1)$ of Π_Ω and analogously for the $(L, \leq)$-possibility measures Π_{f_2} on $(X_2, \mathcal{R}_2)$ and $\Pi_{(f_1,f_2)}$ on $(X_1 \times X_2, \mathcal{R}_1 \times \mathcal{R}_2)$. These possibility measures contain information about the values that the respective possibilistic variables f_1, f_2 and (f_1, f_2) assume in the universes X_1, X_2 and $X_1 \times X_2$ respectively. The distributions of these possibility measures are denoted respectively by π_{f_1}, π_{f_2} and $\pi_{(f_1,f_2)}$. Following the spirit of Zadeh's work and the terminology introduced in Part I, Definition 4.3, we call Π_{f_1}, Π_{f_2} and $\Pi_{(f_1,f_2)}$ the possibility distributions of f_1, f_2 and (f_1, f_2); π_{f_1}, π_{f_2} and $\pi_{(f_1,f_2)}$ are called the possibility distribution functions of these possibilistic variables.

For these possibility measures, a number of interesting results may be derived, provided we use the projection operators proj_1 and proj_2. The foundation for these results is laid down in the next proposition, using the notations established in Part I, subsection 4.1.

Proposition 4.2. $(\mathcal{R}_1 \times \mathcal{R}_2)^{(\mathrm{proj}_1)} = \mathcal{R}_1$ *and* $(\mathcal{R}_1 \times \mathcal{R}_2)^{(\mathrm{proj}_2)} = \mathcal{R}_2$.

Proof. We only give the proof of the first equality. The proof of the second equality is completely analogous. By definition,

$$\begin{aligned}(\mathcal{R}_1 \times \mathcal{R}_2)^{(\mathrm{proj}_1)} &= \{\, A_1 \mid A_1 \in \wp(X_1) \text{ and } \mathrm{proj}_1^{-1}(A_1) \in \mathcal{R}_1 \times \mathcal{R}_2 \,\} \\ &= \{\, A_1 \mid A_1 \in \wp(X_1) \text{ and } A_1 \times X_2 \in \mathcal{R}_1 \times \mathcal{R}_2 \,\}.\end{aligned}$$

Consider an arbitrary A_1 in $\wp(X_1)$. Assume on the one hand that $A_1 \in \mathcal{R}_1$. It follows from the definition of $\mathcal{R}_1 \times \mathcal{R}_2$ (see Part I, Eq. (2)) that $A_1 \times X_2 \in \mathcal{R}_1 \times \mathcal{R}_2$, whence $A_1 \in (\mathcal{R}_1 \times \mathcal{R}_2)^{(\mathrm{proj}_1)}$. Therefore $\mathcal{R}_1 \subseteq (\mathcal{R}_1 \times \mathcal{R}_2)^{(\mathrm{proj}_1)}$.

On the other hand, assume that $A_1 \in (\mathcal{R}_1 \times \mathcal{R}_2)^{(\mathrm{proj}_1)}$. This implies that $A_1 \times X_2 \in \mathcal{R}_1 \times \mathcal{R}_2$. Taking into account Part I, Eq. (1), this implies that

$$A_1 \times X_2 = \bigcup_{(x_1,x_2) \in A_1 \times X_2} [\,(x_1, x_2)\,]_{\mathcal{R}_1 \times \mathcal{R}_2}$$

and, taking into account Part I, Eq. (3) and the associativity of union,

$$
\begin{aligned}
&= \bigcup_{x_1 \in A_1} \bigcup_{x_2 \in X_2} [\, x_1 \,]_{\mathcal{R}_1} \times [\, x_2 \,]_{\mathcal{R}_2} \\
&= \left(\bigcup_{x_1 \in A_1} [\, x_1 \,]_{\mathcal{R}_1} \right) \times \left(\bigcup_{x_2 \in X_2} [\, x_2 \,]_{\mathcal{R}_2} \right) \\
&= \left(\bigcup_{x_1 \in A_1} [\, x_1 \,]_{\mathcal{R}_1} \right) \times X_2.
\end{aligned}
$$

It follows that $A_1 = \bigcup_{x_1 \in A_1} [\, x_1 \,]_{\mathcal{R}_1}$ whence, taking into account Part I, Eq. (1), $A_1 \in \mathcal{R}_1$. This implies that $(\mathcal{R}_1 \times \mathcal{R}_2)^{(\mathrm{proj}_1)} \subseteq \mathcal{R}_1$. □

In very much the same way as $\mathcal{R}_1$ and $\mathcal{R}_2$ can be obtained through 'projection' of $\mathcal{R}_1 \times \mathcal{R}_2$, the projection operators allow us to transform $\Pi_{(f_1,f_2)}$ into Π_{f_1} and Π_{f_2}. Indeed, for any A_1 in $\mathcal{R}_1$

$$
\begin{aligned}
\Pi_{f_1}(A_1) &= \Pi_\Omega{}^{(f_1)}(A_1) \\
&= \Pi_\Omega(f_1^{-1}(A_1)) \\
&= \Pi_\Omega((f_1, f_2)^{-1}(A_1 \times X_2))
\end{aligned}
$$

and, since $A_1 \times X_2 \in \mathcal{R}_1 \times \mathcal{R}_2$,

$$
\begin{aligned}
&= \Pi_{(f_1,f_2)}(A_1 \times X_2) \\
&= \Pi_{(f_1,f_2)}(\mathrm{proj}_1^{-1}(A_1)) \\
&= \Pi_{(f_1,f_2)}{}^{(\mathrm{proj}_1)}(A_1).
\end{aligned}
$$

Similarly, we find for any element A_2 of $\mathcal{R}_2$ that $\Pi_{f_2}(A_2) = \Pi_{(f_1,f_2)}{}^{(\mathrm{proj}_2)}(A_2)$. We conclude, also taking into account the previous proposition, that $\Pi_{f_1} = \Pi_{(f_1,f_2)}{}^{(\mathrm{proj}_1)}$ and $\Pi_{f_2} = \Pi_{(f_1,f_2)}{}^{(\mathrm{proj}_2)}$. This can be reformulated in the following proposition, a formalization of Eqs. (1)–(4).

Proposition 4.3. *(i)* $(\forall A_1 \in \mathcal{R}_1)(\Pi_{f_1}(A_1) = \Pi_{(f_1,f_2)}(A_1 \times X_2))$.

(ii) $(\forall A_2 \in \mathcal{R}_2)(\Pi_{f_2}(A_2) = \Pi_{(f_1,f_2)}(X_1 \times A_2))$.

(iii) $(\forall x_1 \in X_1)(\pi_{f_1}(x_1) = \sup_{x_2 \in X_2} \pi_{(f_1,f_2)}(x_1, x_2))$.

(iv) $(\forall x_2 \in X_2)(\pi_{f_2}(x_2) = \sup_{x_1 \in X_1} \pi_{(f_1,f_2)}(x_1, x_2))$.

4.2 Conditional Possibility

We now have enough information to begin the discussion of conditional possibility of possibilistic variables. In Theorem 4.4, we investigate the existence and uniqueness of the solutions of the integral equation (24). This result leads to the introduction of a new type of conditional possibility in Definition 4.5. It should be noted that (24) is a generalization of (21) from fuzzy variables to possibilistic ones.

Theorem 4.4. *For every A_1 in $\mathcal{R}_1$ there exists a $(L,\leq)$-fuzzy variable h in $(X_2,\mathcal{R}_2)$ satisfying*

$$(\forall A_2 \in \mathcal{R}_2)\left((P)\!\!\int_{A_2} h \mathrm{d}\Pi_{f_2} = \Pi_{(f_1,f_2)}(A_1 \times A_2)\right). \tag{24}$$

Any solution h of this integral equation is unique in the sense of (Π_{f_2}, P)-equivalence.

Proof. Let A_1 be any element of $\mathcal{R}_1$. We define the $\mathcal{R}_2 - L$-mapping Φ_{A_1} as

$$(\forall A_2 \in \mathcal{R}_2)(\Phi_{A_1}(A_2) = \Pi_{(f_1,f_2)}(A_1 \times A_2)).$$

Φ_{A_1} is clearly a $(L,\leq)$-possibility measure on $(X_2,\mathcal{R}_2)$. Furthermore, since $A_1 \times A_2 \subseteq X_1 \times A_2$, we find that

$$\Phi_{A_1}(A_2) = \Pi_{(f_1,f_2)}(A_1 \times A_2) \leq \Pi_{(f_1,f_2)}(X_1 \times A_2) = \Pi_{f_2}(A_2),$$

also using Proposition 4.3(ii). Since P is assumed to be weakly left-invertible, the Radon-Nikodym-like Theorem 7.2 in Part I completes the proof. $\square$

Definition 4.5. Let A_1 be an element of $\mathcal{R}_1$. Any member of the equivalence class of solutions of the integral equation (24) in the $(L,\leq)$-fuzzy variable h in $(X_2,\mathcal{R}_2)$ is denoted by $\Pi_{f_1|f_2}(A_1 \mid \cdot)$. This means that

$$(\forall A_2 \in \mathcal{R}_2)\left((P)\!\!\int_{A_2} \Pi_{f_1|f_2}(A_1 \mid \cdot) \mathrm{d}\Pi_{f_2} = \Pi_{(f_1,f_2)}(A_1 \times A_2)\right).$$

For any x_2 in X_2, $\Pi_{f_1|f_2}(A_1 \mid x_2)$ is called the conditional $(L,\leq,P)$-possibility that f_1 takes a value in A_1 given that f_2 takes a value in $[\,x_2\,]_{\mathcal{R}_2}$. For any x_1 in X_1, $\Pi_{f_1|f_2}([\,x_1\,]_{\mathcal{R}_1} \mid x_2)$ is also written as $\pi_{f_1|f_2}(x_1 \mid x_2)$. If, for whatever reason, we do not want to mention the structure $(L,\leq,P)$ explicitly, we simply use the name 'conditional possibility' instead of 'conditional $(L,\leq,P)$-possibility'.

In the following proposition, we derive a formula for the explicit calculation of these conditional possibilities. Here again, use is made of left-residuals. I want to point out that statement (ii) of this proposition justifies and at the same time generalizes Eq. (14).

Proposition 4.6. *Let A_1 be an element of $\mathcal{R}_1$ and let x_1 be an element of X_1. Then*

(i) $\Pi_{f_1|f_2}(A_1 \mid \cdot) \overset{(\Pi_{f_2},P)}{\triangleq} \Pi_{(f_1,f_2)}(A_1 \times [\,\cdot\,]_{\mathcal{R}_2}) \lhd_P \pi_{f_2}(\cdot)$;

(ii) $\pi_{f_1|f_2}(x_1 \mid \cdot) \overset{(\Pi_{f_2},P)}{\triangleq} \pi_{(f_1,f_2)}(x_1, \cdot) \lhd_P \pi_{f_2}(\cdot)$.

The mappings on the right hand sides of (i) and (ii) are also the greatest members, w.r.t. the relation $\sqsubseteq$ on $\mathcal{G}^{\mathcal{R}_2}_{(L,\leq)}(X_2)$, of the respective equivalence classes of the equivalence relation $\overset{(\Pi_{f_2},P)}{\triangleq}$ they belong to. In other words, they are the greatest solutions of the corresponding integral equations.

Proof. Statement (ii) is an immediate consequence of (i). It therefore suffices to prove (i). Let h be any $(L,\leq)$-fuzzy variable in $(X_2,\mathcal{R}_2)$. It is easily verified that (24) is equivalent with

$$(\forall x_2 \in X_2)(P(h(x_2), \pi_{f_2}(x_2)) = \Pi_{(f_1,f_2)}(A_1 \times [\,x_2\,]_{\mathcal{R}_2})). \tag{25}$$

It should furthermore be remembered that P is weakly left-invertible. On the other hand, for any x_2 in X_2, taking into account Proposition 4.3(iv),

$$\pi_{f_2}(x_2) = \sup_{x_1 \in X_1} \pi_{(f_1,f_2)}(x_1,x_2) \geq \sup_{x_1 \in A_1} \pi_{(f_1,f_2)}(x_1,x_2) = \Pi_{(f_1,f_2)}(A_1 \times [\,x_2\,]_{\mathcal{R}_2}).$$

Propositions 2.1 and 2.2 in Part I now complete the proof. $\square$

Let (x_1,x_2) be an element of $X_1 \times X_2$. For $A_1 = [\,x_1\,]_{\mathcal{R}_1}$ and $A_2 = [\,x_2\,]_{\mathcal{R}_2}$ it follows from Definition 4.5 that

$$\pi_{(f_1,f_2)}(x_1,x_2) = P(\pi_{f_1|f_2}(x_1 \mid x_2), \pi_{f_2}(x_2)). \tag{26}$$

This formula generalizes Eqs. (7) and (11), used respectively by Hisdal, and Dubois and Prade to introduce conditional possibility distributions. It should also be stressed that the $\pi_{f_1|f_2}(x_1 \mid x_2)$ occurring in (26) are generalizations of the $\gamma_{h|g}(\lambda \mid \mu)$ that for instance appear in Theorem 3.8(ii).

In particular, we also deduce from Definition 4.5 and Proposition 4.3(i) that

$$(\forall A_1 \in \mathcal{R}_1)\left((P)\!\!\!\!\!\!\!\!\;\;\;\;\;\;\;\int\!\!\!\!\!\!-_{X_2} \Pi_{f_1|f_2}(A_1 \mid \cdot)\mathrm{d}\Pi_{f_2} = \Pi_{f_1}(A_1)\right),$$

which reminds us of Sugeno's formula (17) for conditional fuzzy measures.

In Theorem 4.7, we show that conditional possibilities of the third type also to a certain extent behave as ordinary normal possibility measures would. The 'equalities' that appear in these results are not pointwise equalities of mappings, but rather (Π_{f_2},P)-equivalences of fuzzy variables in $(X_2,\mathcal{R}_2)$. In this context, I want to remind the reader that precisely in this fact lies my solution to Hisdal's problem, as discussed in section 1.

Theorem 4.7. *(i)* $\Pi_{f_1|f_2}(\emptyset \mid \cdot) \stackrel{(\Pi_{f_2},P)}{=} \underline{0_L}$ *and* $\Pi_{f_1|f_2}(X_1 \mid \cdot) \stackrel{(\Pi_{f_2},P)}{=} \underline{1_L}$.

(ii) Let $\{A_j \mid j \in J\}$ *be a family of elements of* $\mathcal{R}_1$*. Then*

$$\Pi_{f_1|f_2}(\bigcup_{j\in J} A_j \mid \cdot) \stackrel{(\Pi_{f_2},P)}{=} \sup_{j\in J} \Pi_{f_1|f_2}(A_j \mid \cdot).$$

(iii) For any A_1 *in* $\mathcal{R}_1$*:*

$$\Pi_{f_1|f_2}(A_1 \mid \cdot) \stackrel{(\Pi_{f_2},P)}{=} \sup_{x_1 \in A_1} \pi_{f_1|f_2}(x_1 \mid x_2).$$

Proof. Statement (i) immediately follows from Part I, Corollary 6.5 and Proposition 4.3(ii). Statement (iii) is an immediate consequence of (ii). Let us therefore prove (ii). Let B be any element of $\mathcal{R}_2$. Then, by definition, and taking into account Part I, Eq. (11),

$$\begin{aligned}
(P)\int\!\!\!\!\!\!-_{B} \Pi_{f_1|f_2}(\bigcup_{j\in J} A_j \mid \cdot)\mathrm{d}\Pi_{f_2} &= \Pi_{(f_1,f_2)}((\bigcup_{j\in J} A_j) \times B) \\
&= \Pi_{(f_1,f_2)}(\bigcup_{j\in J}(A_j \times B)) \\
&= \sup_{j\in J} \Pi_{(f_1,f_2)}(A_j \times B) \\
&= \sup_{j\in J} (P)\int\!\!\!\!\!\!-_{B} \Pi_{f_1|f_2}(A_j \mid \cdot)\mathrm{d}\Pi_{f_2} \\
&= (P)\int\!\!\!\!\!\!-_{B} \sup_{j\in J} \Pi_{f_1|f_2}(A_j \mid \cdot)\mathrm{d}\Pi_{f_2}.
\end{aligned}$$

Proposition 6.4(iii) in Part I now completes the proof. $\square$

We conclude this section with a theorem that exposes yet another important relationship between the conditional possibilities discussed here, and the second type of conditional possibility, introduced in the previous section.

Theorem 4.8. *Let A_1 be an element of $\mathcal{R}_1$. Consider for any x_2 in X_2 the element*

$$\begin{aligned}\gamma_{\chi_{A_1\times X_2}|\chi_{X_1\times[x_2]_{\mathcal{R}_2}}}(1_L \mid 1_L) &= \Pi_{(f_1,f_2)}(\chi^{-1}_{A_1\times X_2}(\{1_L\}) \mid \chi_{X_1\times[x_2]_{\mathcal{R}_2}} = 1_L)\\ &= \Pi_{(f_1,f_2)}(A_1\times X_2 \mid \chi_{X_1\times[x_2]_{\mathcal{R}_2}} = 1_L)\\ &= \Pi_{(f_1,f_2)}(A_1\times X_2 \mid X_1\times[x_2]_{\mathcal{R}_2})\end{aligned}$$

of L (see also mutatis mutandis Definitions 3.6 and 3.9, where the possibility measure $\Pi_{(f_1,f_2)}$ must be substituted for the possibility measure Π). This enables us to introduce the $X_2 - L$-mapping α:

$$(\forall x_2\in X_2)(\alpha(x_2) = \gamma_{\chi_{A_1\times X_2}|\chi_{X_1\times[x_2]_{\mathcal{R}_2}}}(1_L \mid 1_L)).$$

Then α is a $(L,\leq)$-fuzzy variable in $(X_2,\mathcal{R}_2)$, and

$$\Pi_{f_1|f_2}(A_1\mid\cdot) \overset{(\Pi_{f_2},P)}{=} \alpha = \Pi_{(f_1,f_2)}(A_1\times X_2 \mid X_1\times[\cdot]_{\mathcal{R}_2}).$$

Proof. It is perfectly clear that α is $\mathcal{R}_2$-measurable. Let therefore A_2 be any element of $\mathcal{R}_2$. By definition,

$$(P)\!\!\!\!\!\!\;\;\int_{A_2}\Pi_{f_1|f_2}(A_1\mid\cdot)\mathrm{d}\Pi_{f_2} = \Pi_{(f_1,f_2)}(A_1\times A_2) = \sup_{x_2\in A_2}\Pi_{(f_1,f_2)}(A_1\times[x_2]_{\mathcal{R}_2}).$$

Furthermore,

$$\Pi_{(f_1,f_2)}(A_1\times[x_2]_{\mathcal{R}_2}) = \Pi_{(f_1,f_2)}(A_1\times X_2\cap X_1\times[x_2]_{\mathcal{R}_2})$$

and it now follows from (22), if we substitute $A_1\times X_2$ for D, $X_1\times[x_2]_{\mathcal{R}_2}$ for E and $\Pi_{(f_1,f_2)}$ for Π and take into account Proposition 4.3(ii), that

$$\begin{aligned}&= P(\gamma_{\chi_{A_1\times X_2}|\chi_{X_1\times[x_2]_{\mathcal{R}_2}}}(1\mid 1),\Pi_{f_2}([x_2]_{\mathcal{R}_2}))\\ &= P(\alpha(x_2),\pi_{f_2}(x_2)).\end{aligned}$$

Taking into account Part I, Eq. (5), this implies that

$$(P)\int_{A_2}\Pi_{f_1|f_2}(A_1\mid\cdot)\mathrm{d}\Pi_{f_2} = \sup_{x_2\in A_2}P(\alpha(x_2),\pi_{f_2}(x_2)) = (P)\int_{A_2}\alpha\mathrm{d}\Pi_{f_2}.$$

Proposition 6.4(iii) in Part I now completes the proof. $\square$

5 CONCLUSION

A prominent feature of the conditional possibilities defined here, is that they are not necessarily uniquely defined, but only up to almost everywhere equality. A careful inspection of the measure-theoretic literature on probability theory [Burrill, 1972] will show that the same holds for conditional probabilities and expectations. However, in probability theory, this implies that conditional expectations are (as stochastic variables) uniquely determined, except on a set with zero probability. Because the notion of almost everywhere equality is somewhat different in the possibilistic case (see the discussion in Part I, section 6) the nondeterminacy may be more apparent for conditional possibilities, but *it has essentially the same origin!* In principle, there is no fundamental difference between conditional possibilities, and conditional probabilities and expectations as far as their indeterminacy is concerned. As in probability theory, this indeterminacy is of no consequence, because conditional possibilities are to my knowledge in the end never used *per se*, but are eventually always combined with 'ordinary possibilities'. As a result of this combination, the indeterminacy disappears, as is apparent from the formulas in the previous sections. The nondeterminacy of conditional possibilities should therefore not be seen as something undesirable, that necessarily should be avoided or eliminated by imposing additional conditions in order to ensure uniqueness. In my opinion, it should simply be taken at face value, and dealt with using the proper mathematical care.

Indeed, the nondeterminacy of conditional possibilities has one important consequence, that should never be overlooked (as it should not in probability theory either). In this approach here, whenever, in any definition, we use a conditional possibility, we should allow for the fact that this conditional possibility is only determined up to almost everywhere equality. As a consequence, if we write down equalities between conditional possibilities, these cannot be functional pointwise equalities, but must always be the appropriate almost everywhere equalities (or equivalences). As we have seen, Hisdal's failure to appreciate this has led her to distinguish between Zadeh's notion of noninteractivity [Zadeh, 1978], and her own definition of possibilistic independence [Hisdal, 1978]. Her notion of possibilistic independence, by the way, makes little sense, precisely because in her definition she uses the pointwise equality of conditional and marginal possibilities, and not the proper almost everywhere equality, as she should have done, since her conditional possibilities are not uniquely defined. This example will illustrate that this measure- and integral-theoretic approach to possibility theory can solve a number of problems and inconsistencies in the literature.

On the other hand, it is of course always possible to invoke additional principles in order to eliminate the nonuniqueness of conditional possibilities [Dubois and Prade, 1990] [Ramer, 1989]. The point I have tried to make in this paper, however, is that this is *not necessary*, provided that this nonuniqueness is mathematically taken into account, in the way suggested above.

Besides, as Theorems 2.4, 3.4 and 4.7 and Corollary 3.7 indicate, the conditional possibilities defined here behave as normal possibilities in the sense of equivalence. If additional conditions are imposed to ensure their uniqueness, and equivalence is accordingly turned into strict equality, it will in general be a non-trivial problem to still make the corresponding conditional possibilities behave as normal possibilities. As an example, let us consider that most popular of additional requirements, the principle of minimum specificity [Dubois and Prade, 1990], which tells us that if there is nonunicity, we must take the maximal, and therefore least specific or least committal solution. If we borrow the notations from section 4, this together with proposition 4.6 tells us that, for any x_1 in X_1, x_2 in X_2 and A_1 in $\mathcal{R}_1$: $\Pi_{f_1|f_2}(A_1 \mid x_2) = \Pi_{(f_1,f_2)}(A_1 \times [\, x_2 \,]_{\mathcal{R}_2}) \lhd_P \pi_{f_2}(x_2)$

and $\pi_{f_1|f_2}(x_1 \mid x_2) = \pi_{(f_1,f_2)}(x_1, x_2) \lhd_P \pi_{f_2}(x_2)$. Now, to make these conditional possibilities behave as normal possibilities, it must among other things be that

$$\Pi_{f_1|f_2}(A_1 \mid x_2) = \sup_{x_1 \in A_1} \pi_{f_1|f_2}(x_1 \mid x_2).$$

This will in general be the case *if and only if* for any b in L and any family $\{a_j \mid j \in J\}$ of elements of L:

$$\sup_{j \in J}(a_j \lhd_P b) = (\sup_{j \in J} a_j) \lhd_P b. \tag{27}$$

It is for instance easily verified that (27) does not generally hold for the very popular choice $(L, \leq) = ([0,1], \leq)$, $P = \min$.

Moreover, if extra conditions are indeed imposed to ensure uniqueness, one must also make sure that these are consistent with the definition that is used for possibilistic independence, as I shall among other things explain in the third paper of this series.

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