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TOWARDS A POSSIBILISTIC LOGIC

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ABSTRACT

In this paper, we investigate how linguistic information can be incorporated into classical propositional logic. First, we show that Zadeh's extension principle can be justified and at the same time generalized by considerations about transformation of possibility measures. Using these results, we show how linguistic uncertainty about the truth value of a proposition leads to the introduction of the notion of a possibilistic truth value. Since propositions can be combined into new ones using logical operators, linguistic uncertainty about the truth values of the original propositions leads to linguistic uncertainty about the truth value of the resulting proposition. Furthermore, we show that in a number of special cases there is truth-functionality, i.e., the possibilistic truth value of the resulting proposition is a function of the possibilistic truth values of the original propositions. We are thus led to the introduction of possibilistic-logical functions, combining possibilistic truth values. Important classes of such functions, the possibilistic extension logics, directly result from the above-mentioned investigation, and are studied extensively. Finally, the relation between these logics, and Kleene's strong multi-valued systems is established.

1 INTRODUCTION

Classical propositional logic deals with *propositions*, or in other words, affirmative statements that are either true or false. Because the *truth value* of a proposition can only be either *true* or *false*, classical propositional logic is often called *two-valued*. Propositions can be combined into new ones, using so-called *logical operators*. Classical logic is *truth-functional*, because the behaviour of any logical operator, acting on propositions, can be characterized by a *logical function*, acting on truth values.

In the beginning of this century, Kleene and others (see, for instance, [25]) addressed an interesting problem. They considered propositions, which are *a priori* either true or false, but for which there is insufficient knowledge to determine precisely which value in $\{true, false\}$ the truth value assumes. They asked themselves how a mathematical, logical description of this situation could be constructed. Their solution consisted in the introduction of a number of *multi-valued logics*, i.e., logical systems with more than two truth values, called *strong* multi-valued Kleene logics. At least two aspects of Kleene's approach deserve extra attention in the context of this paper. First of all, the truth values used by Kleene are *epistemological*, because they are not the actual truth values of the propositions involved (*a priori true* or *false*), but rather reflect *our knowledge* about the actual truth values. Ideally therefore, Kleene truth values are mathematical representations of the uncertainty that exists about the actual truth values of the propositions involved. Secondly, Kleene logics are *truth-functional*. This means that according to these systems, the behaviour of a logical operator is mirrored in a logical function, combining Kleene truth values. To give an example, in these systems it is possible, from the Kleene truth values of two propositions, unequivocally to determine the Kleene truth value of their conjunction.

In the present work, we investigate the problem studied by Kleene in a special case, namely when the uncertainty about the actual truth values of propositions is *linguistic*. Linguistic uncertainty is the uncertainty contained in (or conveyed by) affirmative statements such as 'Wietse is less than one year old' or 'Linde's temperature is high'. Such statements give us information about, say, Wietse's age or Linde's temperature, but not enough to determine their actual value completely. Due to the imprecision and/or vagueness of the predicates involved ('less than one year old', 'high'), such statements convey information, but also contain uncertainty. In 1978, Zadeh proposed the use of *possibility measures* to mathematically represent linguistic information and, dually, uncertainty [30]. In our doctoral dissertation [9], we have generalized Zadeh's possibility measures, developed a general measure- and integral-theoretic account of possibility theory, and shown that possibility measures are indeed able to represent linguistic information (see also [4, 5, 6, 11, 12, 13, 14, 15]). On the basis of this work, we show in this paper how a possibilistic logic, representing linguistic uncertainty in classical propositional logic, comes about, and how under a number of additional assumptions, this possibilistic logic leads to Kleene's strong multi-valued systems. In doing so, we at the same time generalize and provide the possibilistic basis for previous work in this field, reported on in [16].

In section 2 we have collected the preliminary definitions and notational conventions, necessary for the proper understanding of the material in this paper. In section 3 we study the transmission of possibilistic (or linguistic) information by mappings, and at the same time give a possibilistic justification for Zadeh's extension principle. Using the material of this section, Kleene's problem is addressed in section 4. We consider propositions of the type '(subject) is (predicate)', and show how possibilistic uncertainty about the value the subject assumes in a universe of possible values leads to uncertainty about the truth value of the proposition. This uncertainty can be represented by a possibility distribution on the set $\{true, false\}$, called a possibilistic truth value. Note that possibilistic truth values are descriptions of the uncertainty about the actual truth value (in $\{true, false\}$), and are therefore of an epistemological nature. They are the possibilistic counterparts of the Kleene truth values. The next step consists in investigating how the possibilistic truth value of a combination of propositions, using a logical operator, can be calculated. It turns out that in a number of cases, we can associate with a logical operator a *possibilistic-logical function*, in such a way that the possibilistic truth value of the combination of propositions is the image under this function of the possibilistic truth values of the participating propositions. In other words, our possibilistic logic is indeed in those special cases truth-functional. It also turns out that the possibilistic-logical function involved is a generalized Zadeh extension of the (classical-)logical function associated with the logical operator. We are thus led in section 5 to the introduction of special systems of possibilistic-logical functions, the so-called possibilistic extension logics. We also study the properties of these logics. An interesting special case is considered in section 6, and leads to a possibilistic justification for the introduction of strong multi-valued Kleene logics. A number of conclusions and open research problems are formulated in section 7.

2 PRELIMINARY DEFINITIONS

Let us start this discussion with a few preliminary definitions and notational conventions, valid in the rest of this paper, unless explicitly stated to the contrary. By $(L, \leq)$ we shall mean a complete lattice that is arbitrary but *fixed throughout the whole text.* The bottom element of $(L, \leq)$ will be denoted by 0 and the top element by 1. We also assume that $0 \neq 1$. The meet of $(L, \leq)$ will be denoted by $\frown$, the join of $(L, \leq)$ by $\smile$.

2.1 Triangular Norms

A *triangular norm* (or, shortly, *t-norm*) T on the complete lattice $(L, \leq)$ is a binary operator on L that is isotonic, associative and commutative, and that furthermore satisfies the boundary condition $(\forall \lambda \in L)(T(1, \lambda) = \lambda)$. As a corollary, we have for such a t-norm T that $(\forall \lambda \in L)(T(0, \lambda) = 0)$. Of course, $\frown$ is a triangular norm on $(L, \leq)$, and more specifically, the only one that is idempotent. For a more involved discussion of triangular norms defined on complete lattices, and more in general, on bounded partially ordered sets, we refer to [14]. A t-norm T on $(L, \leq)$ is called *completely distributive* w.r.t. supremum iff for arbitrary λ in L and an arbitrary family $(\mu_j \mid j \in J)$ of elements of L:

$$T(\lambda, \sup_{j \in J} \mu_j) = \sup_{j \in J} T(\lambda, \mu_j).$$

In this case, the structure $(L, \leq, P)$ is called a *complete lattice with t-norm* [14]. In what follows, we shall always denote by T a t-norm on $(L, \leq)$ that is completely distributive w.r.t. supremum.

2.2 Ample Fields

An *ample field* $\mathcal{R}$ on the universe X is a set of subsets of X that is closed under *arbitrary* unions and intersections, and under complementation in X. A special ample field on X, and at the same time the largest, is the *power class* $\mathcal{P}(X)$ of X, i.e. the set of all the subsets of X. In this sense, ample fields on X can be considered as immediate generalizations of this power class. For a more thorough discussion of this subject, we refer to [13, 24]. The *atom* of $\mathcal{R}$ containing the element x of X will be denoted by $[\,x\,]_\mathcal{R}$ and is defined by:

$$[\,x\,]_\mathcal{R} \stackrel{\text{def}}{=} \bigcap \{\, A \mid x \in A \text{ and } A \in \mathcal{R}\,\}.$$

Remark that the atom of the ample field $\mathcal{P}(X)$ containing x is precisely the singleton $\{x\}$. Therefore, atoms of ample fields can be interpreted as generalizations of singletons. In this light, we also have that

$$(\forall x \in X)(\forall A \in \mathcal{R})(x \in A \Leftrightarrow [\,x\,]_\mathcal{R} \subseteq A).$$

Furthermore, for arbitrary x in X:

$$x \in [\,x\,]_{\mathcal{R}} \text{ and } [\,x\,]_{\mathcal{R}} \in \mathcal{R}. \tag{1.1}$$

A subset E of X will be called $\mathcal{R}$-*measurable* iff $E \in \mathcal{R}$. Interestingly,

$$E \in \mathcal{R} \Leftrightarrow E = \bigcup_{x \in E} [\,x\,]_E. \tag{1.2}$$

If we look at (1.2), we are led to the introduction of a $\mathcal{P}(X) - \mathcal{R}$-mapping $C_{\mathcal{R}}$, such that for arbitrary A in $\mathcal{P}(X)$:

$$C_{\mathcal{R}}(A) = \bigcup_{x \in A} [\,x\,]_{\mathcal{R}}.$$

It turns out that $C_{\mathcal{R}}$ is the closure operator on $\mathcal{P}(X)$, associated with the closure system $\mathcal{R}$ [3, 9, 13].

Consider an arbitrary subset $\mathcal{A}$ of $\mathcal{P}(X)$. Since the intersection of an arbitrary family of ample fields is again an ample field, we know that

$$\tau(\mathcal{A}) \stackrel{\text{def}}{=} \bigcap \{\, \mathcal{R} \mid \mathcal{R} \text{ is an ample field on } X \text{ and } \mathcal{A} \subseteq \mathcal{R} \,\}$$

is an ample field on X, called the *ample field generated by* $\mathcal{A}$. This notion can be used to introduce product ample fields. If we consider the universes X_1 and X_2 provided with the respective ample fields $\mathcal{R}_1$ and $\mathcal{R}_2$, then the *product ample field* on $X_1 \times X_2$ of $\mathcal{R}_1$ and $\mathcal{R}_1$ is defined as

$$\mathcal{R}_1 \times \mathcal{R}_2 \stackrel{\text{def}}{=} \tau(\{\, A_1 \times A_2 \mid A_1 \in \mathcal{R}_1 \text{ and } A_2 \in \mathcal{R}_2 \,\}).$$

Interestingly, for the atoms of $\mathcal{R}_1 \times \mathcal{R}_2$, it can be proven that

$$(\forall (x_1, x_2) \in X_1 \times X_2)([\,(x_1, x_2)\,]_{\mathcal{R}_1 \times \mathcal{R}_2} = [\,x_1\,]_{\mathcal{R}_1} \times [\,x_2\,]_{\mathcal{R}_2}),$$

which confirms our interpretation of an atom as a generalization of a singleton. Of course, these results are easily extended towards products of more than two ample fields.

If we consider two universe X and Y with respective ample fields $\mathcal{R}_X$ and $\mathcal{R}_Y$, then a $X - Y$-mapping f is called $\mathcal{R}_X - \mathcal{R}_Y$-*measurable* iff

$$(\forall B \in \mathcal{R}_Y)(f^{-1}(B) \in \mathcal{R}_X),$$

where $f^{-1}(B) = \{\, x \mid x \in X \text{ and } f(x) \in B \,\}$ is the *inverse image* of B under the mapping f.

2.3 Fuzzy Sets and Fuzzy Variables

With an arbitrary subset A of a universe X, we can associate its *characteristic* $X-L$ *mapping* χ_A, defined by

$$\chi_A(x) \stackrel{\text{def}}{=} \begin{cases} 1 & ; \quad x \in A \\ 0 & ; \quad x \in \text{co}A. \end{cases}$$

In accordance with the terminology introduced by Goguen [20], an arbitrary $X-L$ mapping will be called a $(L,\leq)$-*fuzzy set* (or simply fuzzy set) in X. It is an obvious generalization of a characteristic $X-L$-mapping. The set of the $(L,\leq)$-fuzzy sets in X will be denoted by $\mathcal{F}_{(L,\leq)}(X)$. A $([0,1],\leq)$-fuzzy set in X will also be called a *Zadeh fuzzy set*, and the set of all Zadeh fuzzy sets in X will be denoted by $\mathcal{F}(X)$. A $(L,\leq)$-fuzzy set h will be called *sup-normal* iff $\sup_{x \in X} h(x) = 1$.

A $X-L$ mapping h will be called $\mathcal{R}$-*measurable* iff it is constant on the atoms of $\mathcal{R}$. A $\mathcal{R}$-measurable $X-L$ mapping—or $(L,\leq)$-fuzzy set in X—is also called a $(L,\leq)$-*fuzzy variable* in $(X,\mathcal{R})$. Whenever we want to omit reference to the structures $(L,\leq)$ and $(X,\mathcal{R})$, we shall simply speak of fuzzy variables. A fuzzy variable can therefore be considered as a 'fuzzification' of a measurable set. Indeed, a subset E of X is $\mathcal{R}$-measurable if and only if its characteristic $X-L$-mapping is. The set of the $(L,\leq)$-fuzzy variables in $(X,\mathcal{R})$ is denoted by $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)$. A more detailed treatment of fuzzy variables can be found in [4, 5, 6, 9].

2.4 Possibility Measures

A $(L,\leq)$-*possibility measure* Π on $(X,\mathcal{R})$ is a complete join-morphism between the complete lattices $(\mathcal{R},\subseteq)$ and $(L,\leq)$. By definition, this means that Π satisfies the following requirement: for an arbitrary family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$:

$$\Pi(\bigcup_{j \in J} A_j) = \sup_{j \in J} \Pi(A_j).$$

This definition immediately implies that $\Pi(\emptyset) = 0$. For arbitrary A in $\mathcal{R}$, $\Pi(A)$ will be called the $(L,\leq)$-*possibility of* A. Π will be called *normalized*[1] iff

[1] In very much the same was as for probability measures, normalization is a very natural property for possibility measures, and expresses in some sense that the universe X is adequate, or large enough. We shall therefore in this paper always work with normalized possibility measures.

$\Pi(X) = 1$. Again, whenever we do not want to mention the complete lattice $(L, \leq)$ explicitly, we shall simply speak of possibility and possibility measures. A $\mathcal{R}$-measurable $X - L$-mapping π such that for arbitrary A in $\mathcal{R}$

$$\Pi(A) = \sup_{x \in A} \pi(x), \tag{1.3}$$

is called a *distribution* of Π. Such a distribution is *unique*, and satisfies

$$(\forall x \in X)(\pi(x) \stackrel{\text{def}}{=} \Pi([\,x\,]_{\mathcal{R}}))$$

The distribution of a normalized possibility measure is a sup-normal fuzzy variable. $(L, \leq)$-possibility measures are generalizations towards more general domains and codomains of Zadeh's possibility measures [30], Wang's fuzzy contactabilities [24], and the possibility measures we introduced in [17]. For a more detailed discussion of these generalizations, we refer to [9, 13, 17].

2.5 Possibilistic Variables

First, let us introduce transformation of possibility by a mapping. Let X and Y be universes, and f a $X - Y$-mapping. Let $\mathcal{R}$ be an ample field on X and Π a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$. Then

$$\mathcal{R}^{(f)} \stackrel{\text{def}}{=} \{\, B \mid B \in \mathcal{P}(Y) \text{ and } f^{-1}(B) \in \mathcal{R} \,\}$$

is an ample field on Y, and the $\mathcal{R}^{(f)} - L$ mapping

$$\Pi^{(f)}\colon \mathcal{R}^{(f)} \to L\colon B \mapsto \Pi(f^{-1}(B)) \tag{1.4}$$

is a $(L, \leq)$-possibility measure on $(Y, \mathcal{R}^{(f)})$, called the *transformed* $(L, \leq)$-*possibility measure on* $(Y, \mathcal{R}^{(f)})$ *by the mapping* f. If $\mathcal{R}_Y$ is an ample field on Y, such that f is $\mathcal{R} - \mathcal{R}_Y$-measurable, then

$$\mathcal{R}_Y \subseteq \mathcal{R}^{(f)}, \tag{1.5}$$

and the restriction $\Pi^{(f)}|\mathcal{R}_Y$ of $\Pi^{(f)}$ to $\mathcal{R}_Y$ is called the *transformed* $(L, \leq)$-*possibility measure on* $(Y, \mathcal{R}_Y)$ *by the mapping* f.

It is also possible to define *possibilistic variables*, which are possibilistic equivalents of the stochastic variables in probability theory (see, for instance, [2]). We consider a universe Ω and an ample field $\mathcal{R}_\Omega$ on Ω. This universe is called a *basic space*. X is called a *sample space*. A $\Omega - X$-mapping that is $\mathcal{R}_\Omega - \mathcal{R}$-measurable, is called a *possibilistic variable in* $(X, \mathcal{R})$. If we also consider a normalized $(L, \leq)$-possibility measure Π_Ω on $(\Omega, \mathcal{R}_\Omega)$, we can use the possibilistic variable f to *transform* Π_Ω to a $(L, \leq)$-possibility measure Π_f on $(X, \mathcal{R})$, defined by $\Pi_f \stackrel{\text{def}}{=} {\Pi_\Omega}^{(f)}|\mathcal{R}$, or equivalently,

$$(\forall B \in \mathcal{R})(\Pi_f(B) \stackrel{\text{def}}{=} \Pi_\Omega(f^{-1}(B))).$$

Π_f is called the *possibility distribution*[2] of the possibilistic variable f. The distribution π_f of Π_f is called the *possibility distribution function* of f, and satisfies

$$\pi_f(x) = \sup_{f(\omega) \in [x]_\mathcal{R}} \pi_\Omega(\omega),$$

where x is an element of X and π_Ω is the distribution of Π_Ω. Of course, π_f is a sup-normal $(L, \leq)$-fuzzy variable in $(X, \mathcal{R})$. For a more detailed account of possibilistic variables, we refer to [6, 9].

2.6 Classical Truth Values and Their Combinations

We shall also be working with the set $\mathcal{T} \stackrel{\text{def}}{=} \{\mathit{false}, \mathit{true}\}$ of truth values in classical propositional logic. On $\mathcal{T}$, we define the total order relation

$$\leq \stackrel{\text{def}}{=} \{(\mathit{false}, \mathit{false}), (\mathit{false}, \mathit{true}), (\mathit{true}, \mathit{true})\},$$

i.e., $(\mathcal{T}, \leq)$ is a Boolean chain of length 2, with top element *true* and bottom element *false*. On this Boolean chain, we may define the complement $\neg$, called *negation*; the Boolean multiplication or meet $\wedge$, called *conjunction*; the Boolean addition or join $\vee$, called *disjunction*, the *implication* $\Rightarrow$ and the *equivalence*

[2]Note that there is a clear distinction between the *distribution* of a possibility measure and the *possibility distribution* of a possibilistic variable.

$\Leftrightarrow$. More explicitly, we have

$$\neg\colon \mathcal{T} \to \mathcal{T}\colon \nu \mapsto \begin{cases} \mathit{false} & ; \quad \nu = \mathit{true} \\ \mathit{true} & ; \quad \nu = \mathit{false} \end{cases}$$

$$\wedge\colon \mathcal{T}^2 \to \mathcal{T}\colon (\nu,\mu) \mapsto \begin{cases} \mathit{true} & ; \quad \nu = \mu = \mathit{true} \\ \mathit{false} & ; \quad \text{elsewhere} \end{cases}$$

$$\vee\colon \mathcal{T}^2 \to \mathcal{T}\colon (\nu,\mu) \mapsto \begin{cases} \mathit{false} & ; \quad \nu = \mu = \mathit{false} \\ \mathit{true} & ; \quad \text{elsewhere} \end{cases}$$

$$\Rightarrow\colon \mathcal{T}^2 \to \mathcal{T}\colon (\nu,\mu) \mapsto \begin{cases} \mathit{false} & ; \quad \nu = \mathit{true} \text{ and } \mu = \mathit{false} \\ \mathit{true} & ; \quad \text{elsewhere} \end{cases}$$

$$\Leftrightarrow\colon \mathcal{T}^2 \to \mathcal{T}\colon (\nu,\mu) \mapsto \begin{cases} \mathit{true} & ; \quad \nu = \mu \\ \mathit{false} & ; \quad \nu \neq \mu. \end{cases}$$

3 POSSIBILITY THEORY AND THE EXTENSION PRINCIPLE

3.1 Zadeh's Extension Principle

In 1965 Zadeh proposed a method for extending a mapping from a universe X to a universe Y to a mapping from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$ [28, 29]. This method later received the name '*Zadeh's extension principle*'. It can be formalized as follows.

Definition 1 (Extension principle) *With a $X - Y$-mapping φ we can associate a $\mathcal{F}(X) - \mathcal{F}(Y)$-mapping $\widetilde{\varphi}$, defined as follows:*

$$(\forall h \in \mathcal{F}(X))(\forall y \in Y)(\widetilde{\varphi}(h) \cdot y \stackrel{\text{def}}{=} \sup_{\varphi(x)=y} h(x)), \tag{1.6}$$

and called the Zadeh extension of φ.

Whereas φ maps an arbitrary element x of X into the element $\varphi(x)$ of Y, $\widetilde{\varphi}$ maps an arbitrary $([0,1],\leq)$-fuzzy set h in X into the $([0,1],\leq)$-fuzzy set $\widetilde{\varphi}(h)$ in Y. Let furthermore A be an arbitrary subset of X, then the characteristic

$X-[0,1]$-mapping χ_A of A is a $([0,1],\leq)$-fuzzy set in X. It is easily verified that $\widetilde{\varphi}(\chi_A)=\chi_{\varphi(A)}$, with $\varphi(A)=\{\,\varphi(x)\mid x\in A\,\}$ the direct image of A under φ. We conclude that a Zadeh extension can be interpreted as a generalization—or, more precisely, fuzzification—of the notion of a direct image in classical set theory. In the rest of this section, we want to show that Zadeh's extension method finds its origin in the transformation of possibility measures.

3.2 Possibilistic Extensions

Let us consider a universe X, provided with an ample field $\mathcal{R}$, and a variable ξ that assumes values in X. In this paper, we shall be working with the *formal mathematical*, rather than the intuitive *notion of a variable*. This means that we assume the existence of a *basic space* Ω, provided with an ample field $\mathcal{R}_\Omega$, and a *normalized* $(L,\leq)$-possibility measure Π_Ω on $(\Omega,\mathcal{R}_\Omega)$. The distribution of Π_Ω will be denoted by π_Ω. In the formal picture, X is considered as a *sample space*, and the variable ξ is a $\mathcal{R}_\Omega-\mathcal{R}$*-measurable* $\Omega-X$-mapping, i.e., a possibilistic variable in $(X,\mathcal{R})$. Possibilistic information about the values ξ may take in X is given by the possibility distribution of ξ, or, in other words, by the transformed $(L,\leq)$-possibility measure $\Pi_\xi\overset{\text{def}}{=}{\Pi_\Omega}^{(\xi)}|\mathcal{R}$ of Π_Ω on $(X,\mathcal{R})$ by the mapping ξ. Thus, for arbitrary A in $\mathcal{R}$, taking into account (1.4),

$$\Pi_\xi(A)\overset{\text{def}}{=}\Pi_\Omega(\xi^{-1}(A))=\text{ the possibility that }\xi\text{ takes a value in }A. \tag{1.7}$$

The possibility distribution function of ξ is the distribution of Π_ξ, and will be denoted by π_ξ. We have seen in the previous section (see (1.3)) that it completely characterizes the possibilistic information Π_ξ about the values that ξ can assume in X. Note that for arbitrary x in X

$$\pi_\xi(x)=\sup_{\xi(\omega)\in[\,x\,]_\mathcal{R}}\pi_\Omega(\omega). \tag{1.8}$$

Remark that, since $\xi^{-1}(X)=\Omega$ and since Π_Ω is normalized, Π_ξ is normalized as well. The possibility distribution function π_ξ of ξ moreover is a sup-normal $(L,\leq)$-fuzzy variable in $(X,\mathcal{R})$.

Besides the universe X we shall now also consider a universe Y and a $X-Y$-mapping φ. Since ξ is a variable in X, $\varphi(\xi)=\varphi\circ\xi$ can be considered as a variable in Y, and we wonder if we can deduce possibilistic information about the values that $\varphi(\xi)$ can assume in Y. From a formal mathematical point of view, Y can also be considered as a sample space, and we take a closer look at the $\Omega-Y$-mapping $\varphi(\xi)$. Since, by definition, for arbitrary B in $\mathcal{R}^{(\varphi)}$,

$\varphi^{-1}(B) \in \mathcal{R}$, and since ξ is by assumption $\mathcal{R}_\Omega - \mathcal{R}$-measurable, we deduce that $(\varphi \circ \xi)^{-1}(B) = \xi^{-1}(\varphi^{-1}(B)) \in \mathcal{R}_\Omega$. This means that $\varphi(\xi)$ is a *possibilistic variable* in $(Y, \mathcal{R}^{(\varphi)})$. Since we want information about the values that this variable $\varphi(\xi)$ can assume in Y, let us take a look at the transformed possibility measure $\Pi_{\varphi(\xi)} \stackrel{\text{def}}{=} {\Pi_\Omega}^{(\varphi(\xi))}|\mathcal{R}^{(\varphi)}$ of Π_Ω on $(Y, \mathcal{R}^{(\varphi)})$ by $\varphi(\xi)$, i.e., the possibility distribution of the possibilistic variable $\varphi(\xi)$ in $(Y, \mathcal{R}^{(\varphi)})$. For arbitrary B in $\mathcal{R}^{(\varphi)}$, we have, taking into account (1.4) and (1.7) and the fact that, by definition, $\varphi^{-1}(B) \in \mathcal{R}$,

$$\begin{aligned} \Pi_{\varphi(\xi)}(B) &= \Pi_\Omega((\varphi \circ \xi)^{-1}(B)) = \Pi_\Omega(\xi^{-1}(\varphi^{-1}(B))) \\ &= {\Pi_\Omega}^{(\xi)}(\varphi^{-1}(B)) = \Pi_\xi(\varphi^{-1}(B)) = {\Pi_\xi}^{(\varphi)}(B). \end{aligned}$$

We may therefore conclude that $\Pi_{\varphi(\xi)} = {\Pi_\xi}^{(\varphi)}$. Since furthermore $\varphi^{-1}(Y) = X$ and Π_ξ is normalized, $\Pi_{\varphi(\xi)}$ is normalized as well. For the possibility distribution function $\pi_{\varphi(\xi)}$ of the possibilistic variable $\varphi(\xi)$ in $(Y, \mathcal{R}^{(\varphi)})$ we have, for arbitrary y in Y

$$\pi_{\varphi(\xi)}(y) = \Pi_{\varphi(\xi)}([\,y\,]_{\mathcal{R}^{(\varphi)}}) = \Pi_\xi(\varphi^{-1}([\,y\,]_{\mathcal{R}^{(\varphi)}})) = \sup_{\varphi(x) \in [\,y\,]_{\mathcal{R}^{(\varphi)}}} \pi_\xi(x). \tag{1.9}$$

$\pi_{\varphi(\xi)}$ is clearly a sup-normal $(L, \leq)$-fuzzy variable in $(Y, \mathcal{R}^{(\varphi)})$. Also remark that, with respect to set inclusion, $\mathcal{R}^{(\varphi)}$ is the largest possible ample field $\mathcal{R}_Y$ on Y such that φ is still $\mathcal{R} - \mathcal{R}_Y$-measurable (see (1.5)). In this sense, the choice of the ample field $\mathcal{R}^{(\varphi)}$ on Y makes the transmission of possibilistic information from X to Y as detailed as possible.

We conclude that the mapping φ can be used to transform the possibilistic information π_ξ (or Π_ξ) about the values ξ may take in X, into possibilistic information $\pi_{\varphi(\xi)}$ (or $\Pi_{\varphi(\xi)}$) about the values $\varphi(\xi)$ can assume in Y. If we look at (1.9), we are led to the introduction of a special $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X) - \mathcal{G}^{\mathcal{R}^{(\varphi)}}_{(L,\leq)}(Y)$-mapping, directly transforming π_ξ into $\pi_{\varphi(\xi)}$: $\pi_{\varphi(\xi)} = \widetilde{\varphi}(\pi_\xi)$.

Definition 2 (Possibilistic extensions) *With a $X - Y$-mapping φ we can associate a $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X) - \mathcal{G}^{\mathcal{R}^{(\varphi)}}_{(L,\leq)}(Y)$-mapping $\widetilde{\varphi}$, defined as follows:*

$$(\forall h \in \mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X))(\forall y \in Y)(\widetilde{\varphi}(h) \cdot y \stackrel{\text{def}}{=} \sup_{\varphi(x) \in [\,y\,]_{\mathcal{R}^{(\varphi)}}} h(x)). \tag{1.10}$$

$\widetilde{\varphi}$ is called the $(L, \leq)$-possibilistic extension of φ. If, for whatever reason, we do not want to mention the complete lattice $(L, \leq)$ explicitly, we shall simply call $\widetilde{\varphi}$ a possibilistic extension.

Corollary 1 *If the $(L,\leq)$-fuzzy variable h in $(X,\mathcal{R})$ is sup-normal, then so is the $(L,\leq)$-fuzzy variable $\widetilde{\varphi}(h)$ in $(Y,\mathcal{R}^{(\varphi)})$.*

Let in particular A be an element of $\mathcal{R}$, then the characteristic $X-L$-mapping χ_A of A is of course a $(L,\leq)$-fuzzy variable in $(X,\mathcal{R})$. Furthermore, for arbitrary y in Y

$$\widetilde{\varphi}(\chi_A)\cdot y = \sup_{\varphi(x)\in[\,y\,]_{\mathcal{R}^{(\varphi)}}} \chi_A(x) = \begin{cases} 1 & ; \quad A\cap\varphi^{-1}([\,y\,]_{\mathcal{R}^{(\varphi)}})\neq\emptyset \\ 0 & ; \quad A\cap\varphi^{-1}([\,y\,]_{\mathcal{R}^{(\varphi)}})=\emptyset. \end{cases}$$

It is easily verified that $A\cap\varphi^{-1}([\,y\,]_{\mathcal{R}^{(\varphi)}})\neq\emptyset \Leftrightarrow y\in\bigcup_{x\in A}[\,\varphi(x)\,]_{\mathcal{R}^{(\varphi)}}$, whence, taking into account the definition of the closure operator $C_{\mathcal{R}^{(\varphi)}}$ (see subsection 2.2), $\widetilde{\varphi}(\chi_A)=\chi_{C_{\mathcal{R}^{(\varphi)}}(\varphi(A))}$. We conclude that a possibilistic extension is a generalization—or fuzzification—of the notion of a direct image, taking into account certain measurability aspects.

How does this relate to Zadeh's extension principle? Clearly, when $\mathcal{R}=\mathcal{P}(X)$, and therefore also $\mathcal{R}^{(\varphi)}=\mathcal{P}(Y)$, we find for arbitrary y in Y that $[\,y\,]_{\mathcal{R}^{(\varphi)}}=\{y\}$. Furthermore, $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)=\mathcal{F}_{(L,\leq)}(X)$ and $\mathcal{G}^{\mathcal{R}^{(\varphi)}}_{(L,\leq)}(Y)=\mathcal{F}_{(L,\leq)}(Y)$. This tells us that Zadeh's extension method (1.6) is a special case of the possibilistic extension method (1.10) for $(L,\leq)=([0,1],\leq)$ and $\mathcal{R}=\mathcal{P}(X)$. We have thus deduced Zadeh's extension principle from considerations about the transformation of possibility measures by a mapping. This also provides us with an interpretation for the Zadeh extension of such a mapping.

Another question now comes to mind: what happens if two or more such transformations of possibility are concatenated? The following theorem provides an answer. It has a graphical representation in figure 1.

Theorem 1 (Chain rule for possibilistic extensions) *Let X, Y and Z be universes. Let $\mathcal{R}$ be an ample field on X. φ is a $X-Y$-mapping and ψ a $Y-Z$-mapping. We define the following $(L,\leq)$-possibilistic extensions:*

$$\widetilde{\varphi}\colon \mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)\to\mathcal{G}^{\mathcal{R}^{(\varphi)}}_{(L,\leq)}(Y)\colon h\mapsto\widetilde{\varphi}(h),$$

$$\widetilde{\psi}\colon \mathcal{G}^{\mathcal{R}^{(\varphi)}}_{(L,\leq)}(Y)\to\mathcal{G}^{(\mathcal{R}^{(\varphi)})^{(\psi)}}_{(L,\leq)}(Z)\colon g\mapsto\widetilde{\varphi}(g)$$

$$\widetilde{\psi\circ\varphi}\colon \mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X)\to\mathcal{G}^{\mathcal{R}^{(\psi\circ\varphi)}}_{(L,\leq)}(Z)\colon h\mapsto(\widetilde{\psi\circ\varphi})(h)$$

with, of course,

$$
\begin{aligned}
&(\forall y \in Y)(\widetilde{\varphi}(h) \cdot y \stackrel{\text{def}}{=} \sup_{x \in \varphi^{-1}([\,y\,]_{\mathcal{R}^{(\varphi)}})} h(x))\\
&(\forall z \in Z)(\widetilde{\psi}(g) \cdot z \stackrel{\text{def}}{=} \sup_{y \in \psi^{-1}([\,z\,]_{(\mathcal{R}^{(\varphi)})^{(\psi)}})} g(y))\\
&(\forall z \in Z)((\widetilde{\psi \circ \varphi})(h) \cdot z \stackrel{\text{def}}{=} \sup_{x \in (\psi \circ \varphi)^{-1}([\,z\,]_{\mathcal{R}^{(\psi \circ \varphi)}})} h(x)).
\end{aligned}
$$

Then (i) $\mathcal{R}^{(\psi \circ \varphi)} = (\mathcal{R}^{(\varphi)})^{(\psi)}$*; and (ii)* $\widetilde{\psi \circ \varphi} = \widetilde{\psi} \circ \widetilde{\varphi}$*. In particular, this means that the right-most diagram in figure 1 commutes*[3]*.*

Proof. Let us first show that (i) holds. Consider an arbitrary $C \subseteq Z$. Then, taking into account (1.4),

$$
\begin{aligned}
C \in \mathcal{R}^{(\psi \circ \varphi)} &\Leftrightarrow (\psi \circ \varphi)^{-1}(C) \in \mathcal{R} \Leftrightarrow \varphi^{-1}(\psi^{-1}(C)) \in \mathcal{R}\\
&\Leftrightarrow \psi^{-1}(C) \in \mathcal{R}^{(\varphi)} \Leftrightarrow C \in (\mathcal{R}^{(\varphi)})^{(\psi)}.
\end{aligned}
$$

This indeed implies that $\mathcal{R}^{(\psi \circ \varphi)} = (\mathcal{R}^{(\varphi)})^{(\psi)}$. We now proceed to prove (ii). Consider an arbitrary $(L, \leq)$-fuzzy variable h in $(X, \mathcal{R})$ and an arbitrary z in Z. We know, taking into account (i) and (1.1), that $[\,z\,]_{\mathcal{R}^{(\psi \circ \varphi)}} \in (\mathcal{R}^{(\varphi)})^{(\psi)}$ which is, by definition, equivalent with $\psi^{-1}([\,z\,]_{\mathcal{R}^{(\psi \circ \varphi)}}) \in \mathcal{R}^{(\varphi)}$, and, taking into account (1.2), also equivalent with $\psi^{-1}([\,z\,]_{\mathcal{R}^{(\psi \circ \varphi)}}) = \bigcup_{y \in \psi^{-1}([\,z\,]_{\mathcal{R}^{(\psi \circ \varphi)}})} [\,y\,]_{\mathcal{R}^{(\varphi)}}$.
This implies, taking into account the properties of inverse images, that

$$
\begin{aligned}
(\psi \circ \varphi)^{-1}([\,z\,]_{\mathcal{R}^{(\psi \circ \varphi)}}) &= \varphi^{-1}(\psi^{-1}([\,z\,]_{\mathcal{R}^{(\psi \circ \varphi)}}))\\
&= \bigcup_{y \in \psi^{-1}([\,z\,]_{\mathcal{R}^{(\psi \circ \varphi)}})} \varphi^{-1}([\,y\,]_{\mathcal{R}^{(\varphi)}}).
\end{aligned}
$$

[3] To use the lingo of fuzzy set theory, this result tells us that the composition of mappings can be 'fuzzified'.

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\psi\circ\varphi"'] & & Y \arrow[dl, "\psi"] \\ & Z & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X) \arrow[rr, "\widetilde{\varphi}"] \arrow[dr, "\widetilde{\psi\circ\varphi}"'] & & \mathcal{G}^{\mathcal{R}(\varphi)}_{(L,\leq)}(Y) \arrow[dl, "\widetilde{\psi}"] \\ & \mathcal{G}^{\mathcal{R}(\psi\circ\varphi)}_{(L,\leq)}(Z) & \end{tikzcd}$$

Figure 1 Commutative diagrams: the chain rule for possibilistic extensions.

If we take into account the associativity of supremum in the complete lattice $(L, \leq)$, (i) and the definitions of $\widetilde{\varphi}$ and $\widetilde{\psi}$, this means that

$$\begin{aligned}
(\widetilde{\psi\circ\varphi})(h)\cdot z &= \sup_{x\in(\psi\circ\varphi)^{-1}([z]_{\mathcal{R}(\psi\circ\varphi)})} h(x) \\
&= \sup_{x\in \bigcup_{y\in\psi^{-1}([z]_{\mathcal{R}(\psi\circ\varphi)})} \varphi^{-1}([y]_{\mathcal{R}(\varphi)})} h(x) \\
&= \sup_{y\in\psi^{-1}([z]_{\mathcal{R}(\psi\circ\varphi)})} \ \sup_{x\in\varphi^{-1}([y]_{\mathcal{R}(\varphi)})} h(x) \\
&= \sup_{y\in\psi^{-1}([z]_{\mathcal{R}(\psi\circ\varphi)})} \widetilde{\varphi}(h)\cdot y \\
&= \widetilde{\psi}(\widetilde{\varphi}(h))\cdot z \\
&= (\widetilde{\psi}\circ\widetilde{\varphi})(h)\cdot z,
\end{aligned}$$

whence indeed $\widetilde{\psi\circ\varphi} = \widetilde{\psi}\circ\widetilde{\varphi}$. □

3.3 Possibilistic t-norm-extensions

We shall now turn to the discussion of an interesting special case, that will play a prominent role in the discussion of possibilistic logic in the following sections. Let us consider, besides the basic space Ω, n ($n \in \mathbb{N}\setminus\{0\}$) universes X_k, each provided with an ample field $\mathcal{R}_k$ ($k = 1\ldots, n$). Furthermore, we shall denote by ξ_k a possibilistic variable in $(X_k, \mathcal{R}_k)$, i.e. a $\mathcal{R}_\Omega - \mathcal{R}_k$-measurable $\Omega - X_k$-mapping. We may also consider the product universe $X_1\times\cdots\times X_n$, provided with the ample field $\mathcal{R}_1\times\cdots\times\mathcal{R}_n$, i.e., the product of the ample fields $\mathcal{R}_1, \ldots, \mathcal{R}_n$ (see subsection 2.2). It is easily proven that the $\Omega - X_1\times\cdots\times X_n$-mapping $(\xi_1,\ldots,\xi_n)$ is a possibilistic variable in $(X_1\times\cdots\times X_n, \mathcal{R}_1\times\cdots\times\mathcal{R}_n)$ (see [4, 9]). On the one hand, for $k = 1,\ldots,n$ we may now consider the normalized $(L,\leq)$-possibility measure $\Pi_{\xi_k} \stackrel{\text{def}}{=} {\Pi_\Omega}^{(\xi_k)}|\mathcal{R}_k$ with distribution π_{ξ_k}. Of course, π_{ξ_k} is the possibility distribution function of the possibilistic variable ξ_k. On the other hand, we may also consider the normalized $(L,\leq)$-possibility measure $\Pi_{(\xi_1,\ldots,\xi_n)} \stackrel{\text{def}}{=} {\Pi_\Omega}^{((\xi_1,\ldots,\xi_n))}|\mathcal{R}_1 \times \cdots \times \mathcal{R}_n$ with distribution $\pi_{(\xi_1,\ldots,\xi_n)}$. $\pi_{(\xi_1,\ldots,\xi_n)}$ is the possibility distribution function of the possibilistic variable $(\xi_1,\ldots,\xi_n)$. We have shown in our treatment of *possibilistic independence* [5, 9, 15] that the possibilistic variables $\xi_1, \ldots, \xi_n$ are (Π_Ω, T)-independent (or, shortly, possibilistically independent) if and only if

$$(\forall(x_1,\ldots,x_n) \in X_1 \times \cdots \times X_n)(\pi_{(\xi_1,\ldots,\xi_n)}(x_1,\ldots,x_n) = T_{k=1}^n \pi_{\xi_k}(x_k)) \quad (1.11)$$

As a next step in this course of reasoning, we consider the $X_1 \times \cdots \times X_n - Y$-mapping φ. On the basis of the discussion in the previous subsection, with in particular $X = X_1 \times \cdots \times X_n$ and $\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_n$, we know that the $\Omega - Y$-mapping $\varphi(\xi_1,\ldots,\xi_n) = \varphi \circ (\xi_1,\ldots,\xi_n)$ is a possibilistic variable in $(Y, (\mathcal{R}_1 \times \cdots \times \mathcal{R}_n)^{(\varphi)})$ and furthermore $\Pi_{\varphi(\xi_1,\ldots,\xi_n)} = {\Pi_{(\xi_1,\ldots,\xi_n)}}^{(\varphi)}$. For arbitrary y in Y

$$\pi_{\varphi(\xi_1,\ldots,\xi_n)}(y) = \sup_{\varphi(x_1,\ldots,x_n) \in [y]_{(\mathcal{R}_1 \times \cdots \times \mathcal{R}_n)^{(\varphi)}}} \pi_{(\xi_1,\ldots,\xi_n)}(x_1,\ldots,x_n).$$

When the possibilistic variables $\xi_1, \ldots, \xi_n$ are (Π_Ω, T)-independent, we find for arbitrary y in Y that, taking into account (1.11),

$$\pi_{\varphi(\xi_1,\ldots,\xi_n)}(y) = \sup_{\varphi(x_1,\ldots,x_n) \in [y]_{(\mathcal{R}_1 \times \cdots \times \mathcal{R}_n)^{(\varphi)}}} T_{k=1}^n \pi_{\xi_k}(x_k).$$

As in the previous subsection, we are thus led to the following definition.

Definition 3 (t-norm-extensions) *With a $X_1 \times \cdots \times X_n - Y$-mapping φ we can associate a $\mathcal{G}_{(L,\leq)}^{\mathcal{R}_1}(X_1) \times \ldots \times \mathcal{G}_{(L,\leq)}^{\mathcal{R}_n}(X_n) - \mathcal{G}_{(L,\leq)}^{(\mathcal{R}_1 \times \cdots \times \mathcal{R}_n)^{(\varphi)}}(Y)$-mapping $\widetilde{\varphi}_T$, defined as follows:*

$$(\forall(h_1,\ldots,h_n) \in \mathcal{G}_{(L,\leq)}^{\mathcal{R}_1}(X_1) \times \ldots \times \mathcal{G}_{(L,\leq)}^{\mathcal{R}_n}(X_n))$$
$$(\forall y \in Y)(\widetilde{\varphi}_T(h_1,\ldots,h_n) \cdot y \stackrel{\text{def}}{=} \sup_{\varphi(x_1,\ldots,x_n) \in [y]_{(\mathcal{R}_1 \times \cdots \times \mathcal{R}_n)^{(\varphi)}}} T_{k=1}^n h_k(x_k)).$$

$\widetilde{\varphi}_T$ is called the $(L,\leq)$-possibilistic T-extension of φ. If, for whatever reason, we do not want to mention the complete lattice $(L,\leq)$ and/or the triangular norm T on $(L,\leq)$ explicitly, we shall simply call $\widetilde{\varphi}_T$ a possibilistic t-norm-extension.

Possibilistic t-norm-extensions are used when we have at our disposal possibilistic information in the form of the possibility distribution functions π_{ξ_k}—essentially sup-normal $(L, \leq)$-fuzzy variables in $(X_k, \mathcal{R}_k)$—of the *possibilistically independent* possibilistic variables ξ_k. This information can be transmitted from the universe $X_1 \times \cdots \times X_n$ to the universe Y by a $X_1 \times \cdots \times X_n - Y$-mapping φ. The possibility distribution function of the possibilistic variable $\varphi(\xi_1, \ldots, \xi_n)$ is then given by $\pi_{\varphi(\xi_1,\ldots,\xi_n)} = \widetilde{\varphi}_T(\pi_{\xi_1}, \ldots \pi_{\xi_n})$, and constitutes information about the values that $\varphi(\xi_1, \ldots, \xi_n)$ may assume in the universe Y.

Corollary 2 *When the $(L, \leq)$-fuzzy variables $h_1, \ldots, h_n$ in $(X_1, \mathcal{R}_1), \ldots, (X_n, \mathcal{R}_n)$ respectively are sup-normal, $\widetilde{\varphi}_T(h_1, \ldots, h_n)$ is a sup-normal $(L, \leq)$-fuzzy variable in $(Y, (\mathcal{R}_1 \times \cdots \times \mathcal{R}_n)^{(\varphi)})$.*

Let us now give a few examples in order better to understand the notions introduced thus far.

Example 1 Let $(L, \leq) = ([0,1], \leq)$, $X_1 = X_2 = Y = \mathbb{R}$ and $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{P}(\mathbb{R})$. A $([0,1], \leq)$-fuzzy variable in $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$, or equivalently, a $([0,1], \leq)$-fuzzy set in $\mathbb{R}$ is also called a *fuzzy quantity* (see, for instance, [21] section 3.1). The $X_1 \times X_2 - Y$-mapping φ we consider here, is the additive operation on the reals, i.e., $\varphi = +$, with $([0,1], \leq)$-possibilistic T-extension $\widetilde{+}_T$:

$$\widetilde{+}_T : \mathcal{F}(\mathbb{R})^2 \to \mathcal{F}(\mathbb{R}) : (h, g) \mapsto h \mathbin{\widetilde{+}_T} g,$$

with, for arbitrary c in $\mathbb{R}$,

$$(h \mathbin{\widetilde{+}_T} g) \cdot c = \sup_{a+b=c} T(h(a), g(b)) = \sup_{a \in \mathbb{R}} T(h(a), g(c-a)). \tag{1.12}$$

When π_ξ and π_ζ represent possibilistic information about the values that two $([0,1], \leq)$-possibilistically T-independent possibilistic variables ξ and ζ may assume in $\mathbb{R}$, then $\pi_\xi \mathbin{\widetilde{+}_T} \pi_\zeta$ represents possibilistic information about the values that the possibilistic variable $\xi + \zeta$ may assume in $\mathbb{R}$.

Example 2 Besides possibilistic information, we may also consider probabilistic information. Let f_ξ and f_ζ be the probability density functions of two continuous real stochastic variables ξ en ζ, that are furthermore stochastically independent. It is well known that the probability density function $f_{\xi+\zeta}$ of the sum $\xi+\zeta$ can be written as (see, for instance, [2] section 12-4 and exercise 12-29)

$$f_{\xi+\zeta} : \mathbb{R} \to [0,1] : c \mapsto (f_\xi * f_\zeta) \cdot c$$

with

$$(f_\xi * f_\zeta) \cdot c = \int_{-\infty}^{+\infty} f_\xi(a) f_\zeta(c-a)\mathrm{d}a. \tag{1.13}$$

In other words, $f_{\xi+\zeta}$ is the *convolution product* $f_\xi * f_\zeta$ of f_ξ and f_ζ. Notice the striking formal analogy between (1.12) and (1.13), which is certainly not a coincidence. Both formulas are derived from considerations about transformations of measures. This again indicates that Zadeh's extension principle is by no means an isolated *ad hoc* principle, but that it rather has its natural place in a much broader measure-theoretical context.

4 POSSIBILISTIC TRUTH VALUES AND THEIR COMBINATIONS

4.1 Possibilistic Truth Values

Let us apply the results of the previous section to the problem of representing linguistic uncertainty in classical propositional logic, briefly described in the introduction. As far as we know, Van Schooten was the first to study this problem in his doctoral dissertation [27]. It must however also be mentioned that Gaines has briefly discussed it in his important article about approximate reasoning [19].

We shall first give a fairly general description of the problem. For a start, consider a *property* (or predicate) p of the elements of a universe X, that is *clear*: for every object x in X we have that x either completely satisfies or completely does not satisfy p (for more details, we refer to [7, 8, 9]). With every x in X, we may therefore associate a *proposition* $P_p(x) \stackrel{\text{def}}{=}$ 'x is p', i.e., an affirmative sentence that is either true or false. P_p can be considered as a mapping from the universe X to an appropriate set of propositions, and is therefore also called a *proposition function*. With P_p we can also associate the set A_{P_p} of the objects that satisfy p: $A_{P_p} \stackrel{\text{def}}{=} \{\, x \mid x \in X \text{ and } x \text{ is } p \,\}$. Moreover, A_{P_p} can also be characterized by its characteristic $X - \mathcal{T}$-mapping:

$$\chi_{A_{P_p}} : X \to \mathcal{T} : x \mapsto \begin{cases} \mathit{true} & ; \quad x \text{ is } p \\ \mathit{false} & ; \quad x \text{ is not } p. \end{cases}$$

For arbitrary x in X, $\chi_{A_{P_p}}(x)$ is the *truth value* of the proposition 'x is p'.

Example 3 Let $X = \mathbb{R}$ and let p be the clear predicate 'greater than or equal to 10'. Then, of course, $A_{P_p} = \{ a \mid a \in \mathbb{R} \text{ and } a \geq 10 \}$ and for arbitrary a in $\mathbb{R}$

$$\chi_{A_{P_p}}(a) = \begin{cases} \mathit{true} & ; \quad a \geq 10 \\ \mathit{false} & ; \quad a < 10. \end{cases}$$

So far, we are able unequivocally to associate with every object x in X the truth value $\chi_{A_{P_p}}(x)$ of the proposition $P_p(x)$. In the next step of our course of reasoning, we shall introduce *linguistic uncertainty* into the picture. Let us assume that on the universe X there is defined an ample field $\mathcal{R}$ of measurable subsets of X. We consider a possibilistic variable ξ in $(X, \mathcal{R})$. As in the previous section, this means that we assume the existence of a basic space Ω, provided with an ample field $\mathcal{R}_\Omega$ and a normalized possibility measure Π_Ω. X is considered as a sample space, and ξ is a $\mathcal{R}_\Omega - \mathcal{R}$-measurable $\Omega - X$-mapping. Possibilistic information about the values that the possibilistic variable ξ may assume in X is given in the form of the possibility distribution Π_ξ (see (1.7)) or the possibility distribution function π_ξ of ξ (see (1.8)). Given this information, it is fairly natural to ask what information may be derived about the truth value $\chi_{A_{P_p}}(\xi)$ of the *proposition variable* $P_p \circ \xi = P_p(\xi) \overset{\text{def}}{=}$ 'ξ is p'. From the results of the previous section, we now deduce as a special case, with $Y = \mathcal{T}$ and $\varphi = \chi_{A_{P_p}}$, the following conclusions. First of all, the truth value $\chi_{A_{P_p}}(\xi)$ of the proposition variable $P_p(\xi)$ is a possibilistic variable in $(\mathcal{T}, \mathcal{R}^{(\chi_{A_{P_p}})})$. Secondly, its possibility distribution $\Pi_{\chi_{A_{P_p}}(\xi)} = \Pi_\xi{}^{(\chi_{A_{P_p}})}$ with distribution $\pi_{\chi_{A_{P_p}}(\xi)}$ is a normalized $(L, \leq)$-possibility measure on $(\mathcal{T}, \mathcal{R}^{(\chi_{A_{P_p}})})$, that gives possibilistic information about the values that the variable $\chi_{A_{P_p}}(\xi)$ may assume in $\mathcal{T}$, or in other words, about the truth value of the proposition variable 'ξ is p'. Finally, the $(L, \leq)$-possibilistic extension $\widetilde{\chi_{A_{P_p}}}$ of $\chi_{A_{P_p}}$ is a $\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X) - \mathcal{G}^{\mathcal{R}^{(\chi_{A_{P_p}})}}_{(L,\leq)}(\mathcal{T})$-mapping that transforms possibilistic information—in the form of $(L, \leq)$-fuzzy variables in $(X, \mathcal{R})$—about the values ξ may assume in X, into possibilistic information—in the form of $(L, \leq)$-fuzzy variables in $(\mathcal{T}, \mathcal{R}^{(\chi_{A_{P_p}})})$—about the values that $\chi_{A_{P_p}}(\xi)$ may assume in $\mathcal{T}$: $\pi_{\chi_{A_{P_p}}(\xi)} = \widetilde{\chi_{A_{P_p}}}(\pi_\xi)$. We are thus led to the following important definition.

Definition 4 (Possibilistic truth values) *If the values a variable ξ may assume in X are restricted by the possibilistic information h—in the form of a sup-normal $(L, \leq)$-fuzzy variable in $(X, \mathcal{R})$—then $\widetilde{\chi_{A_{P_p}}}(h)$—a sup-normal $(L, \leq)$-fuzzy variable in $(\mathcal{T}, \mathcal{R}^{(\chi_{A_{P_p}})})$—is called the $(L, \leq)$-possibilistic truth value of the proposition variable 'ξ is p'.*

In the next subsection, we shall take the next step in our course of reasoning, and investigate how the logical combination of propositions (and proposition variables) may lead to an appropriate *combination of possibilistic truth values.* In the rest of this subsection, we shall add a little more detail to the picture that has been sketched thus far.

First of all, it is important to note that we only work with properties p that are clear. This means that this discussion lies well within the province of *classical propositional logic.* What we are trying to do here is to add linguistic uncertainty to the classical description. Secondly, it should be clear that the information we derive about the values that $\chi_{A_{P_p}}(\xi)$ takes in $\mathcal{T}$ is really only useful if we are able to separate the information about 'ξ is p' being true on the one hand, and about its being false on the other hand. In other words, $\{true\}$ and $\{false\}$ must each be $\mathcal{R}^{(\chi_{A_{P_p}})}$-measurable sets, or equivalently, we must have that $\mathcal{R}^{(\chi_{A_{P_p}})} = \mathcal{P}(\mathcal{T})$. If not, then clearly $\mathcal{R}^{(\chi_{A_{P_p}})} = \{\emptyset, \mathcal{T}\}$ and we find for an arbitrary sup-normal $(L, \leq)$-fuzzy variable h only the following, hardly relevant, information:

$$\begin{cases} \widetilde{\chi_{A_{P_p}}}(h) \cdot true = 1 \\ \widetilde{\chi_{A_{P_p}}}(h) \cdot false = 1, \end{cases}$$

since in this case $[\,true\,]_{\mathcal{R}^{(\chi_{A_{P_p}})}} = [\,false\,]_{\mathcal{R}^{(\chi_{A_{P_p}})}} = \mathcal{T}$. The following proposition sheds more light on this observation.

Proposition 1 *Let X be a universe and let $\mathcal{R}$ be an ample field on X. Let furthermore A be an arbitrary subset of X, with characteristic $X - \mathcal{T}$-mapping χ_A. Then $\mathcal{R}^{(\chi_A)} = \mathcal{P}(\mathcal{T}) \Leftrightarrow A \in \mathcal{R}$.*

Proof. It is clear that '$\mathcal{R}^{(\chi_A)} = \mathcal{P}(\mathcal{T})$' is equivalent with '$\{true\} \in \mathcal{R}^{(\chi_A)}$', which is in turn equivalent with $\chi_A^{-1}(\{true\}) \in \mathcal{R}$. Moreover, $\chi_A^{-1}(\{true\}) = A$. $\square$

This means that the separation of information about the proposition variable 'ξ is p' being true or its being false is only possible if A_{P_p} is $\mathcal{R}$-measurable, or, in other words, if, looking through the glasses of $\mathcal{R}$, we may distinguish between objects in X that satisfy p and objects that do not. *In the rest of this paper, we shall assume that this is indeed the case*, which in particular implies that

$$\mathcal{G}_{(L,\leq)}^{\mathcal{R}^{(\chi_{A_{P_p}})}}(\mathcal{T}) = \mathcal{G}_{(L,\leq)}^{\mathcal{P}(\mathcal{T})}(\mathcal{T}) = \mathcal{F}_{(L,\leq)}(\mathcal{T}),$$

and that $\widetilde{\chi_{A_{P_p}}}$ is a $\mathcal{G}_{(L,\leq)}^{\mathcal{R}}(X) - \mathcal{F}_{(L,\leq)}(\mathcal{T})$-mapping.

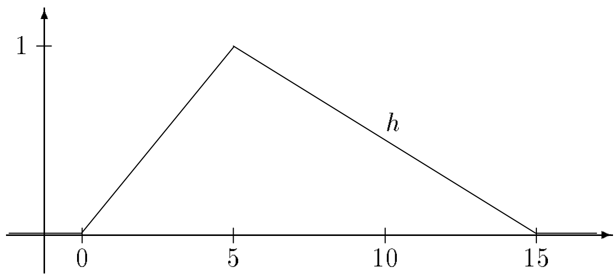

Figure 2 The fuzzy quantity h from example 4.

To summarize, if the information about the values that ξ may assume in X is given by a sup-normal $(L,\leq)$-fuzzy variable π_ξ in $(X,\mathcal{R})$, we now know that for arbitrary ν in $\mathcal{T}$

$$\pi_{\chi_{A_{P_p}}(\xi)}(\nu) = \widetilde{\chi_{A_{P_p}}}(\pi_\xi)\cdot\nu = \sup_{\chi_{A_{P_p}}(x)=\nu} \pi_\xi(x).$$

The following example will illustrate the course of reasoning established thus far.

Example 4 Let $X=\mathbb{R}$, $\mathcal{R}=\mathcal{P}(\mathbb{R})$, $(L,\leq)=([0,1],\leq)$ and, as in example 3, let p be the predicate 'greater than or equal to 10'. Let furthermore ξ be a real possibilistic variable, with possibility distribution function h, the triangular fuzzy quantity depicted in figure 2: for arbitrary a in $\mathbb{R}$, $h(a)$ is the $([0,1],\leq)$-possibility that ξ assumes the value a. We now ask ourselves what is the $([0,1],\leq)$-possibilistic truth value of the proposition 'ξ is greater then or equal to 10'? From the considerations above we deduce that

$$\begin{cases} \widetilde{\chi_{A_{P_p}}}(h)\cdot \mathit{true} = \sup\limits_{a\geq 10} h(a) = 1/2 \\ \widetilde{\chi_{A_{P_p}}}(h)\cdot \mathit{false} = \sup\limits_{a<10} h(a) = 1. \end{cases}$$

The $([0,1],\leq)$-possibilistic truth value of the proposition '$\xi\geq 10$' is therefore $\{(\mathit{true},1/2),(\mathit{false},1)\}$, and this must be interpreted as follows: given the possibilistic information h about the values ξ can assume in $\mathbb{R}$, the $([0,1],\leq)$-possibility that the proposition '$\xi\geq 10$' is true is equal to $1/2$ and the $([0,1],\leq)$-possibility that it is false is equal to 1.

To close this subsection, let us note that, until now, we have always considered the case of just one variable ξ assuming values in a universe X. It is of course possible to extend these observations to the case of more than one, say n, variables $\xi_1, \ldots, \xi_n$ assuming values in the respective universes $X_1, \ldots, X_n$ ($n \in \mathbb{N} \setminus \{0\}$). This is not at all difficult since the treatment of n variables can be considered as a special case of the treatment of one variable by putting $\xi = (\xi_1, \ldots, \xi_n)$ and $X = X_1 \times \cdots \times X_n$. Due to limitations of space, we shall however in this paper leave such an extension implicit. For more details, we refer to our doctoral dissertation [9].

4.2 Combination of Possibilistic Truth Values

In the previous subsection, we have shown how possibilistic information about the values a variable may assume in a universe can be transformed into possibilistic information about the truth value of a proposition about this variable. To go to the next stage in our discussion, we observe that, in general, propositions can be combined to form new propositions, using so-called *logical operators*. In this way, a proposition P can be transformed by the *logical negation operator* into a new proposition NOT P. By extension, the proposition function P_p discussed in the previous subsection can be transformed into a new proposition function NOT P_p by the pointwise application of the logical negation operator:

$$(\forall x \in X)((\text{NOT } P_p)(x) \stackrel{\text{def}}{=} \text{NOT}(P_p(x)) = \text{`}x \text{ is not } p\text{'}).$$

In a completely similar way, the proposition variable $P_p(\xi) \stackrel{\text{def}}{=}$ 'ξ is p' is transformed by the logical negation operator into the new proposition variable (NOT P_p)(ξ), defined as 'ξ is not p'. Analogously, the proposition variables 'ξ is p' and 'ξ is q' can be transformed into the new proposition variable (P_p AND P_q)(ξ), defined as 'ξ is p AND ξ is q', using the logical conjunction operator AND, and can be transformed into the new proposition variable (P_p OR P_q)(ξ), defined as 'ξ is p OR ξ is q', using the logical disjunction operator OR.

As is well known, classical propositional logic is *truth-functional*, and the behaviour of logical operators can be characterized by what we shall call further on classical-logical functions, i.e., in general $\mathcal{T}^n - \mathcal{T}$-mappings, with $n \in \mathbb{N}\setminus\{0\}$. To give an example, the behaviour of the *logical negation operator* is mirrored in the behaviour of the complement operator $\neg$ on the Boolean chain $(\mathcal{T}, \leq)$, in the following sense:

$$(\forall x \in X)(\chi_{A_{\text{NOT } P_p}}(x) = \neg\chi_{A_{P_p}}(x)),$$

where, of course $A_{\text{NOT } P_p} = \{\, x \mid x \in X \text{ and } x \text{ is not } p \,\} = \text{co}A_{P_p}$. In a completely analogous way, the behaviour of the *logical disjunction operator* is mirrored in the behaviour of the join $\vee$ of the Boolean chain $(\mathcal{T}, \leq)$, in the following sense:

$$(\forall x \in X)(\chi_{A_{P_p \text{ OR } P_q}}(x) = \chi_{A_{P_p}}(x) \vee \chi_{A_{P_q}}(x)),$$

where $A_{P_p \text{ OR } P_q} = \{\, x \mid x \in X \text{ and } (x \text{ is } p \text{ OR } x \text{ is } q) \,\} = A_{P_p} \cup A_{P_q}$.

Generally speaking, we can start with n ($n \in \mathbb{N} \setminus \{0\}$) clear predicates p_1, $\ldots$, p_n with associated sets $A_{P_{p_1}}$, $\ldots$, $A_{P_{p_n}}$, and with a n-ary logical operator LOP. This logical operator transforms the proposition variables $P_{p_1}(\xi)$, $\ldots$, $P_{p_n}(\xi)$ into the new proposition variable $\text{LOP}(P_{p1}, \ldots, P_{p_n})(\xi)$, defined as LOP('$\xi$ is p_1', $\ldots$, 'ξ is p_n'), with associated set $A_{\text{LOP}(P_{p1},\ldots,P_{p_n})}$. The behaviour of LOP is mirrored by a $\mathcal{T}^n - \mathcal{T}$-mapping ϕ, in the following sense:

$$(\forall x \in X)(\chi_{A_{\text{LOP}(P_{p1},\ldots,P_{p_n})}}(x) = \phi(\chi_{A_{P_{p_1}}}(x), \ldots, \chi_{A_{P_{p_n}}}(x))),$$

or, shortly, $\chi_{A_{\text{LOP}(P_{p_1},\ldots,P_{p_n})}} = \phi \circ (\chi_{A_{P_{p_1}}}, \ldots, \chi_{A_{P_{p_n}}})$, where, as usual, the characteristic $X - \mathcal{T}$-mapping of $A_{P_{p_k}}$ is denoted by $\chi_{A_{P_{p_k}}}$ ($k = 1, \ldots, n$), and $\chi_{A_{\text{LOP}(P_{p_1},\ldots,P_{p_n})}}$ is the characteristic $X - \mathcal{T}$-mapping of $A_{\text{LOP}(P_{p1},\ldots,P_{p_n})}$.

The problem we will treat in this subsection can now be briefly formulated as follows: *how, starting with possibilistic information about the values that a possibilistic variable ξ may assume in X, can we derive the possibilistic truth value of the combined proposition variable* LOP('ξ is p_1', $\ldots$, 'ξ is p_n')? In order to give an answer to this question, we again apply the general results of the previous section.

For a start, the sets $A_{P_{p_1}}$, $\ldots$, $A_{P_{p_n}}$ will, as explained in the previous subsection, be assumed $\mathcal{R}$-measurable. Let us now define the $X - \mathcal{T}^n$-mapping χ as follows: $\chi \stackrel{\text{def}}{=} (\chi_{A_{P_{p_1}}}, \ldots, \chi_{A_{P_{p_n}}})$. Since the sets $A_{P_{p_1}}$, $\ldots$, $A_{P_{p_n}}$ are assumed to be $\mathcal{R}$-measurable, it is easily verified that

$$\mathcal{R}^{(\chi)} = \mathcal{P}(\mathcal{T}^n). \tag{1.14}$$

Using the results of the previous section with $Y = \mathcal{T}^n$ and $\varphi = \chi$, we find that the $\Omega - \mathcal{T}^n$-mapping $\chi \circ \xi = \chi(\xi)$ is a possibilistic variable in $(\mathcal{T}^n, \mathcal{P}(\mathcal{T}^n))$. For the possibility distribution $\Pi_{\chi(\xi)}$ of this variable, we find that

$$\Pi_{\chi(\xi)} = \Pi_\xi{}^{(\chi)}, \tag{1.15}$$

and for the possibility distribution function $\pi_{\chi(\xi)}$ of $\chi(\xi)$ we have that $\pi_{\chi(\xi)} = \widetilde{\chi}(\pi_\xi)$, or, for arbitrary $(\nu_1, \ldots, \nu_n)$ in $\mathcal{T}^n$:

$$\begin{aligned} \pi_{\chi(\xi)}(\nu_1, \ldots, \nu_n) &= \sup_{\chi(x)=(\nu_1,\ldots,\nu_n)} \pi_\xi(x) \\ &= \sup_{(\chi_{A_{P_{p_1}}}(x),\ldots,\chi_{A_{P_{p_n}}}(x))=(\nu_1,\ldots,\nu_n)} \pi_\xi(x) \\ &= \sup_{x\in \bigcap_{k=1}^{n} \chi_{A_{P_{p_k}}}{}^{-1}(\{\nu_k\})} \pi_\xi(x). \end{aligned} \tag{1.16}$$

In a second stage, we also bring the $\mathcal{T}^n - \mathcal{T}$-mapping ϕ into the picture, and apply the results of the previous section in the following special case: $\varphi = \chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}} = \phi \circ \chi$ and $Y = \mathcal{T}$. $\phi \circ \chi$ is a $X - \mathcal{T}$-mapping, for which, taking into account theorem 1 and (1.14), $\mathcal{R}^{(\phi\circ\chi)} = (\mathcal{R}^{(\chi)})^{(\phi)} = (\mathcal{P}(\mathcal{T}^n))^{(\phi)} = \mathcal{P}(\mathcal{T})$. It now follows from the course of reasoning in the previous section that the $\Omega - \mathcal{T}$-mapping $\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi) = (\phi\circ\chi)\circ\xi = (\phi\circ\chi)(\xi)$ is a possibilistic variable in $(\mathcal{T}, \mathcal{P}(\mathcal{T}))$. For the possibility distribution $\Pi_{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi)}$ of $\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi)$ we find that $\Pi_{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi)} = \Pi_{(\phi\circ\chi)(\xi)} = \Pi_\xi{}^{(\phi\circ\chi)}$, or equivalently, for arbitrary B in $\mathcal{P}(\mathcal{T})$, taking into account (1.15):

$$\begin{aligned} \Pi_{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi)}(B) &= \Pi_\xi{}^{(\phi\circ\chi)}(B) \\ &= \Pi_\xi((\phi\circ\chi)^{-1}(B)) \\ &= \Pi_\xi(\chi^{-1}(\phi^{-1}(B))) \\ &= \Pi_{\chi(\xi)}(\phi^{-1}(B)) \\ &= \Pi_{\chi(\xi)}{}^{(\phi)}(B), \end{aligned}$$

since clearly $\phi^{-1}(B) \in \mathcal{P}(\mathcal{T}^n)$. Hence, $\Pi_{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi)} = \Pi_{\chi(\xi)}{}^{(\phi)}$. For the possibility distribution function $\pi_{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi)}$ of the possibilistic variable $\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi)$ we find, for arbitrary ν in $\mathcal{T}$:

$$\begin{aligned} \pi_{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi)}(\nu) &= \widetilde{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}}(\pi_\xi)\cdot\nu \\ &= \Pi_{\chi(\xi)}{}^{(\phi)}(\{\nu\}) \\ &= \Pi_{\chi(\xi)}(\phi^{-1}(\{\nu\})) \\ &= \sup_{(\nu_1,\ldots,\nu_n)\in\phi^{-1}(\{\nu\})} \pi_{\chi(\xi)}(\nu_1,\ldots,\nu_n) \\ &= \sup_{\phi(\nu_1,\ldots,\nu_n)=\nu} \pi_{\chi(\xi)}(\nu_1,\ldots,\nu_n) \end{aligned} \tag{1.17}$$

$$
\begin{tikzcd}
X \arrow[rr, "(\chi_{A_{P_{p_1}}}, \ldots, \chi_{A_{P_{p_n}}})"] \arrow[dr, "\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}"'] & & \mathcal{T}^n \arrow[dl, "\varphi"] \\
& \mathcal{T} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X) \arrow[rr, "\widetilde{(\chi_{A_{P_{p_1}}}, \ldots, \chi_{A_{P_{p_n}}})}"] \arrow[dr, "\widetilde{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}}"'] & & \mathcal{F}_{(L,\leq)}(\mathcal{T}^n) \arrow[dl, "\widetilde{\varphi}"] \\
& \mathcal{F}_{(L,\leq)}(\mathcal{T}) &
\end{tikzcd}
$$

Figure 3 Commutative diagrams for the course of reasoning in subsection 4.2

By combining (1.16) and (1.17), we find that

$$\pi_{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi)}(\nu) = \sup_{\phi(\nu_1,\ldots,\nu_n)=\nu} \; \sup_{x \in \bigcap_{k=1}^{n} \chi_{A_{P_{p_k}}}{}^{-1}(\{\nu_k\})} \pi_\xi(x). \tag{1.18}$$

Using the appropriate possibilistic extensions, this result may also be written as

$$\pi_{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}(\xi)}(\nu) = \widetilde{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}}(\pi_\xi) = (\widetilde{\phi \circ \chi})(\pi_\xi) = \widetilde{\phi}(\widetilde{\chi}(\pi_\xi))$$

This line of reasoning is pictorially summarized in the commuting diagrams of figure 3.

4.3 A Truth-Functional Approximation

In his doctoral dissertation [27], Van Schooten uses a different way[4] to calculate the possibilistic truth value of the propositional variable $\mathrm{LOP}(P_{p1}, \ldots, P_{p_n})(\xi)$. His approach could be called *truth-functional.* We may briefly summarize the ideas behind his method as follows. Starting with the possibilistic information π_ξ about the values ξ may assume in X, he separately calculates the $(L, \leq)$-possibilistic[5] truth value $\pi_{\chi_{A_{P_{p_k}}}(\xi)}$ of every proposition variable 'ξ is p_k' ($k =$

[4] This statement must be seen as an *a posteriori* evaluation of Van Schooten's approach, based upon the possibility-theoretic framework we are constructing here. First of all, Van Schooten only considers the special case of unary and binary logical operators, and Zadeh's possibility measures. Secondly, his approach is more *ad hoc*, and certainly does not draw upon the rigorous mathematical account of possibility measures and possibilistic variables we have developed in the previous section, and in other papers [4, 6, 10, 15].

[5] Van Schooten only works with the special case $(L, \leq) = ([0,1], \leq)$, and the minimum operator on $[0,1]$ as triangular norm, but his approach can easily be extended towards the

$1,\ldots,n$). Since $\mathcal{R}^{(\chi_{A_{P_{p_k}}})} = \mathcal{P}(\mathcal{T})$ and therefore $\mathcal{G}_{(L,\leq)}^{\mathcal{R}^{(\chi_{A_{P_{p_k}}})}}(\mathcal{T}) = \mathcal{F}_{(L,\leq)}(\mathcal{T})$, we have that $\widetilde{\chi_{A_{P_{p_k}}}}$ is a $\mathcal{G}_{(L,\leq)}^{\mathcal{R}}(X) - \mathcal{F}_{(L,\leq)}(\mathcal{T})$-mapping, with, for arbitrary ν_k in $\mathcal{T}$:

$$\pi_{\chi_{A_{P_{p_k}}}(\xi)}(\nu_k) = \widetilde{\chi_{A_{P_{p_k}}}}(\pi_\xi) \cdot \nu_k = \sup_{\chi_{A_{P_{p_k}}}(x)=\nu_k} \pi_\xi(x). \tag{1.19}$$

These n $(L,\leq)$-possibilistic truth values are then combined into a new $(L,\leq)$-possibilistic truth value, using the *$(L,\leq)$-possibilistic T-extension $\widetilde{\phi}_T$ of ϕ*:

$$\widetilde{\phi}_T : \mathcal{F}_{(L,\leq)}(\mathcal{T})^n \to \mathcal{F}_{(L,\leq)}(\mathcal{T}) : (t_1,\ldots,t_n) \mapsto \widetilde{\phi}_T(t_1,\ldots,t_n)$$

with, for arbitrary ν in $\mathcal{T}$

$$\widetilde{\phi}_T(t_1,\ldots,t_n) \cdot \nu = \sup_{\phi(\nu_1,\ldots,\nu_n)=\nu} T_{k=1}^n t_k(\nu_k). \tag{1.20}$$

If we denote this new $(L,\leq)$-possibilistic truth value by $\widetilde{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}}'(\pi_\xi)$, then we find, by combining (1.19) and (1.20)

$$\begin{aligned}\widetilde{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}}'(\pi_\xi) &= \widetilde{\phi}_T(\pi_{\chi_{A_{P_{p_1}}}(\xi)},\ldots,\pi_{\chi_{A_{P_{p_n}}}(\xi)}) \\ &= \widetilde{\phi}_T(\widetilde{\chi_{A_{P_{p_1}}}}(\pi_\xi),\ldots,\widetilde{\chi_{A_{P_{p_n}}}}(\pi_\xi)),\end{aligned}$$

or, equivalently, for arbitrary ν in $\mathcal{T}$

$$\widetilde{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}}'(\pi_\xi) \cdot \nu = \sup_{\phi(\nu_1,\ldots,\nu_n)=\nu} T_{k=1}^n \sup_{\chi_{A_{P_{p_k}}}(x)=\nu_k} \pi_\xi(x). \tag{1.21}$$

This course of reasoning is summarized in the commuting diagram of figure 4.

In the following counter-example, we show that this truth-functional approach does not necessarily lead to the same result as the possibilistic method described in the previous subsection.

Example 5 We use the notations and conventions of examples 3 and 4. Besides the predicate p, we shall also consider the complementary predicate q, defined as 'smaller than 10'. The $(L,\leq)$-possibilistic truth value $\widetilde{\chi_{A_{P_q}}}(h)$ of the proposition variable '$\xi < 10$' is then given by

$$\begin{cases} \widetilde{\chi_{A_{P_q}}}(h) \cdot \mathit{true} = \sup_{a<10} h(a) = 1 \\ \widetilde{\chi_{A_{P_q}}}(h) \cdot \mathit{false} = \sup_{a\geq 10} h(a) = 1/2. \end{cases}$$

more general $(L,\leq)$ and arbitrary t-norms T on $(L,\leq)$. It is precisely this generalization that we are discussing here.

$$
\begin{tikzcd}
\mathcal{G}^{\mathcal{R}}_{(L,\leq)}(X) \arrow[rr, "{(\widetilde{\chi_{A_{P_{p_1}}}}, \ldots, \widetilde{\chi_{A_{P_{p_n}}}})}"] \arrow[dr, swap, "\widetilde{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}}"] & & \mathcal{F}_{(L,\leq)}(\mathcal{T})^n \arrow[dl, "\widetilde{\varphi}_T"] \\
& \mathcal{F}_{(L,\leq)}(\mathcal{T}) &
\end{tikzcd}
$$

Figure 4 Commutative diagram for the course of reasoning in subsection 4.3

Let us furthermore consider the logical disjunction operator OR with associated logical function $\phi = \vee$, then clearly $A_{P_p \text{ OR } P_q} = \mathbb{R}$, since, by definition, for arbitrary a in $\mathbb{R}$, $\chi_{A_{P_p \text{ OR } P_q}}(a) = \chi_{A_{P_p}}(a) \vee \chi_{A_{P_q}}(a) = \mathit{true}$. Therefore $\widetilde{\chi_{A_{P_p \text{ OR } P_q}}}(h) = \{(\mathit{true}, 1), (\mathit{false}, 0)\}$, irrespective of h, which was, of course, to be expected, since the predicates p and q are complementary.

In order to apply the truth-functional method, we must calculate $\widetilde{\vee}_T$. For arbitrary (t_1, t_2) in $\mathcal{F}_{(L,\leq)}(\mathcal{T})^2$:

$$
\begin{aligned}
&(t_1 \,\widetilde{\vee}_T\, t_2) \cdot \mathit{true} \\
&\quad = \sup_{\nu_1 \vee \nu_2 = \mathit{true}} T(t_1(\nu_1), t_2(\nu_2)) \\
&\quad = \sup(T(t_1(\mathit{true}), t_2(\mathit{true})), T(t_1(\mathit{true}), t_2(\mathit{false})), T(t_1(\mathit{false}), t_2(\mathit{true})))
\end{aligned}
$$

and

$$
(t_1 \,\widetilde{\vee}_T\, t_2) \cdot \mathit{false} = \sup_{\nu_1 \vee \nu_2 = \mathit{false}} T(t_1(\nu_1), t_2(\nu_2)) = T(t_1(\mathit{false}), t_2(\mathit{false})).
$$

If we substitute $t_1 = \widetilde{\chi_{A_{P_p}}}(h)$ and $t_2 = \widetilde{\chi_{A_{P_q}}}(h)$ in these expressions, we find

$$
\begin{cases}
\widetilde{\chi_{A_{P_p \text{ OR } P_q}}}'(h) \cdot \mathit{true} = \sup(T(1/2, 1), T(1/2, 1/2), T(1, 1)) = 1 \\
\widetilde{\chi_{A_{P_p \text{ OR } P_q}}}'(h) \cdot \mathit{false} = T(1, 1/2) = 1/2,
\end{cases}
$$

for every triangular norm T, which clearly cannot be correct.

The fact that the truth-functional method is not necessarily correct should not surprise us. Indeed, we find by comparing (1.17) and (1.21), taking into account

$$
\pi_{\chi_{A_{P_{p_k}}}(\xi)}(\nu_k) = \widetilde{\chi_{A_{P_{p_k}}}}(\pi_\xi) \cdot \nu_k = \sup_{\chi_{A_{P_{p_k}}} = \nu_k} \pi_\xi(x),
$$

that it is only correct if

$$(\forall(\nu_1,\ldots,\nu_n)\in\mathcal{T}^n)(\pi_{\chi(\xi)}(\nu_1,\ldots,\nu_n)=T_{k=1}^n\pi_{\chi_{A_{P_{p_k}}}(\xi)}(\nu_k)),$$

which is equivalent with (see [9, 15]) the (Π_Ω,T)-independence of the possibilistic variables $\chi_{A_{P_{p_1}}}(\xi),\ldots,\chi_{A_{P_{p_n}}}(\xi)$ in $(\mathcal{T},\mathcal{P}(\mathcal{T}))$, and, taking into account $\chi_{A_{P_{p_k}}}(\xi)=\chi_{\xi^{-1}(A_{P_{p_k}})}$, also equivalent with the (Π_Ω,T)-independence of the events $\xi^{-1}(A_{P_{p_1}}),\ldots,\xi^{-1}(A_{P_{p_n}})$ in Ω. From a possibilistic point of view, this imposes certain restrictions on the truth-functional approach, as proposed by Van Schooten. Nevertheless, in the next proposition we show that the truth-functional approach is to a certain extent defendable. Its proof is obvious, taking into account the definition and properties of infimum and supremum.

Proposition 2 *Let X be a universe, $\mathcal{R}$ an ample field on X, n an element of $\mathbb{N}\setminus\{0\}$, ϕ a $\mathcal{T}^n-\mathcal{T}$-mapping, A_k an element of $\mathcal{R}$ ($k=1,\ldots,n$), h a $(L,\leq)$-fuzzy variable in $(X,\mathcal{R})$ and ν an element of $\mathcal{T}$. Then*

$$\sup_{\phi(\nu_1,\ldots,\nu_n)=\nu}\ \sup_{x\in\bigcap_{k=1}^n\chi_{A_k}^{-1}(\{\nu_k\})}h(x)\leq\sup_{\phi(\nu_1,\ldots,\nu_n)=\nu}\ \inf_{k=1}^n\ \sup_{x\in\chi_{A_k}^{-1}(\{\nu_k\})}h(x). \tag{1.22}$$

This proposition implies that in the case $(L,\leq)=([0,1],\leq)$ and $T=\frown$ considered by Van Schooten, the truth-functional approach results in a *conservative approximation*, because it pushes the possibilistic truth values towards $\{(\mathit{true},1),(\mathit{false},1)\}$, i.e., both truth values *true* and *false* are equally possible. To put it more concretely, assume that the possibilistic method results in a $([0,1],\leq)$-possibilistic truth value[6] $t_{pos}=\{(\mathit{true},1),(\mathit{false},a)\}$, with $a\in[0,1]$. The proposition above then tells us that the truth-functional approach must yield a $([0,1],\leq)$-possibilistic truth value $t_{tf}=\{(\mathit{true},1),(\mathit{false},b)\}$ with $b\in[0,1]$ and $b\geq a$. This means that the information, obtained in the truth-functional way, is only less restrictive, or in other words, there can be no contradiction, only loss of specificity. This, together with the results derived in section 6, makes the truth-functional approach for $T=\frown$ surely defendable.

[6] Normalization implies that at least one of both numbers $t_{pos}(\mathit{true})$ and $t_{pos}(\mathit{false})$ must be equal to 1.

5 POSSIBILISTIC EXTENSION LOGICS

5.1 Towards a Possibilistic Logic

In this section, we shall take the discussion of the previous section one step further, and look at its results from the standpoint of multi-valued logic (see, for instance, [25]). In other words, we want to investigate how the introduction of possibilistic uncertainty in classical propositional logic leads to the introduction of a special multi-valued logic, with a proper set of truth values and logical functions combining these truth values.

As in the previous sections, we consider a universe X, provided with an ample field $\mathcal{R}$ of measurable sets. Let us briefly summarize what we already know. In the previous sections, we have seen that possibilistic information about the values that a variable ξ may assume in X, can be represented by a sup-normal $(L, \leq)$-fuzzy variable in $(X, \mathcal{R})$, interpreted as the distribution of a normalized $(L, \leq)$-possibility measure on $(X, \mathcal{R})$. When p is a clear property, and the associated subset A_{P_p} of X is $\mathcal{R}$-measurable, this possibilistic information can be transformed into possibilistic information about the truth value of the proposition variable 'ξ is p'. This information can be represented by a sup-normal $(L, \leq)$-fuzzy variable in $(\mathcal{T}, \mathcal{P}(\mathcal{T}))$. Such a $(L, \leq)$-fuzzy variable in $(\mathcal{T}, \mathcal{P}(\mathcal{T}))$, in other words a $(L, \leq)$-fuzzy set in $\mathcal{T}$, can be generally called a $(L, \leq)$-possibilistic truth value. This is formalized in the following definition.

Definition 5 (Possibilistic truth values) *We call $(L, \leq)$-possibilistic truth value any sup-normal $(L, \leq)$-fuzzy set in $\mathcal{T} = \{true, false\}$. The set of the $(L, \leq)$-possibilistic truth values will be denoted by $\widetilde{\mathcal{T}}$. If, for whatever reason, we do not want to mention the complete lattice $(L, \leq)$ explicitly, we shall simply speak of possibilistic truth values.*

Since the complete lattice $(L, \leq)$ is bounded, we immediately arrive at the following general definition.

Definition 6 *We introduce three $(L, \leq)$-possibilistic truth values with a special meaning: $\widetilde{false} \stackrel{\text{def}}{=} \{(true, 0), (false, 1)\}$, $\widetilde{true} \stackrel{\text{def}}{=} \{(true, 1), (false, 0)\}$ and $\widetilde{unknown} \stackrel{\text{def}}{=} \{(true, 1), (false, 1)\}$.*

These special possibilistic truth values can be interpreted as follows. When a proposition variable 'ξ is p' has the $(L, \leq)$-possibilistic truth value $\widetilde{true}$, this means that it cannot be false, and is therefore *necessarily true*, taking into account the information we have about the values that ξ may assume in X. An analogous (dual) interpretation can be given to $\widetilde{false}$. When, on the other hand, the proposition variable 'ξ is p' has the $(L, \leq)$-possibilistic truth value $\widetilde{unknown}$, this means that, taking into account the information we have about the values that ξ may assume, it is completely possible that the proposition variable is true, and equally possible that it is false. In other words, the truth value of this proposition variable is *completely unknown*, because of insufficient information about the values that ξ may assume in X.

An important property of classical propositional logic is what could be called its *truth-functionality*. This means that propositions can be combined to form new propositions using *logical operators*, the behaviour of which is mirrored in *logical functions* that turn the *truth values* of those propositions into the truth values of the new, combined propositions. In other words, with every logical operator, acting on propositions, there can be associated a unique logical function, acting on truth values, that completely characterizes its behaviour. The study of logical functions is of course an important part of classical propositional logic. In the rest of this section, we shall concentrate on the introduction and study of logical functions for the new type of (possibilistic) logic we are creating here, and that is used to model possibilistic uncertainty in classical logic. In the following definition we explicitly repeat the classical definition of a logical function, using our notations and terminology (see, for instance, [23] sections 1.6 and 1.7).

Definition 7 (Classical-logical functions) *Let n be an element of $\mathbb{N} \setminus \{0\}$. A $\mathcal{T}^n - \mathcal{T}$-mapping is called a classical-logical function of arity n. The set of the classical-logical functions of arity n is denoted by $\mathcal{L}_n$. The set of classical-logical functions of arbitrary arity is given the notation $\mathcal{L}$.*

Example 6 The conjunction $\wedge$, the disjunction $\vee$ and the implication $\Rightarrow$, defined on $\mathcal{T}$, are classical-logical functions of arity 2, characterizing the truth-functional behaviour of respectively the logical conjunction, disjunction and implication operator in classical propositional logic. The negation $\neg$, defined on $\mathcal{T}$, is a classical-logical function of arity 1, characterizing the truth-functional behaviour of the logical negation operator in that logic.

What we now want to do is to extend the classical, truth-functional approach: we formally consider $\widetilde{\mathcal{T}}$ as a set of truth values, and look at how such possibilistic truth values can be combined into new ones. After that, we intend to show that at least for some of these combinations, there is a clear and definite link with combinations of propositions. In this way, we intend to prove that, in some cases, our possibilistic logic is also truth-functional.

Definition 8 (Possibilistic-logical functions) *Let n be an element of $\mathbb{N} \setminus \{0\}$. A $(\widetilde{\mathcal{T}})^n - \widetilde{\mathcal{T}}$-mapping is called a $(L, \leq)$-possibilistic-logical function of arity n. The set of the $(L, \leq)$-possibilistic-logical functions of arity n is denoted by $\widetilde{\mathcal{L}}_n$. The set of the $(L, \leq)$-possibilistic-logical functions of arbitrary arity is given the notation $\widetilde{\mathcal{L}}$. If, for whatever reason, we do not want to mention the complete lattice $(L, \leq)$ explicitly, we shall simply speak of possibilistic-logical functions.*

If we look at the previous section, we at once see that we can associate a $(L, \leq)$-possibilistic-logical function with every classical-logical function, simply by looking at its $(L, \leq)$-possibilistic T-extension. Of course, this extension must be properly restricted, because we only work with elements of $\widetilde{\mathcal{T}}$—and not $\mathcal{F}_{(L,\leq)}(\mathcal{T})$—as possibilistic truth values.

Definition 9 *Let n be an element of $\mathbb{N} \setminus \{0\}$ and let ϕ be a classical-logical function of arity n. The $(L, \leq)$-possibilistic-logical T-extension $\widetilde{\phi}_{\ell T}$ of ϕ is defined as the restriction of the $(L, \leq)$-possibilistic T-extension $\widetilde{\phi}_T$ of ϕ to the set $(\widetilde{\mathcal{T}})^n$, i.e., $\widetilde{\phi}_{\ell T} \stackrel{\text{def}}{=} \widetilde{\phi}_T|(\widetilde{\mathcal{T}})^n$.*

Corollary 3 *$(L, \leq)$-possibilistic-logical T-extensions of classical-logical functions are $(L, \leq)$-possibilistic-logical functions: $(\forall \phi \in \mathcal{L})(\widetilde{\phi}_{\ell T} \in \widetilde{\mathcal{L}})$.*

Definition 10 *We call $(L, \leq)$-possibilistic T-extension logic the set $\widetilde{\mathcal{L}}_T$ of the $(L, \leq)$-possibilistic-logical T-extensions of the classical-logical functions of any arity, i.e., $\widetilde{\mathcal{L}}_T \stackrel{\text{def}}{=} \{\, \widetilde{\phi}_{\ell T} \mid \phi \in \mathcal{L} \,\}$. If, for whatever reason, we do not want to mention the complete lattice $(L, \leq)$ and/or the t-norm T explicitly, we shall simply speak of possibilistic extension logics.*

The rationale for the introduction of these extension logics has been given in subsection 4.3. Borrowing the notations from that subsection, we know that if the events $\xi^{-1}(A_{P_{p_1}})$, ..., $\xi^{-1}(A_{P_{p_n}})$ in Ω are (Π_Ω, T)-independent, the $(L, \leq)$-possibilistic truth value $\pi_{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}}(\xi) = \widetilde{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}}(\pi_\xi)$ of the proposition variable LOP('ξ is p_1', ..., 'ξ is p_n'), is given by

$$\pi_{\chi_{A_{\mathrm{LOP}(P_{p_1},\ldots,P_{p_n})}}}(\xi) = \widetilde{\phi}_{\ell T}(\pi_{\chi_{A_{P_{p_1}}}}(\xi), \ldots, \pi_{\chi_{A_{P_{p_n}}}}(\xi)),$$

where $\pi_{\chi_{A_{P_{p_k}}}}(\xi) = \widetilde{\chi_{A_{P_{p_k}}}}(\pi_\xi)$ is the $(L, \leq)$-possibilistic truth value of the proposition variable 'ξ is p_k' $(k = 1, \ldots, n)$. Indeed, in the case of possibilistic independence, there is truth-functionality for our possibilistic logic.

5.2 Some Interesting Restrictions

We have already introduced three special possibilistic truth values $\widetilde{\mathit{true}}$, $\widetilde{\mathit{false}}$ and $\widetilde{\mathit{unknown}}$, and have briefly discussed their meaning. If we define the sets $\mathcal{W}_1 \stackrel{\text{def}}{=} \{\widetilde{\mathit{true}}, \widetilde{\mathit{false}}\}$ and $\mathcal{W}_2 \stackrel{\text{def}}{=} \{\widetilde{\mathit{true}}, \widetilde{\mathit{false}}, \widetilde{\mathit{unknown}}\}$, and properly restrict a possibilistic extension logic to these sets, a number of interesting observations can be made. The proofs of these observations are straightforward, and will be omitted here. These proofs, and more details, can be found in [9], and for the special case $(L, \leq) = ([0,1], \leq)$ in [16]. For a start, all elements of $\widetilde{\mathcal{L}}_T$ are internal in the sets $\mathcal{W}_1$ and $\mathcal{W}_2$, or in other words,

$$\begin{cases} (\forall n \in \mathbb{N} \setminus \{0\})(\forall \phi \in \mathcal{L}_n)(\forall t \in (\mathcal{W}_1)^n)(\widetilde{\phi}_{\ell T}(t) \in \mathcal{W}_1) \\ (\forall n \in \mathbb{N} \setminus \{0\})(\forall \phi \in \mathcal{L}_n)(\forall t \in (\mathcal{W}_2)^n)(\widetilde{\phi}_{\ell T}(t) \in \mathcal{W}_2). \end{cases}$$

Furthermore, $\widetilde{\mathcal{L}}_T$ restricted to $\mathcal{W}_1$ is essentially the same as—or isomorphic to—$\mathcal{L}$, via an isomorphism that identifies $\widetilde{\mathit{true}}$ and *true* on the one hand, and $\widetilde{\mathit{false}}$ en *false* on the other hand. In particular, this also implies that the Boolean algebras $(\mathcal{T}, \wedge, \vee, \neg)$ and $(\mathcal{W}_1, \widetilde{\wedge}_{\ell T}|(\mathcal{W}_1)^2, \widetilde{\vee}_{\ell T}|(\mathcal{W}_1)^2, \widetilde{\neg}_{\ell T}|\mathcal{W}_1)$ are isomorphic.

Since all elements of $\widetilde{\mathcal{L}}_T$ are internal in $\mathcal{W}_2$, restriction of the truth domain of these possibilistic-logical functions to $\mathcal{W}_2$ yields a *three-valued logic*. Since all triangular norms on $(L, \leq)$ have the same behaviour in the subset $\{0,1\}^2$ of L^2, we will find *the same ternary logic* for every choice of T. For different choices of $(L, \leq)$, the corresponding ternary logics are furthermore isomorphic. It is easily shown [16] that the truth tables for $\widetilde{\neg}_{\ell T}|\mathcal{W}_2$, $\widetilde{\wedge}_{\ell T}|(\mathcal{W}_2)^2$, $\widetilde{\vee}_{\ell T}|(\mathcal{W}_2)^2$ and $\widetilde{\Rightarrow}_{\ell T}|(\mathcal{W}_2)^2$ are identical to the corresponding truth tables of the so-called *strong ternary logic of Kleene* (see, for instance, [25] section 2.5). We shall return to this interesting fact in the following section.

5.3 A Few Properties

In the rest of this section, we shall study the most important properties of some special $(L,\leq)$-possibilistic-logical functions of arity 1 and 2: $\widetilde{\neg}_{\ell T}$, $\widetilde{\wedge}_{\ell T}$, $\widetilde{\vee}_{\ell T}$ and $\widetilde{\Rightarrow}_{\ell T}$. First of all, it will help us if we can find simple expressions for these operators. This is the subject of the next proposition. Its proof is straightforward, and is therefore omitted.

Proposition 3 *(i)* $\widetilde{\neg}_{\ell T} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}} : t \mapsto \widetilde{\neg}_{\ell T}\, t$, *with*

$$\begin{cases} (\widetilde{\neg}_{\ell T}\, t) \cdot \mathit{true} = t(\mathit{false}) \\ (\widetilde{\neg}_{\ell T}\, t) \cdot \mathit{false} = t(\mathit{true}). \end{cases}$$

(ii) $\widetilde{\wedge}_{\ell T} : (\widetilde{\mathcal{T}})^2 \to \widetilde{\mathcal{T}} : (t_1, t_2) \mapsto t_1 \,\widetilde{\wedge}_{\ell T}\, t_2$, *with*

$$\begin{cases} (t_1 \,\widetilde{\wedge}_{\ell T}\, t_2) \cdot \mathit{true} = T(t_1(\mathit{true}), t_2(\mathit{true})) \\ (t_1 \,\widetilde{\wedge}_{\ell T}\, t_2) \cdot \mathit{false} = t_1(\mathit{false}) \smile t_2(\mathit{false})). \end{cases}$$

(iii) $\widetilde{\vee}_{\ell T} : (\widetilde{\mathcal{T}})^2 \to \widetilde{\mathcal{T}} : (t_1, t_2) \mapsto t_1 \,\widetilde{\vee}_{\ell T}\, t_2$, *with*

$$\begin{cases} (t_1 \,\widetilde{\vee}_{\ell T}\, t_2) \cdot \mathit{true} = t_1(\mathit{true}) \smile t_2(\mathit{true}) \\ (t_1 \,\widetilde{\vee}_{\ell T}\, t_2) \cdot \mathit{false} = T(t_1(\mathit{false}), t_2(\mathit{false})). \end{cases}$$

(iv) $\widetilde{\Rightarrow}_{\ell T} : (\widetilde{\mathcal{T}})^2 \to \widetilde{\mathcal{T}} : (t_1, t_2) \mapsto t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_2$, *with*

$$\begin{cases} (t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_2) \cdot \mathit{true} = t_1(\mathit{false}) \smile t_2(\mathit{true}) \\ (t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_2) \cdot \mathit{false} = T(t_1(\mathit{true}), t_2(\mathit{false})). \end{cases}$$

Let us now give a brief survey of the most important properties of the above-mentioned possibilistic-logical functions. The proofs of these properties are fairly simple, and we have consequently omitted them. It should nevertheless be noted that the equalities that appear in these properties, are equalities of $(L,\leq)$-possibilistic truth values, and therefore pointwise equalities of $\mathcal{T} - L$-mappings.

Property 1 (Commutativity) *For arbitrary* t_1 *and* t_2 *in* $\widetilde{\mathcal{T}}$*:*

$$\begin{cases} t_1 \,\widetilde{\wedge}_{\ell T}\, t_2 = t_2 \,\widetilde{\wedge}_{\ell T}\, t_1 \\ t_1 \,\widetilde{\vee}_{\ell T}\, t_2 = t_2 \,\widetilde{\vee}_{\ell T}\, t_1. \end{cases}$$

Property 2 (Neutral elements) *For arbitrary t in $\widetilde{T}$:*
$$\begin{cases} t \,\widetilde{\wedge}_{\ell T}\, \widetilde{true} = t \\ t \,\widetilde{\vee}_{\ell T}\, \widetilde{false} = t. \end{cases}$$

Property 3 (Associativity) *For arbitrary t_1, t_2 and t_3 in $\widetilde{T}$:*
$$\begin{cases} t_1 \,\widetilde{\wedge}_{\ell T}\, (t_2 \,\widetilde{\wedge}_{\ell T}\, t_3) = (t_1 \,\widetilde{\wedge}_{\ell T}\, t_2) \,\widetilde{\wedge}_{\ell T}\, t_3 \\ t_1 \,\widetilde{\vee}_{\ell T}\, (t_2 \,\widetilde{\vee}_{\ell T}\, t_3) = (t_1 \,\widetilde{\vee}_{\ell T}\, t_2) \,\widetilde{\vee}_{\ell T}\, t_3. \end{cases}$$

Property 4 (De Morgan's Laws) *For arbitrary t_1 and t_2 in $\widetilde{T}$:*
$$\begin{cases} \widetilde{\neg}_{\ell T}\, (t_1 \,\widetilde{\wedge}_{\ell T}\, t_2) = (\widetilde{\neg}_{\ell T}\, t_1) \,\widetilde{\vee}_{\ell T}\, (\widetilde{\neg}_{\ell T}\, t_2) \\ \widetilde{\neg}_{\ell T}\, (t_1 \,\widetilde{\vee}_{\ell T}\, t_2) = (\widetilde{\neg}_{\ell T}\, t_1) \,\widetilde{\wedge}_{\ell T}\, (\widetilde{\neg}_{\ell T}\, t_2). \end{cases}$$

Property 5 (Absorbing elements) *For arbitrary t in $\widetilde{T}$:*
$$\begin{cases} t \,\widetilde{\wedge}_{\ell T}\, \widetilde{false} = \widetilde{false} \\ t \,\widetilde{\vee}_{\ell T}\, \widetilde{true} = \widetilde{true}. \end{cases}$$

Property 6 (Involutivity) *For arbitrary t in $\widetilde{T}$:*
$\widetilde{\neg}_{\ell T}\, (\widetilde{\neg}_{\ell T}\, t) = t.$

Property 7 (Implication) *For arbitrary t_1 and t_2 in $\widetilde{T}$:*
$t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_2 = (\widetilde{\neg}_{\ell T}\, t_1) \,\widetilde{\vee}_{\ell T}\, t_2.$

Property 8 (Complementation) *For arbitrary t in $\widetilde{T}$:*
$$\begin{cases} t \,\widetilde{\wedge}_{\ell T}\, (\widetilde{\neg}_{\ell T}\, t) = \{(true, T(t(true), t(false))), (false, 1)\} \\ t \,\widetilde{\vee}_{\ell T}\, (\widetilde{\neg}_{\ell T}\, t) = \{(true, 1), (false, T(t(true), t(false)))\} \end{cases}$$

Property 9 (Contrapositive symmetry) *For arbitrary t_1 and t_2 in $\widetilde{T}$:*
$t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_2 = (\widetilde{\neg}_{\ell T}\, t_2) \,\widetilde{\Rightarrow}_{\ell T}\, (\widetilde{\neg}_{\ell T}\, t_1).$

Property 10 (Neutrality principle) *For arbitrary t in $\widetilde{T}$:*
$(\widetilde{true} \,\widetilde{\Rightarrow}_{\ell T}\, t) = t.$

Property 11 (Exchange principle) *For arbitrary t_1, t_2 and t_3 in $\widetilde{T}$:*
$t_1 \,\widetilde{\Rightarrow}_{\ell T}\, (t_2 \,\widetilde{\Rightarrow}_{\ell T}\, t_3) = t_2 \,\widetilde{\Rightarrow}_{\ell T}\, (t_1 \,\widetilde{\Rightarrow}_{\ell T}\, t_3).$

Property 12 (Boundary conditions) *For arbitrary t_1 and t_2 in $\widetilde{\mathcal{T}}$:*

$$\begin{cases} (t_1 \,\widetilde{\wedge}_{\ell T}\, t_2 = \widetilde{true}) \Leftrightarrow (t_1 = \widetilde{true} \text{ and } t_2 = \widetilde{true}) \\ (t_1 \,\widetilde{\vee}_{\ell T}\, t_2 = \widetilde{false}) \Leftrightarrow (t_1 = \widetilde{false} \text{ and } t_2 = \widetilde{false}) \\ (\widetilde{\neg}_{\ell T}\, t = \widetilde{true}) \Leftrightarrow (t = \widetilde{false}) \\ (\widetilde{\neg}_{\ell T}\, t = \widetilde{false}) \Leftrightarrow (t = \widetilde{true}) \end{cases}$$

It is important to note that $\widetilde{\wedge}_{\ell T}$ and $\widetilde{\vee}_{\ell T}$ are idempotent if and only if T is, or, in other words, if and only if $T = \frown$ (see, for instance, [9, 14]). Furthermore, $\widetilde{\wedge}_{\ell T}$ and $\widetilde{\vee}_{\ell T}$ are mutually distributive if and only if T and sup are mutually distributive. This is only possible if $T = \frown$ (see, for instance, [9, 14]). Thus, it appears that the choice $T = \frown$ is a rather special[7] one. We therefore devote the next section to the study of this special case.

6 AN INTERESTING SPECIAL CASE

6.1 A Few Algebraic Results

In this section, we intend to take a closer look at the notions, introduced in the previous section, in the special case $T = \frown$. This means that we shall assume that $(L, \leq)$ is a complete Brouwerian lattice (see [1] section V.10). In particular, this implies that the binary operators $\frown$ and $\smile$ are mutually distributive (see, for instance, [1] section I.6). In this subsection, we have collected a number of algebraic and order-theoretic results, the first of which is given is proposition 4.

Proposition 4 *$(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown})$ is a bounded distributive lattice (as an algebra) with top element $\widetilde{true}$ and bottom element $\widetilde{false}$. The natural partial order relation $\widetilde{\leq}$ on $\widetilde{\mathcal{T}}$ that corresponds with this structure, satisfies*

$$(\forall (t_1, t_2) \in (\widetilde{\mathcal{T}})^2) \left(t_1 \,\widetilde{\leq}\, t_2 \Leftrightarrow \begin{cases} t_1(true) \leq t_2(true) \\ t_1(false) \geq t_2(false) \end{cases} \right). \tag{1.23}$$

[7] In this respect, it should also be noted that if we consider the lattice $(\widetilde{\mathcal{T}}, \widetilde{\leq})$, where $\widetilde{\leq}$ is the partial order relation on $\widetilde{\mathcal{T}}$, introduced in the following section, then $\widetilde{\wedge}_{\ell T}$ is a t-norm and $\widetilde{\vee}_{\ell T}$ is a t-conorm [9, 14] on this structure. These operators are dual [9, 14] w.r.t. the negation $\widetilde{\neg}_{\ell T}$ on $(\widetilde{\mathcal{T}}, \widetilde{\leq})$. Remark that $\widetilde{\wedge}_{\ell T}$ is the meet and $\widetilde{\vee}_{\ell T}$ the join of the lattice $(\widetilde{\mathcal{T}}, \widetilde{\leq})$ if and only if $T = \frown$.

Proof. Let us first show that $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown})$ is a lattice (as an algebra) (see, for instance, [1] section I.5). It must be proven that $\widetilde{\wedge}_{\ell\frown}$ and $\widetilde{\vee}_{\ell\frown}$ satisfy the fundamental properties of meet and join in lattices (see [1] theorem I.8). Indeed, making use of the simplified expressions for $\widetilde{\wedge}_{\ell\frown}$ and $\widetilde{\vee}_{\ell\frown}$, derived in proposition 3 (with the special choice $T = \frown$), it is easy to prove that $\widetilde{\wedge}_{\ell\frown}$ and $\widetilde{\vee}_{\ell\frown}$ are idempotent, commutative and associative, and that they satisfy the absorption laws. Next, it is a well-known result from lattice theory (see, for instance, [1] theorem I.8 and lemma I.1) that for any lattice as an algebra, and in particular also for the structure $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown})$, a natural partial order relation $\widetilde{\leq}$ can be defined on $\widetilde{\mathcal{T}}$, in such a way that $(\widetilde{\mathcal{T}}, \widetilde{\leq})$ is an order-theoretic lattice with meet $\widetilde{\wedge}_{\ell\frown}$ and join $\widetilde{\vee}_{\ell\frown}$. Let t_1 and t_2 be elements of $\widetilde{\mathcal{T}}$, then we must have, taking into account the consistency property of the meet $\widetilde{\wedge}_{\ell\frown}$ in the lattice $(\widetilde{\mathcal{T}}, \widetilde{\leq})$ (see, for instance, [1] section I.5), that

$$\begin{aligned} t_1 \,\widetilde{\leq}\, t_2 \quad & \Leftrightarrow t_1 \,\widetilde{\wedge}_{\ell\frown}\, t_2 = t_1 \\ & \Leftrightarrow \begin{cases} t_1(\mathit{true}) \frown t_2(\mathit{true}) = t_1(\mathit{true}) \\ t_1(\mathit{false}) \smile t_2(\mathit{false}) = t_1(\mathit{false}), \end{cases} \\ & \Leftrightarrow \begin{cases} t_1(\mathit{true}) \leq t_2(\mathit{true}) \\ t_1(\mathit{false}) \geq t_2(\mathit{false}), \end{cases} \end{aligned}$$

taking into account the consistency property of $\frown$ and $\smile$ in $(L, \leq)$. This proves (1.23). From property 5 for $T = \frown$ and the consistency property of the meet $\widetilde{\wedge}_{\ell\frown}$ and the join $\widetilde{\vee}_{\ell\frown}$ in $(\widetilde{\mathcal{T}}, \widetilde{\leq})$ it then follows that the lattice (as an algebra) $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown})$ is bounded, with top element $\widetilde{\mathit{true}}$ and bottom element $\widetilde{\mathit{false}}$. We proceed to show that the lattice (as an algebra) $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown})$ is distributive (see, for instance [1] section I.6). Let t_1, t_2 and t_3 be elements of $\widetilde{\mathcal{T}}$. Then, since $(L, \leq)$ is by assumption in particular a distributive lattice,

$$\begin{aligned} (t_1 \,\widetilde{\wedge}_{\ell\frown}\, (t_2 \,\widetilde{\vee}_{\ell\frown}\, t_3)) \cdot \mathit{true} \quad & = t_1(\mathit{true}) \frown (t_2 \,\widetilde{\vee}_{\ell\frown}\, t_3) \cdot \mathit{true} \\ & = t_1(\mathit{true}) \frown (t_2(\mathit{true}) \smile t_3(\mathit{true})) \\ & = (t_1(\mathit{true}) \frown t_2(\mathit{true})) \smile (t_1(\mathit{true}) \frown t_3(\mathit{true})) \\ & = (t_1 \,\widetilde{\wedge}_{\ell\frown}\, t_2) \cdot \mathit{true} \smile (t_1 \,\widetilde{\wedge}_{\ell\frown}\, t_3) \cdot \mathit{true} \\ & = ((t_1 \,\widetilde{\wedge}_{\ell\frown}\, t_2) \,\widetilde{\vee}_{\ell\frown}\, (t_1 \,\widetilde{\wedge}_{\ell\frown}\, t_3)) \cdot \mathit{true}. \end{aligned}$$

Analogously,

$$(t_1 \,\widetilde{\wedge}_{\ell\frown}\, (t_2 \,\widetilde{\vee}_{\ell\frown}\, t_3)) \cdot \mathit{false} = ((t_1 \,\widetilde{\wedge}_{\ell\frown}\, t_2) \,\widetilde{\vee}_{\ell\frown}\, (t_1 \,\widetilde{\wedge}_{\ell\frown}\, t_3)) \cdot \mathit{false},$$

whence $t_1 \,\widetilde{\wedge}_{\ell\frown}\, (t_2 \,\widetilde{\vee}_{\ell\frown}\, t_3) = (t_1 \,\widetilde{\wedge}_{\ell\frown}\, t_2) \,\widetilde{\vee}_{\ell\frown}\, (t_1 \,\widetilde{\wedge}_{\ell\frown}\, t_3)$. This implies that the structure $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown})$ is a distributive lattice (as an algebra). □

Besides the binary operators meet $\widetilde{\wedge}_{\ell\frown}$ and join $\widetilde{\vee}_{\ell\frown}$ of the bounded distributive lattice $(\widetilde{\mathcal{T}}, \widetilde{\leq})$, there also exists the unary operator $\widetilde{\neg}_{\ell\frown}$. The properties of this operator are studied in the next proposition. By a negation operator on a bounded poset, we shall mean a dual order-automorphism on that structure (for more details, see [9, 14]).

Proposition 5 $\tilde{\neg}_{\ell\frown}$ *is an involutive negation operator on* $(\widetilde{\mathcal{T}}, \tilde{\leq})$*, but not a complement operator on* $(\widetilde{\mathcal{T}}, \tilde{\leq})$*.*

Proof. Let us first show that $\tilde{\neg}_{\ell\frown}$ is an involutive negation operator on $(\widetilde{\mathcal{T}}, \tilde{\leq})$. The involutivity of $\tilde{\neg}_{\ell\frown}$ follows from property 6 for $T = \frown$. An involutive transformation is furthermore always a permutation. Let t_1 and t_2 be elements of $\widetilde{\mathcal{T}}$. Then, taking into account (1.23) and proposition 3(i),

$$\begin{aligned}
(\tilde{\neg}_{\ell\frown}\, t_1) \,\tilde{\leq}\, (\tilde{\neg}_{\ell\frown}\, t_2) \quad & \Leftrightarrow \begin{cases} (\tilde{\neg}_{\ell\frown}\, t_1) \cdot \mathit{true} \leq (\tilde{\neg}_{\ell\frown}\, t_2) \cdot \mathit{true} \\ (\tilde{\neg}_{\ell\frown}\, t_1) \cdot \mathit{false} \geq (\tilde{\neg}_{\ell\frown}\, t_2) \cdot \mathit{false} \end{cases} \\
& \Leftrightarrow \begin{cases} t_1(\mathit{false}) \leq t_2(\mathit{false}) \\ t_1(\mathit{true}) \geq t_2(\mathit{true}) \end{cases} \\
& \Leftrightarrow t_2 \,\tilde{\leq}\, t_1.
\end{aligned}$$

We conclude that $\tilde{\neg}_{\ell\frown}$ is a dual order-automorphism of $(\widetilde{\mathcal{T}}, \tilde{\leq})$, or equivalently, a negation operator on $(\widetilde{\mathcal{T}}, \tilde{\leq})$. As mentioned above, $\tilde{\neg}_{\ell\frown}$ is furthermore involutive. In order to complete the proof, we must show that $\tilde{\neg}_{\ell\frown}$ is not a complement operator on $(\widetilde{\mathcal{T}}, \tilde{\leq})$. $\tilde{\neg}_{\ell\frown}$ is a complement operator on $(\widetilde{\mathcal{T}}, \tilde{\leq})$ if and only if $(\forall t \in \widetilde{\mathcal{T}})(\tilde{\neg}_{\ell\frown}\, t$ is a complement of $t)$. Taking into account the definition of a complement (see, for instance, [1] section I.9), this is equivalent with $(\forall t \in \widetilde{\mathcal{T}})(t \,\tilde{\wedge}_{\ell\frown}\, (\tilde{\neg}_{\ell\frown}\, t) = \widetilde{\mathit{false}}$ and $t \,\tilde{\vee}_{\ell\frown}\, (\tilde{\neg}_{\ell\frown}\, t) = \widetilde{\mathit{true}})$, which, taking into account property 12 for $T = \frown$, is also equivalent with $(\forall t \in \widetilde{\mathcal{T}})(t(\mathit{true}) \frown t(\mathit{false}) = 0)$. Since, surely, $\widetilde{\mathit{unknown}} \in \widetilde{\mathcal{T}}$ and furthermore $\widetilde{\mathit{unknown}}(\mathit{true}) \frown \widetilde{\mathit{unknown}}(\mathit{false}) = 1 \frown 1 = 1 \neq 0$, we deduce that $\tilde{\neg}_{\ell\frown}$ cannot be a complement operator on $(\widetilde{\mathcal{T}}, \tilde{\leq})$. $\square$

The following proposition immediately follows from proposition 5 and property 4 for $T = \frown$.

Proposition 6 $(\widetilde{\mathcal{T}}, \tilde{\wedge}_{\ell\frown}, \tilde{\vee}_{\ell\frown}, \tilde{\neg}_{\ell\frown})$ *is a Morgan algebra*[8]*, i.e.,* $(\widetilde{\mathcal{T}}, \tilde{\wedge}_{\ell\frown}, \tilde{\vee}_{\ell\frown})$ *is a bounded distributive lattice (as an algebra), with a unary operator* $\tilde{\neg}_{\ell\frown}$ *satisfying (i)* $\tilde{\neg}_{\ell\frown}$ *is involutive; and (ii)* $\tilde{\wedge}_{\ell\frown}$*,* $\tilde{\vee}_{\ell\frown}$ *and* $\tilde{\neg}_{\ell\frown}$ *satisfy de Morgan's laws.*

[8] For the introduction a Morgan algebra, we refer to [26].

In the next proposition, we establish the relationship between our possibilistic $\frown$-extension logics and a class of *multi-valued logics*, studied extensively in the literature (see, for instance, [25]).

Proposition 7 *$(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown}, \widetilde{\neg}_{\ell\frown})$ is a Kleene algebra*[9]*, i.e., the structure $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown}, \widetilde{\neg}_{\ell\frown})$ is a Morgan algebra with furthermore*

$$(\forall(t_1,t_2)\in(\widetilde{\mathcal{T}})^2)(t_1 \,\widetilde{\wedge}_{\ell\frown}\, (\widetilde{\neg}_{\ell\frown}\, t_1) \,\widetilde{\leq}\, t_2 \,\widetilde{\vee}_{\ell\frown}\, (\widetilde{\neg}_{\ell\frown}\, t_2)).$$

Proof. We know from proposition 6 that $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown}, \widetilde{\neg}_{\ell\frown})$ is indeed a Morgan algebra. Let furthermore t_1 and t_2 be elements of $\widetilde{\mathcal{T}}$. Then, taking into account property 8 for $T = \frown$,

$$\begin{cases} t_1 \,\widetilde{\wedge}_{\ell\frown}\, (\widetilde{\neg}_{\ell\frown}\, t_1) = \{(true, t_1(true) \frown t_1(\mathit{false})), (\mathit{false}, 1)\} \\ t_2 \,\widetilde{\vee}_{\ell\frown}\, (\widetilde{\neg}_{\ell\frown}\, t_2) = \{(true, 1), (\mathit{false}, t_2(true) \frown t_2(\mathit{false}))\}. \end{cases}$$

Since, for $k = 1, 2$, $t_k(true) \frown t_k(\mathit{false}) \leq 1$, we find that

$$\begin{cases} (t_1 \,\widetilde{\wedge}_{\ell\frown}\, (\widetilde{\neg}_{\ell\frown}\, t_1)) \cdot true \leq (t_2 \,\widetilde{\vee}_{\ell\frown}\, (\widetilde{\neg}_{\ell\frown}\, t_2)) \cdot true \\ (t_1 \,\widetilde{\wedge}_{\ell\frown}\, (\widetilde{\neg}_{\ell\frown}\, t_1)) \cdot \mathit{false} \geq (t_2 \,\widetilde{\vee}_{\ell\frown}\, (\widetilde{\neg}_{\ell\frown}\, t_2)) \cdot \mathit{false}, \end{cases}$$

whence $t_1 \,\widetilde{\wedge}_{\ell\frown}\, (\widetilde{\neg}_{\ell\frown}\, t_1) \,\widetilde{\leq}\, t_2 \,\widetilde{\vee}_{\ell\frown}\, (\widetilde{\neg}_{\ell\frown}\, t_2)$. □

This proposition can be interpreted as follows: the operators $\widetilde{\neg}_{\ell\frown}$, $\widetilde{\wedge}_{\ell\frown}$ and $\widetilde{\vee}_{\ell\frown}$ on $\widetilde{\mathcal{T}}$ satisfy the characteristic properties of the negation, conjunction and disjunction operator in the *multi-valued strong Kleene logics*[10] with truth domain $(\widetilde{\mathcal{T}}, \leq)$ (see, for instance, [25] section 2.5; appendix section 11). This correspondence is also apparent in the definitions of a number of other important operators, combining possibilistic truth values in possibilistic $\frown$-extension logics: for the *conjunction*, the meet $\widetilde{\wedge}_{\ell\frown}$ of $(\widetilde{\mathcal{T}}, \widetilde{\leq})$ is used; for the *disjunction*, the join $\widetilde{\vee}_{\ell\frown}$ of $(\widetilde{\mathcal{T}}, \widetilde{\leq})$ is used; for the *implication* we have, taking into account property 7 for $T = \frown$, that $t_1 \,\widetilde{\Rightarrow}_{\ell\frown}\, t_2 = (\widetilde{\neg}_{\ell\frown}\, t_1) \,\widetilde{\vee}_{\ell\frown}\, t_2$ for arbitrary t_1 and t_2 in $\widetilde{\mathcal{T}}$, which implies that this implication operator is a typical instance of what is called a *Kleene-Dienes implication* in the literature (see, for

[9] For the introduction of a Kleene algebra and a discussion of its meaning, we refer to [26].

[10] Kleene [22] was the first to introduce the ternary logic satisfying these properties. The extension towards general multi-valued logics is mainly due to Dienes [18]. That explains why these logics are often called 'Kleene-Dienes logics'. It must be noted that on the one hand the multi-valued logics of Kleene and Dienes, and on the other hand for instance the Łukasiewicz logics do not differ as far as the negation, conjunction and disjunction operators are concerned. They do differ, however, in their implication operator (see, for instance, [25], sections 2.6 and 2.7).

instance, [25] appendix section 11, [21] section 5.3); for the *equivalence*, it is easily verified, using the mutual distributivity of $\frown$ and $\smile$ in $(L,\leq)$, that $t_1 \mathrel{\widetilde{\Leftrightarrow}_{\ell\frown}} t_2 = (t_1 \mathrel{\widetilde{\Rightarrow}_{\ell\frown}} t_2) \mathbin{\widetilde{\wedge}_{\ell\frown}} (t_2 \mathrel{\widetilde{\Rightarrow}_{\ell\frown}} t_1)$ for arbitrary t_1 and t_2 in $\widetilde{\mathcal{T}}$. At the same time, it should be noted that if $(L,\leq)$ is a Boolean chain (of length 2), we recover Kleene's strong ternary logic (see, in this respect, subsection 5.2). The exact relationship between our possibilistic $\frown$-extension logics and Kleene's strong ternary logic is studied in detail in the following subsection.

6.2 Classical Possibility and Kleene's Strong Ternary Logic

Let us consider a universe X and two clear properties p and q. As always, we consider an ample field $\mathcal{R}$ of measurable subsets of X. We also assume that the sets $A_{P_p} \stackrel{\text{def}}{=} \{\, x \mid x \in X \text{ and } x \text{ is } p \,\}$ and $A_{P_q} \stackrel{\text{def}}{=} \{\, x \mid x \in X \text{ and } x \text{ is } q \,\}$ are $\mathcal{R}$-measurable. Finally, we consider a possibilistic variable ξ in $(X,\mathcal{R})$. Let us assume that we have the following information about the values that ξ may assume in X: ξ must be an element of A, with $A \in \mathcal{R} \setminus \{\emptyset\}$. This information can be represented in the form of the normalized $(\{0,1\},\leq)$-possibility measure (or classical possibility measure, see [9]) Π_A on $(X,\mathcal{R})$, with, for arbitrary B in $\mathcal{R}$:

$$\Pi_A(B) = \begin{cases} 1 & ; \quad B \cap A \neq \emptyset \\ 0 & ; \quad B \cap A = \emptyset \end{cases}$$

the $(\{0,1\},\leq)$-possibility (or simply possibility) that ξ belongs to B. Indeed, if $B \cap A = \emptyset$, then ξ cannot belong to B, since we already know that $\xi \in A$. With Π_A we can associate the dual $(\{0,1\},\leq)$-necessity measure (or classical necessity measure, see [9]) N_A, with, for arbitrary B in $\mathcal{R}$:

$$\mathrm{N}_A(B) = \begin{cases} 1 & ; \quad A \subseteq B \\ 0 & ; \quad A \not\subseteq B \end{cases}$$

the $(\{0,1\},\leq)$-necessity (or simply necessity) that ξ belongs to B. Indeed, if $A \subseteq B$, then ξ must belong to B, since we already know that $\xi \in A$. Remark that the distribution of Π_A, and therefore also the possibility distribution function of ξ, is the characteristic $X - \{0,1\}$-function χ_A of A.

Starting from this possibilistic information χ_A, we now ask ourselves what can be deduced about the truth values of the proposition variables 'ξ is p', 'ξ is q' and a few of their combinations. In order to answer this question, we simply apply the theory, developed in the previous sections, in the special case $(L,\leq) = (\{0,1\},\leq)$. The only triangular norm on $(\{0,1\},\leq)$ is the binary infimum operator or meet $\frown$ [14], which immediately leads us to the special case, discussed in this section. It should be noted that in this particular case

$$\begin{aligned}\widetilde{\mathcal{T}} &= \{\widetilde{\mathit{false}}, \widetilde{\mathit{unknown}}, \widetilde{\mathit{true}}\}\\ \tilde{\leq} &= \{(\widetilde{\mathit{false}}, \widetilde{\mathit{false}}), (\widetilde{\mathit{false}}, \widetilde{\mathit{unknown}}), (\widetilde{\mathit{false}}, \widetilde{\mathit{true}}),\\ &\quad (\widetilde{\mathit{unknown}}, \widetilde{\mathit{unknown}}), (\widetilde{\mathit{unknown}}, \widetilde{\mathit{true}}), (\widetilde{\mathit{true}}, \widetilde{\mathit{true}})\},\end{aligned}$$

which implies that $(\widetilde{\mathcal{T}}, \tilde{\leq})$ is a chain of length 3, with bottom element $\widetilde{\mathit{false}}$, top element $\widetilde{\mathit{true}}$ and in between $\widetilde{\mathit{unknown}}$. In this chain, $\widetilde{\wedge}_{\ell\frown}$ is the binary infimum operator or meet, $\widetilde{\vee}_{\ell\frown}$ is the binary supremum operator or join, and $\widetilde{\neg}_{\ell\frown}$ is the unique, involutive negation operator. The structure $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown}, \widetilde{\neg}_{\ell\frown})$ is for this choice of $(L, \leq)$ and T isomorphic to the structure $(\mathcal{W}_2, \widetilde{\wedge}_{\ell T}|(\mathcal{W}_2)^2, \widetilde{\vee}_{\ell T}|(\mathcal{W}_2)^2, \widetilde{\neg}_{\ell T}|\mathcal{W}_2)$, mentioned in subsection 5.2. The structure $(\widetilde{\mathcal{T}}, \widetilde{\wedge}_{\ell\frown}, \widetilde{\vee}_{\ell\frown}, \widetilde{\neg}_{\ell\frown})$ is a Kleene algebra and is as such isomorphic to the corresponding structure of the strong ternary logic introduced by Kleene (see, for instance, [25]).

The $(\{0,1\}, \leq)$-possibilistic truth value $t_{P_p} \stackrel{\text{def}}{=} \widetilde{\chi_{A_{P_p}}}(\chi_A)$ of the proposition variable 'ξ is p' is determined by

$$\begin{cases} t_{P_p}(\mathit{true}) = \sup\limits_{\chi_{A_{P_p}}(x)=\mathit{true}} \chi_A(x) = \begin{cases} 1\,; & A_{P_p} \cap A \neq \emptyset \\ 0\,; & A_{P_p} \cap A = \emptyset \end{cases} = \Pi_A(A_{P_p}) \\ t_{P_p}(\mathit{false}) = \sup\limits_{\chi_{A_{P_p}}(x)=\mathit{false}} \chi_A(x) = \begin{cases} 1\,; & \mathrm{co}A_{P_p} \cap A \neq \emptyset \\ 0\,; & \mathrm{co}A_{P_p} \cap A = \emptyset \end{cases} = \Pi_A(\mathrm{co}A_{P_p}), \end{cases}$$

where χ_A is the characteristic $X-\{0,1\}$-mapping of A and $\chi_{A_{P_p}}$ the characteristic $X-\mathcal{T}$-mapping of A_{P_p}[11]. For the $(\{0,1\}, \leq)$-possibilistic truth value t_{P_p} there are *three possibilities*, since $t_{P_p} \in \widetilde{\mathcal{T}}$. We have that

$$t_{P_p} = \widetilde{\mathit{true}} \Leftrightarrow \begin{cases} \Pi_A(A_{P_p}) = 1 \\ \mathrm{N}_A(A_{P_p}) = 1 \end{cases} \Leftrightarrow \mathrm{N}_A(A_{P_p}) = 1 \Leftrightarrow A \subseteq A_{P_p},$$

or equivalently, it is *necessary* that ξ is p. On the other hand,

$$t_{P_p} = \widetilde{\mathit{false}} \Leftrightarrow \begin{cases} \Pi_A(A_{P_p}) = 0 \\ \mathrm{N}_A(A_{P_p}) = 0 \end{cases} \Leftrightarrow \Pi_A(A_{P_p}) = 0 \Leftrightarrow A \cap A_{P_p} = \emptyset,$$

or equivalently, it is *impossible* that ξ is p. Finally, we have that

$$t_{P_p} = \widetilde{\mathit{unknown}} \Leftrightarrow \begin{cases} \Pi_A(A_{P_p}) = 1 \\ \mathrm{N}_A(A_{P_p}) = 0 \end{cases} \Leftrightarrow \begin{cases} A \cap A_{P_p} \neq \emptyset \\ A \cap \mathrm{co}A_{P_p} \neq \emptyset, \end{cases}$$

[11] The reader will have noticed that we work with two Boolean chains: $(\{0,1\}, \leq)$ for the representation of the uncertainty about the values ξ can assume in X, and $(\mathcal{T}, \leq)$ for the truth values of propositions.

or equivalently, it is possible but not necessary that ξ is p, in other words, it is *uncertain* whether ξ is p. These observations completely agree with the interpretation of the possibilistic truth values $\widetilde{true}$, $\widetilde{unknown}$ and $\widetilde{false}$, given in section 5.

For the possibilistic truth value t_{P_q} of the proposition variable 'ξ is q', completely analogous observations can be made. Let us now turn our attention to the $(\{0,1\},\leq)$-possibilistic truth value of the proposition variable 'NOT(ξ is p)', or equivalently, (NOT P_p)(ξ), or 'ξ is not p'. It is obvious that $A_{\text{NOT } P_p} = \text{co}A_{P_p}$, whence

$$\begin{cases} t_{\text{NOT } P_p}(true) = \Pi_A(A_{\text{NOT } P_p}) = \Pi_A(\text{co}A_{P_p}) = t_{P_p}(false) \\ t_{\text{NOT } P_p}(false) = \Pi_A(\text{co}A_{\text{NOT } P_p}) = \Pi_A(A_{P_p}) = t_{P_p}(true). \end{cases}$$

We may therefore write, taking into account proposition 3(i) for $T = \frown$, that $t_{\text{NOT } P_p} = \tilde{\neg}_{\ell\frown}\, t_{P_p}$. We conclude that for the logical negation operator of classical propositional logic, there is *always truth-functionality* as far as the $(\{0,1\},\leq)$-possibilistic truth values are concerned.

Let us now investigate the proposition variable 'ξ is p AND ξ is q', or equivalently, (P_p AND P_q)(ξ), where P_p AND P_q is a proposition function that is the pointwise conjunction of the proposition functions P_p and P_q. It is obvious that $A_{P_p \text{ AND } P_q} = A_{P_p} \cap A_{P_q}$, whence

$$t_{P_p \text{ AND } P_q}(true) = \Pi_A(A_{P_p \text{ AND } P_q}) = \Pi_A(A_{P_p} \cap A_{P_q})$$

and, also taking into account proposition 3(ii) for $T = \frown$,

$$\begin{aligned} t_{P_p \text{ AND } P_q}(false) &= \Pi_A(\text{co}A_{P_p \text{ AND } P_q}) \\ &= \Pi_A(\text{co}(A_{P_p} \cap A_{P_q})) \\ &= \Pi_A(\text{co}A_{P_p} \cup \text{co}A_{P_q}) \\ &= \Pi_A(\text{co}A_{P_p}) \smile \Pi_A(\text{co}A_{P_q}) \\ &= t_{P_p}(false) \smile t_{P_q}(false) \\ &= (t_{P_p} \tilde{\wedge}_{\ell\frown}\, t_{P_q}) \cdot false. \end{aligned}$$

Only if

$$\Pi_A(A_{P_p} \cap A_{P_q}) = \Pi_A(A_{P_p}) \frown \Pi_A(A_{P_q}) \tag{1.24}$$

we have, taking into account proposition 3(ii) for $T = \frown$,

$$\begin{aligned} t_{P_p \text{ AND } P_q}(true) &= \Pi_A(A_{P_p} \cap A_{P_q}) = \Pi_A(A_{P_p}) \frown \Pi_A(A_{P_q}) \\ &= t_{P_p}(true) \frown t_{P_q}(true) = (t_{P_p} \tilde{\wedge}_{\ell\frown}\, t_{P_q}) \cdot true. \end{aligned}$$

We conclude that only in this case there is *truth-functionality* for the logical conjunction operator in classical propositional logic as far as the possibilistic truth values are concerned, or equivalently,

$$t_{P_p \text{ AND } P_q} = t_{P_p} \widetilde{\wedge}_{\ell\frown} t_{P_q}, \tag{1.25}$$

Let us also investigate the proposition variable 'ξ is p OR ξ is q', or equivalently, $(P_p$ OR $P_q)(\xi)$, where P_p OR P_q is a proposition function that is the pointwise disjunction of the proposition functions P_p and P_q. It is obvious that $A_{P_p \text{ OR } P_q} = A_{P_p} \cup A_{P_q}$, whence, taking into account proposition 3(iii) for $T = \frown$,

$$\begin{aligned} t_{P_p \text{ OR } P_q}(true) &= \Pi_A(A_{P_p \text{ OR } P_q}) \\ &= \Pi_A(A_{P_p} \cup A_{P_q}) \\ &= \Pi_A(A_{P_p}) \smile \Pi_A(A_{P_q}) \\ &= t_{P_p}(true) \smile t_{P_q}(true) \\ &= (t_{P_p} \widetilde{\vee}_{\ell\frown} t_{P_q}) \cdot true. \end{aligned}$$

and

$$t_{P_p \text{ OR } P_q}(false) = \Pi_A(\text{co}A_{P_p \text{ OR } P_q}) = \Pi_A(\text{co}A_{P_p} \cap \text{co}A_{P_q}).$$

Only if

$$\Pi_A(\text{co}A_{P_p} \cap \text{co}A_{P_q}) = \Pi_A(\text{co}A_{P_p}) \frown \Pi_A(\text{co}A_{P_q}) \tag{1.26}$$

we have, also taking into account proposition 3(iii) for $t = \frown$, that

$$\begin{aligned} t_{P_p \text{ OR } P_q}(false) &= \Pi_A(\text{co}A_{P_p} \cap \text{co}A_{P_q}) = \Pi_A(\text{co}A_{P_p}) \frown \Pi_A(\text{co}A_{P_q}) \\ &= t_{P_p}(false) \frown t_{P_q}(false) = (t_{P_p} \widetilde{\vee}_{\ell\frown} t_{P_q}) \cdot false \end{aligned}$$

We conclude that only in this case there is *truth-functionality* for the logical disjunction operator in classical propositional logic as far as the possibilistic truth values are concerned, or equivalently,

$$t_{P_p \text{ OR } P_q} = t_{P_p} \widetilde{\vee}_{\ell\frown} t_{P_q}. \tag{1.27}$$

Let us now briefly discuss the meaning of (1.24)–(1.27). It is easily shown that (1.24), and therefore also (1.25), does not hold if and only if $A \cap A_{P_p \text{ AND } P_q} = \emptyset$ and at the same time

$$A \cap A_{P_p} \neq \emptyset \text{ and } A \cap \text{co}A_{P_p} \neq \emptyset \text{ and } A \cap A_{P_q} \neq \emptyset \text{ and } A \cap \text{co}A_{P_q} \neq \emptyset,$$

in other words, if and only if it is *uncertain* (i.e., not impossible and not necessary) whether ξ is p and whether ξ is q, and at the same time *impossible* that ξ is p AND ξ is q. Indeed, in that case, we have that $t_{P_p \text{ AND } P_q} = \widetilde{false}$, whereas $t_{P_p} \widetilde{\wedge}_{\ell\frown} t_{P_q} = \widetilde{unknown} \widetilde{\wedge}_{\ell\frown} \widetilde{unknown} = \widetilde{unknown}$. A similar argument can be given for the disjunction. We conclude that there is *not necessarily truth-functionality* for the logical disjunction and conjunction operators of classical propositional logic, as far as the $(\{0,1\},\leq)$-possibilistic truth values are concerned. In other words, (1.25) and (1.27) are not necessarily valid for arbitrary clear properties p and q, with $\mathcal{R}$-measurable A_{P_P} and A_{P_q}. Our possibilistic approach therefore only results in a strong ternary Kleene logic if a number of independence properties are satisfied[12]. In some cases these conditions are not satisfied, and our possibilistic approach is therefore *not truth-functional*, and therefore does not lead to a strong ternary Kleene logic. In these cases however, the strong ternary Kleene logic does provide us with a conservative approximation, since wherever it goes wrong, it will result in the possibilistic truth value $\widetilde{unknown}$, where our possibilistic approach would yield the possibilistic truth values $\widetilde{true}$ or $\widetilde{false}$ (see also proposition 2).

7 CONCLUSION

In the previous sections, we have shown how a possibilistic logic can be constructed. Possibilistic logic can be described as a set of techniques that enable us to incorporate linguistic (possibilistic) uncertainty in classical propositional logic. It turns out that under a number of independence assumptions, possibilistic logic leads to the special case of a possibilistic extension logic. A special subclass of these, the possibilistic-logical $\frown$-extensions, are related with strong multi-valued Kleene logics. Thus, a possibilistic justification is given for the introduction and use of these Kleene systems.

There are a number of problems, however, which have not been dealt with in this paper. Among them, we explicitly mention the decomposability problem. In classical propositional logic, there exist basic sets of classical-logical functions, such that all other classical-logical functions can be expressed in terms of these functions. The following question can then be asked: is a similar result valid in a possibilistic extension logic, i.e., can any member of this logic be decomposed in terms of a basic set? And if so, what is the relationship between the decomposition of a classical-logical function and the decomposition of its possibilistic-logical extension? A partial answer in the special case $(L,\leq)=([0,1],\leq)$ and $T=\min$ has been given in [16], theorem 4.1, although it must be mentioned that the conditions imposed in this theorem are to weak. For a better and more general formulation of this theorem, we refer to [9] chapter 8. Also, in the related domain of reliability theory, we have proven a decomposability property for possibilistic structure functions, which are, for

[12] It is shown in [9, 15] that conditions (1.24) and (1.26) are related to the conditions for the possibilistic (or logical) independence of the events A_{P_p} and A_{P_q}.

fixed $(L, \leq)$ and T, isomorphic to an isotonic subclass of the corresponding $(L, \leq)$-possibilistic T-extension logic. These results will be reported on elsewhere (see, however, also [10]).

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