Convection, Stability, and Turbulence

Charles R. Doering

Department of Mathematics, Department of Physics,
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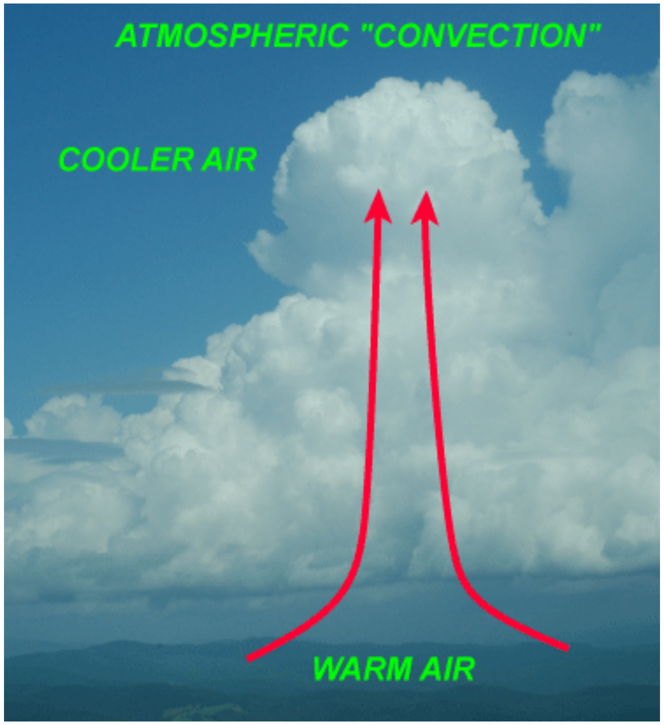
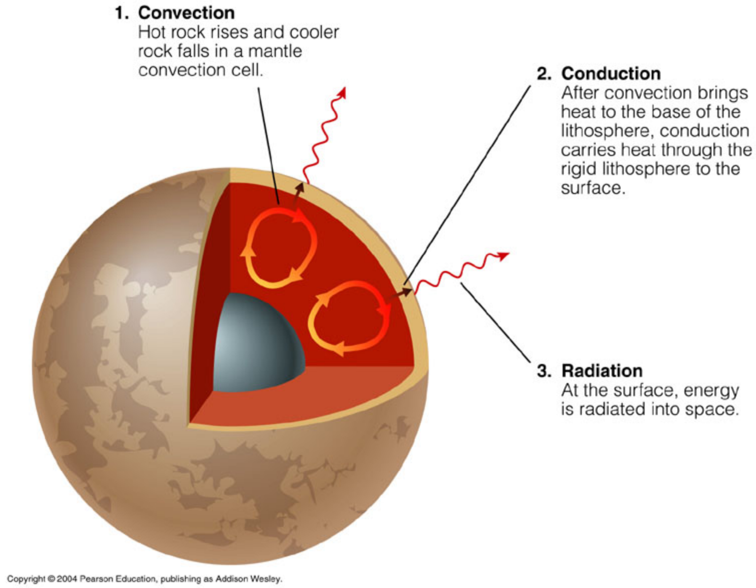
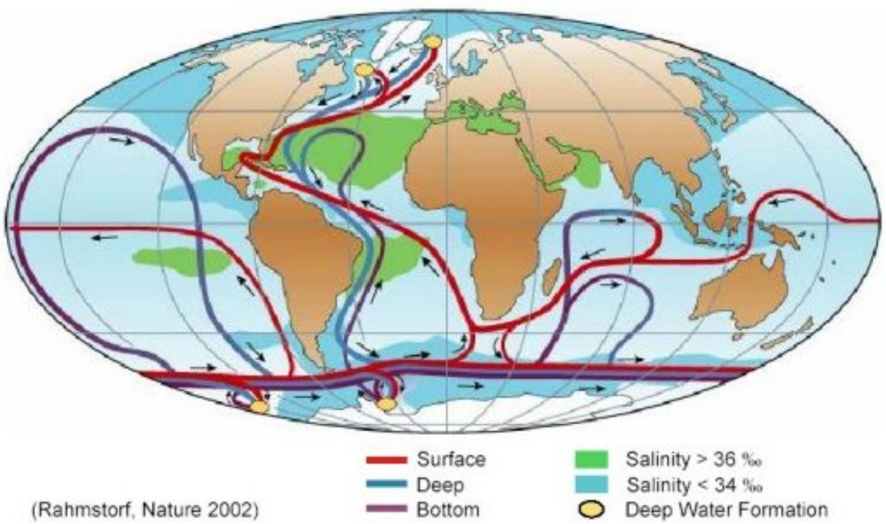
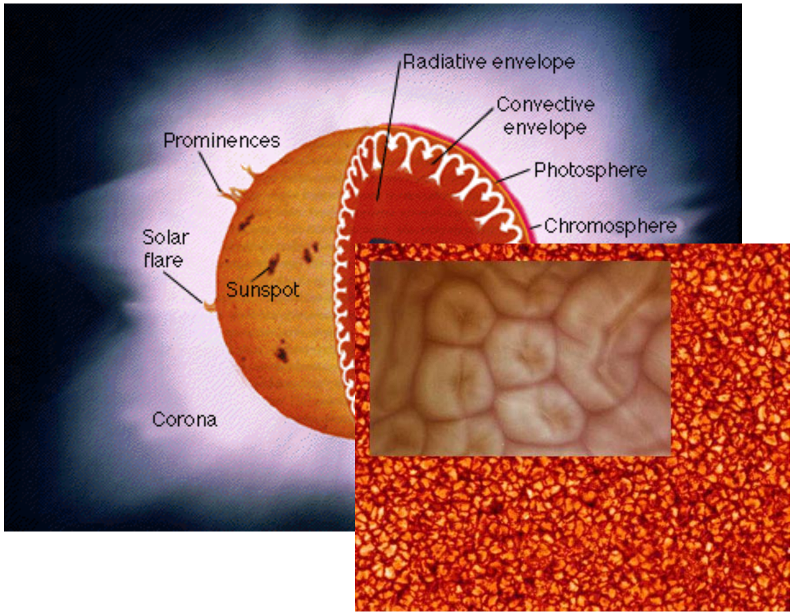






(Rahmstorf, Nature 2002)



Bénard 1900:

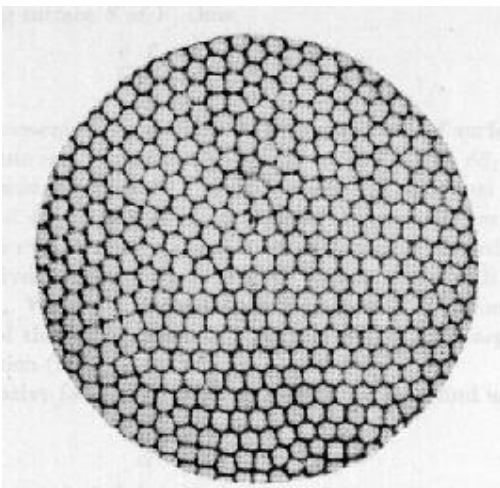


FIG. 1. Bénard cells in spermaceti. A reproduction of one of Bénard's original photographs.

THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

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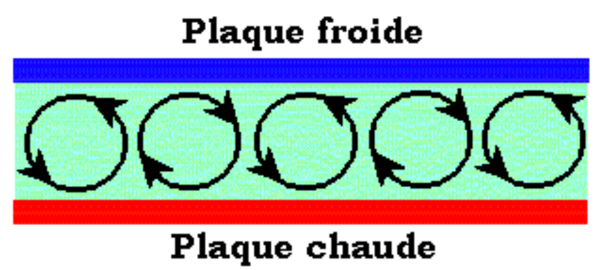
JOURNAL OF SCIENCE.

[SIXTH SERIES]

DECEMBER 1916.

LIX. *On Convection Currents in a Horizontal Layer of Fluid, when the Higher Temperature is on the Under Side.* *By* Lord RAYLEIGH, *O.M., F.R.S.**



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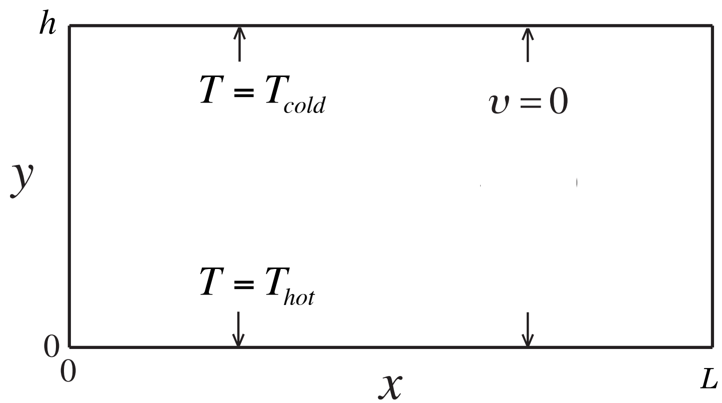
The calculations which follow are based upon equations given by Boussinesq, who has applied them to one or two particular problems. The special limitation which characterizes them is the neglect of variations of density, *except in so far as they modify the action of gravity.* Of course, such neglect can be justified only under certain conditions, which Boussinesq has discussed. They are not so restrictive as to exclude the approximate treatment of many problems of interest.

In the present problem the case is much more complicated, unless we arbitrarily limit it to two dimensions.

Minimal mathematical model:



Dynamical variables: Temperature field $T(\vec{x},t)$

Velocity field $\vec{u}(\vec{x},t) = \hat{i}\,u + \hat{j}\,\upsilon + \hat{k}\,w$ ✗

Pressure field $p(\vec{x},t)$

Boundary conditions: $T = T_{hot}$ and $\hat{j} \cdot \vec{u} = \upsilon = 0$ at $y = 0$

$T = T_{cold}$ and $\hat{j} \cdot \vec{u} = \upsilon = 0$ at $y = h$

Boussinesq equations:

$$\dot{T} + \vec{u} \cdot \vec{\nabla} T = \kappa \Delta T$$

$$\dot{\vec{u}} + \vec{u} \cdot \vec{\nabla} \vec{u} + \frac{1}{\rho} \vec{\nabla} p = \nu \Delta \vec{u} + g \alpha \hat{j} \left(T - T_0 \right)$$

$$0 = \vec{\nabla} \cdot \vec{u}$$

We want to compute the vertical heat flux :

$$J_y = \left\langle \rho c \left(-\kappa \frac{\partial T}{\partial y} + \upsilon T \right) \right\rangle$$

$$= \rho c \kappa \frac{T_{hot} - T_{cold}}{h} + \rho c \langle \upsilon T \rangle$$

space-time average

conduction heat flux

convection heat flux

Boussinesq equations:

$$\dot{T} + \vec{u}\cdot\vec{\nabla}T = \kappa\Delta T$$

$$\dot{\vec{u}} + \vec{u}\cdot\vec{\nabla}\vec{u} + \frac{1}{\rho}\vec{\nabla}p = \nu\Delta\vec{u} + g\alpha\hat{j}\left(T - T_0\right)$$

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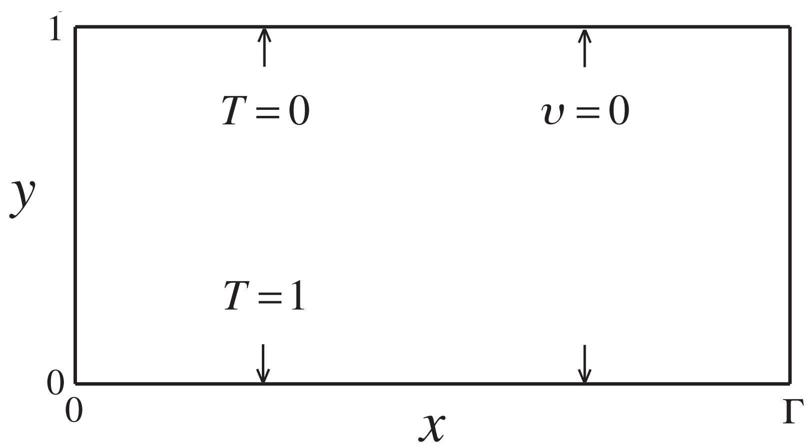
$$= \rho c\,\kappa\frac{T_{hot} - T_{cold}}{h} + \rho c\langle \upsilon T\rangle$$

Lots of parameters! $h,\ L,\ T_0,\ T_{hot}\text{ - }T_{cold},\ g,\ \kappa,\ \rho,\ \nu,\ \alpha,\ c$

Dimensionless variables:

Rayleigh number : $\mathrm{Ra} = \dfrac{g\alpha(T_{hot} - T_{cold})h^3}{\nu\kappa}$ Prandtl number : $\mathrm{Pr} = \dfrac{\nu}{\kappa}$



$$\dot{T} + \vec{u}\cdot\vec{\nabla}T = \Delta T$$

$$\frac{1}{\mathrm{Pr}}\left(\dot{\vec{u}} + \vec{u}\cdot\vec{\nabla}\,\vec{u}\right) + \vec{\nabla}p = \Delta\,\vec{u} + \mathrm{Ra}\,\hat{j}\,T$$

$$0 = \vec{\nabla}\cdot\vec{u}$$

Nusselt number : $\mathrm{Nu} \equiv \dfrac{J_y}{J_{conduction}} = 1 + \langle \upsilon T\rangle$

Facts : $\mathrm{Nu} = \left\langle \left|\vec{\nabla}T\right|^2 \right\rangle = 1 + \dfrac{1}{Ra}\left\langle \left|\vec{\nabla}\vec{u}\right|^2 \right\rangle \geq 1$

Challenge:

find Nu(Ra,Pr)

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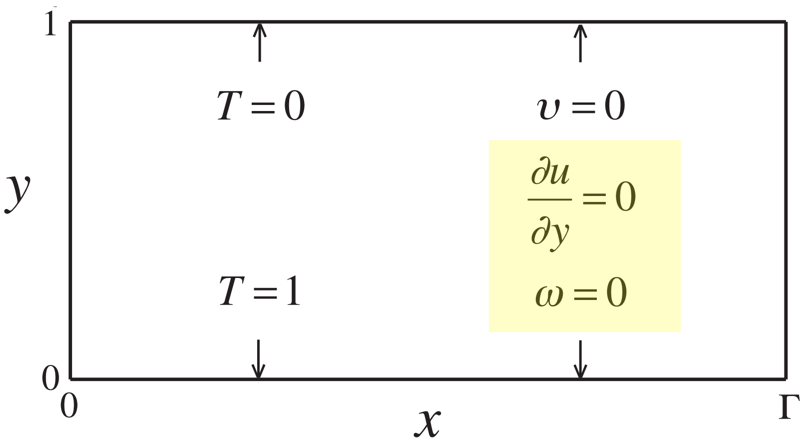
LIX. *On Convection Currents in a Horizontal Layer of Fluid, when the Higher Temperature is on the Under Side.* *By* Lord RAYLEIGH, *O.M.*, *F.R.S.**

We have also to reconsider the boundary conditions at $z=0$ and $z=\zeta$. We may still suppose $\theta=0$ and $w=0$; but for a further condition we should probably prefer $dw/dz=0$, corresponding to a fixed solid wall. But this entails much complication, and we may content ourselves with the supposition $d^2w/dz^2=0$, which (with $w=0$) is satisfied by taking as before w proportional to $\sin sz$ with $s=q\pi/\zeta$. This is equivalent to the annulment of lateral *forces* at the wall.

Dimensionless variables:

Rayleigh number : $\mathrm{Ra} = \dfrac{g\alpha(T_{hot} - T_{cold})h^3}{\nu\kappa}$ Prandtl number : $\mathrm{Pr} = \dfrac{\nu}{\kappa}$



$$\dot{T} + \vec{u}\cdot\vec{\nabla}T = \Delta T$$

$$\frac{1}{\mathrm{Pr}}\left(\dot{\vec{u}} + \vec{u}\cdot\vec{\nabla}\,\vec{u}\right) + \vec{\nabla}p = \Delta\,\vec{u} + \mathrm{Ra}\,\hat{j}\,T$$

$$0 = \vec{\nabla}\cdot\vec{u}$$

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Facts : $\mathrm{Nu} = \left\langle \left|\vec{\nabla}T\right|^2 \right\rangle = 1 + \dfrac{1}{Ra}\left\langle \left|\vec{\nabla}\vec{u}\right|^2 \right\rangle \geq 1$

Stability & *instability*

Conduction solution: $\vec{u} = 0$ $T = 1 - y$ $\mathrm{Nu} = 1$

- Linear analysis → sufficient condition for *instability*.
- Write $T(x,y,t) = 1 - y + \theta(x,y,t)$ and linearize in θ ...
- with θ and $\upsilon \sim (\theta_k, \upsilon_k) \cdot e^{-\lambda t} e^{ikx}$ → eigenvalue problem:

$$-\lambda \hat{\theta}_k(y) = (\partial_y^2 - k^2)\hat{\theta}_k + \hat{\upsilon}_k(y) \qquad -\lambda(\partial_y^2 - k^2)\hat{\upsilon}_k = \Pr(\partial_y^2 - k^2)^2 \hat{\upsilon}_k - \Pr \mathrm{Ra}\, k^2\, \hat{\theta}_k$$

- with $\theta_k = 0$ & $\upsilon_k = 0 = \partial_y^{\,2} \upsilon_k$ at boundaries $y = 0,1$.
- If *any* λ has real part < 0, then there is an *instability*.
- Lord R. '16: $\mathrm{Ra} > \mathrm{Ra}_c = 27\pi^4/4$ → $\lambda_{min} < 0$ → *convection*

Stability & instability

Conduction solution: $\vec{u} = 0 \qquad T = 1 - y \qquad \mathrm{Nu} = 1$

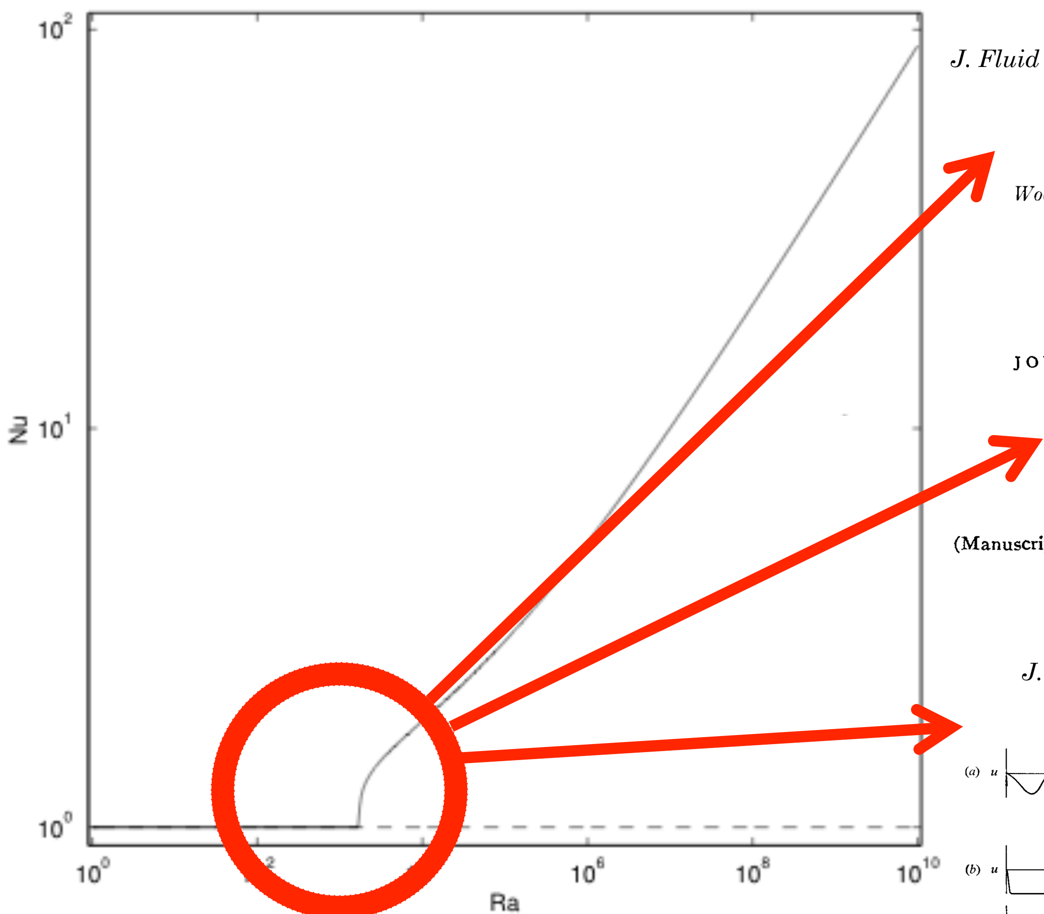
- "Energy" analysis → sufficient condition for *stability*.
- Let $T(x,y,t) = 1 - y + \theta(x,y,t)$ … then *without* linearization,

$$\frac{d}{dt}\frac{1}{2}\int\left[\theta^2 + \frac{1}{\mathrm{Pr\,Ra}}\left|\vec{u}\right|^2\right]dx\,dy = -\int\left[\left|\vec{\nabla}\theta\right|^2 + \frac{1}{\mathrm{Ra}}\left|\vec{\nabla}\vec{u}\right|^2 - 2v\theta\right]dx\,dy$$

$$= -Q\{\theta,v\}$$

- $Q\{\theta,v\} = \int(\theta,v)\cdot S\cdot(\theta,v)$ with symmetric linear operator S.
- If $Q\{\theta,v\} > 0$, i.e., *all* $\lambda > 0$ for $S\cdot(\theta,v) = \lambda(\theta,v)$ → *stability*.
- **Fact**: $\mathrm{Ra} < \mathrm{Ra}_c = 27\pi^4/4 \rightarrow \lambda_{min} > 0$ → *no convection*.

Nu vs. Ra … the big picture:

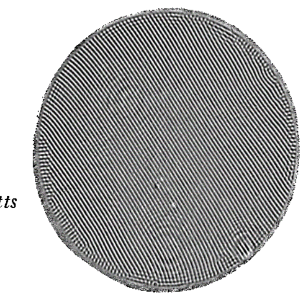


J. Fluid Mech. (1958), *vol.* 4, *part* 3, *pp.* 225–260

Finite amplitude cellular convection

By W. V. R. MALKUS and G. VERONIS
Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

(*Received* 22 *November* 1957)

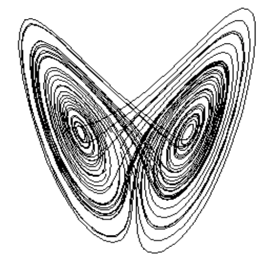


JOURNAL OF THE ATMOSPHERIC SCIENCES

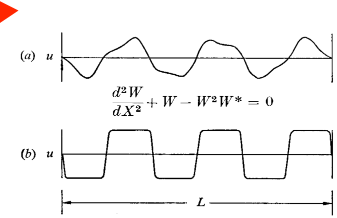
Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ
Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)



J. Fluid Mech. (1969), *vol.* 38, *part* 2, *pp.* 225–260



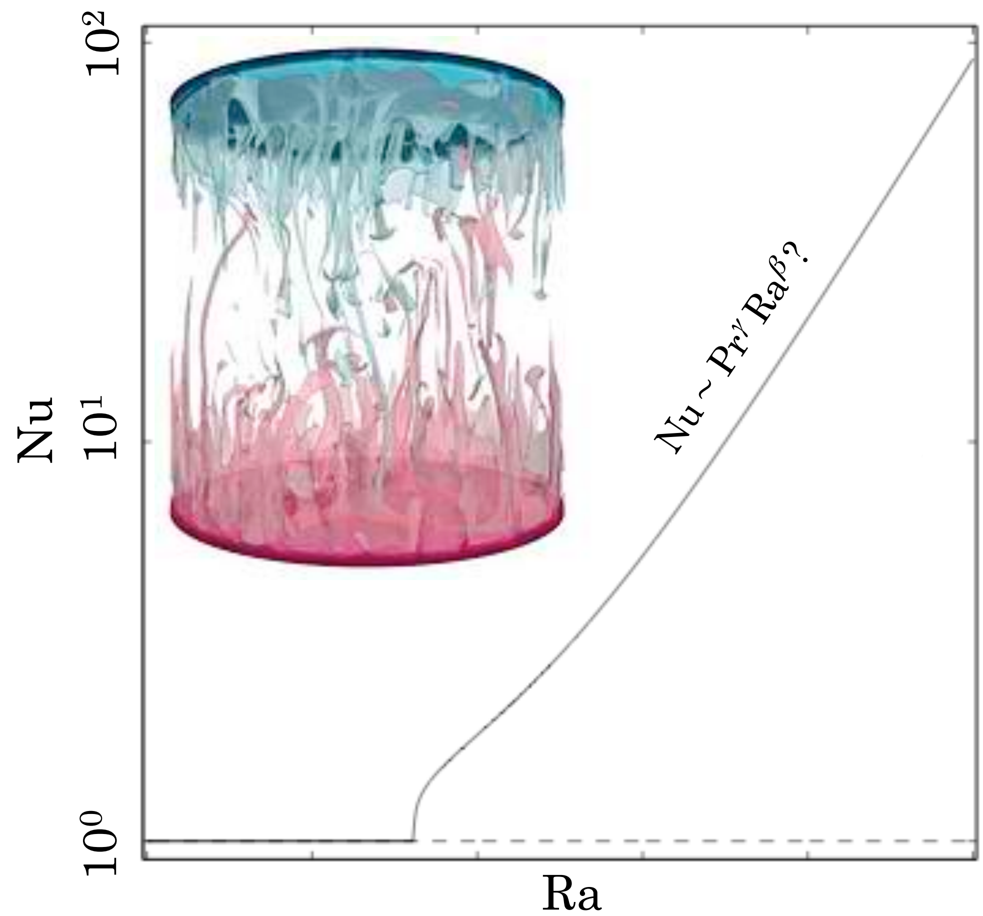
Finite bandwidth, finite amplitude convection

By ALAN C. NEWELL
Department of Planetary and Space Science,
Department of Mathematics
AND J. A. WHITEHEAD
Institute of Geophysics and Planetary Physics,
University of California, Los Angeles

(Received 19 July 1968 and in revised form 4 March 1969)

- $\mathrm{Nu} \geq 1$ for all Ra
- $\mathrm{Nu} = 1$ for all $\mathrm{Ra} < \mathrm{Ra}_c \approx 657$
- What's the behavior of Nu for $\mathrm{Ra} > \mathrm{Ra}_c$?

Nu vs. Ra … the big picture:



- **Nu ≥ 1** for all Ra
- **Nu = 1** for all Ra < $Ra_c \approx 657$
- What is the behavior of Nu for Ra >>> Ra_c?

Source: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 225, No. 1161 (Aug. 31, 1954), pp. 196-212

The heat transport and spectrum of thermal turbulence*

By W. V. R. Malkus

Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

(*Communicated by S. Chandrasekhar, F.R.S.—Received* 26 *November* 1953—
Revised 13 *April* 1954)

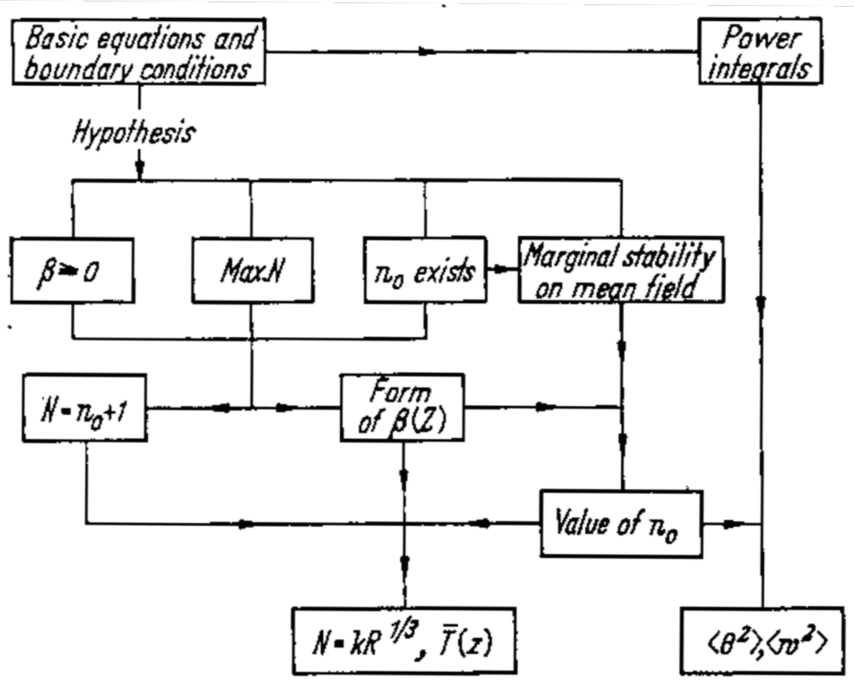
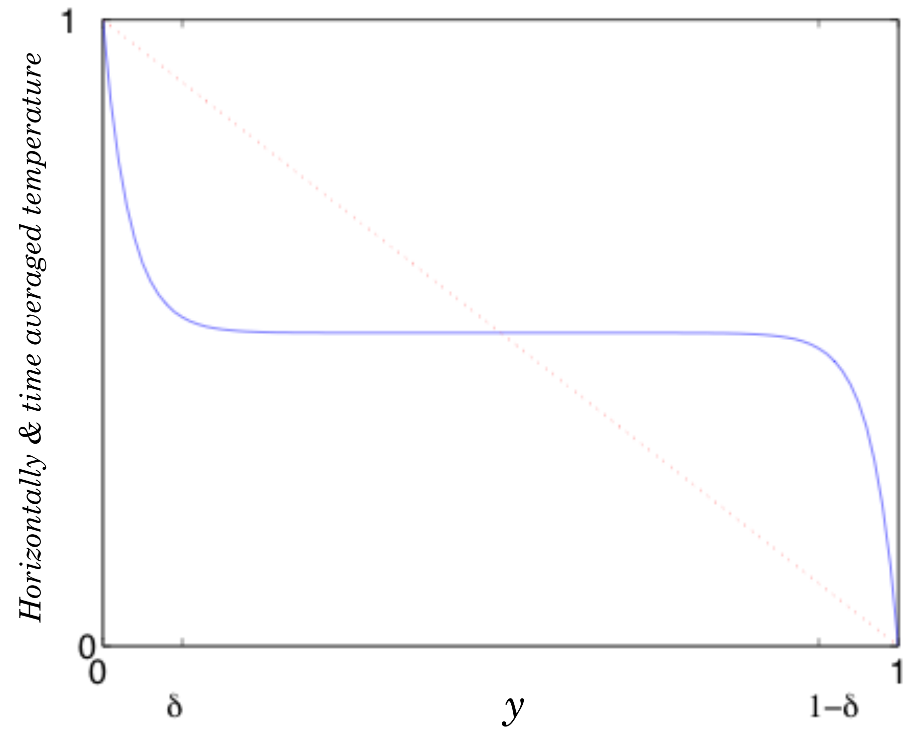


Fig. 1. Structure of Malkus' 1954 theory

Malkus' argument:

For turbulent convection, the mean temperature profile should look like:



$$\mathrm{Nu} \sim \delta^{-1} \quad \ldots \quad \delta = f(\mathrm{Ra})$$

Assume boundary layer thickness is determined by a ***marginal stability condition***:

$$\mathrm{Ra}_c = \mathrm{Ra}_\delta = \frac{g\alpha\,\frac{1}{2}(T_{hot} - T_{cold})\,(\delta h)^3}{\nu\kappa} = \frac{1}{2}\,\frac{g\,\alpha(T_{hot} - T_{cold})\,h^3}{\nu\kappa} \times \delta^3 = \frac{1}{2}\,\mathrm{Ra}\,\delta^3$$

$$\Rightarrow \quad \delta \sim \mathrm{Ra}^{-1/3} \quad \Rightarrow \quad \mathrm{Nu} \sim \mathrm{Ra}^{1/3}$$ ***uniformly*** in Pr

JOURNAL OF GEOPHYSICAL RESEARCH VOLUME 67, NO. 8 JULY 1962

Thermal Turbulence at Very Small Prandtl Number[1]

EDWARD A. SPIEGEL

Courant Institute of Mathematical Sciences
New York University, New York

Further, in the limit as ~~σ~~ Pr $\to 0$ ...

... the well-known $R^{1/3}$ dependence of heat transfer should not hold. Yet this does not seem deducible from the Malkus approach, which shows no Prandtl-number dependence ...

THE PHYSICS OF FLUIDS VOLUME 5, NUMBER 11 NOVEMBER 1962

Turbulent Thermal Convection at Arbitrary Prandtl Number

ROBERT H. KRAICHNAN
Courant Institute of Mathematical Sciences, New York University, New York
(Received May 24, 1962)

The mixing-length theory of turbulent thermal convection in a gravitationally unstable fluid is extended to yield the dependence of Nusselt number H/H_0 on both Prandtl number σ and Rayleigh number Ra. The analysis assumes a layer of Boussinesq fluid contained between infinite, horizontal, perfectly conducting, rigid plates. Also obtained is the dependence of mean temperature deviation $\bar{T}(z)$, rms temperature fluctuation $\tilde{\psi}(z)$, and rms velocity upon height z above the bottom plate. The theory gives $H/H_0 \propto \mathrm{Ra}^{1/3}$ (high σ), $H/H_0 \propto (\sigma\,\mathrm{Ra})^{1/3}$ (low σ), and $H/H_0 \sim 1$ (very low σ). The boundaries of the several σ ranges are determined. At one intermediate Prandtl number only, the behavior of $\bar{T}(z)$ and $\tilde{\psi}(z)$ reduces to that previously found by Priestley. At high σ, there is a range of z, outside the molecular conduction region, where $\bar{T}(z) \propto z^{-1}$, $\tilde{\psi}(z) \propto z^{-1}$. The results at very low σ reduce to those of Ledoux, Schwarzschild, and Spiegel. The dynamics are found to be importantly modified at extremely large Ra because of the stirring action of small-scale turbulence generated in shear boundary layers attached to the eddies of largest scale. The consequent corrected asymptotic law of heat transport at fixed σ is $H/H_0 \propto [\mathrm{Ra}/(\ln \mathrm{Ra})^3]^{1/2}$.

Postulated "***ultimate***" high-Ra scaling: $\mathrm{Nu} \sim \mathrm{Ra}^{1/2}$

Spiegel's argument:

- Assume ***transport across the bulk*** is rate-limiting factor
- … so fluid elements 'free-fall' w/acceleration $\sim g\alpha\Delta T$
- … so vertical velocity scale is $v \sim [g\alpha\Delta T h]^{1/2}$
- … so convective heat flux $J_{conv} \sim \rho v c \Delta T$
- … and therefore Nu = $1 + J_{conv}/J_{cond}$
- $\sim \rho c \Delta T [g\alpha\Delta T h]^{1/2} \div (\rho c \kappa \Delta T/h)$
- … so that Nu ~ (Pr Ra)$^{1/2}$

J. Fluid Mech. (2000), *vol.* 407, *pp.* 27–56. Printed in the United Kingdom

Scaling in thermal convection: a unifying theory

By SIEGFRIED GROSSMANN[1] AND DETLEF LOHSE[2]

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[2]University of Twente, Department of Applied Physics, P.O. Box 217, 7500 AE Enschede, The Netherlands
e-mail: lohse@tn.utwente.nl

(Received 30 April 1998 and in revised form 8 November 1999)

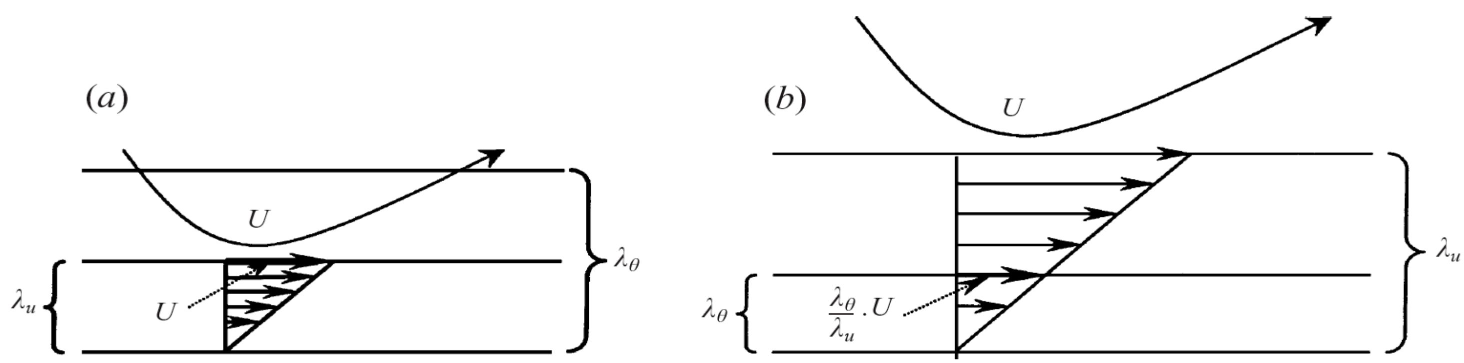
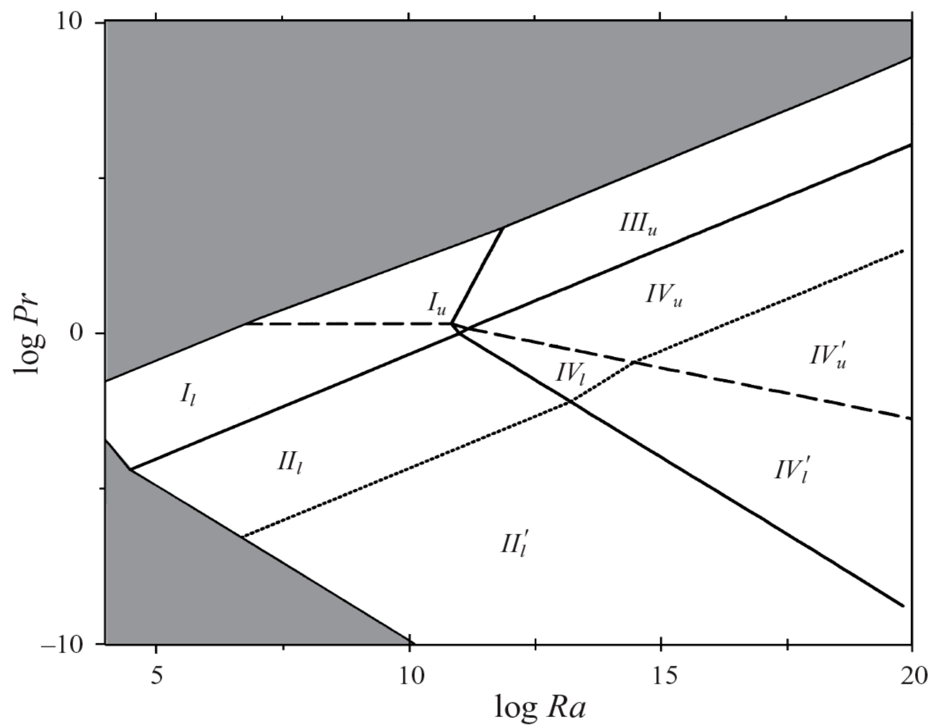


FIGURE 1. Sketch of the boundary layers, (*a*) for low *Pr* where $\lambda_u < \lambda_\theta$ and (*b*) for large *Pr* where $\lambda_u > \lambda_\theta$.



Regime	Dominance of	BL	Nu
I_l	$\epsilon_{u,BL}$, $\epsilon_{\theta,BL}$	$\lambda_u < \lambda_\theta$	$0.27 Ra^{1/4} Pr^{1/8}$
I_u		$\lambda_u > \lambda_\theta$	$0.33 Ra^{1/4} Pr^{-1/12}$
II_l	$\epsilon_{u,bulk}$, $\epsilon_{\theta,BL}$	$\lambda_u < \lambda_\theta$	$0.97 Ra^{1/5} Pr^{1/5}$
(II_u)		$\lambda_u > \lambda_\theta$	$(\sim Ra^{1/5})$
III_l	$\epsilon_{u,BL}$, $\epsilon_{\theta,bulk}$	$\lambda_u < \lambda_\theta$	$6.43 \times 10^{-6} Ra^{2/3} Pr^{1/3}$
III_u		$\lambda_u > \lambda_\theta$	$3.43 \times 10^{-3} Ra^{3/7} Pr^{-1/7}$
IV_l	$\epsilon_{u,bulk}$, $\epsilon_{\theta,bulk}$	$\lambda_u <$	$4.43 \times 10^{-4} Ra^{1/2} Pr^{1/2}$
IV_u		$\lambda_u >$	$0.038 Ra^{1/3}$

Search for the "Ultimate State" in Turbulent Rayleigh-Bénard Convection

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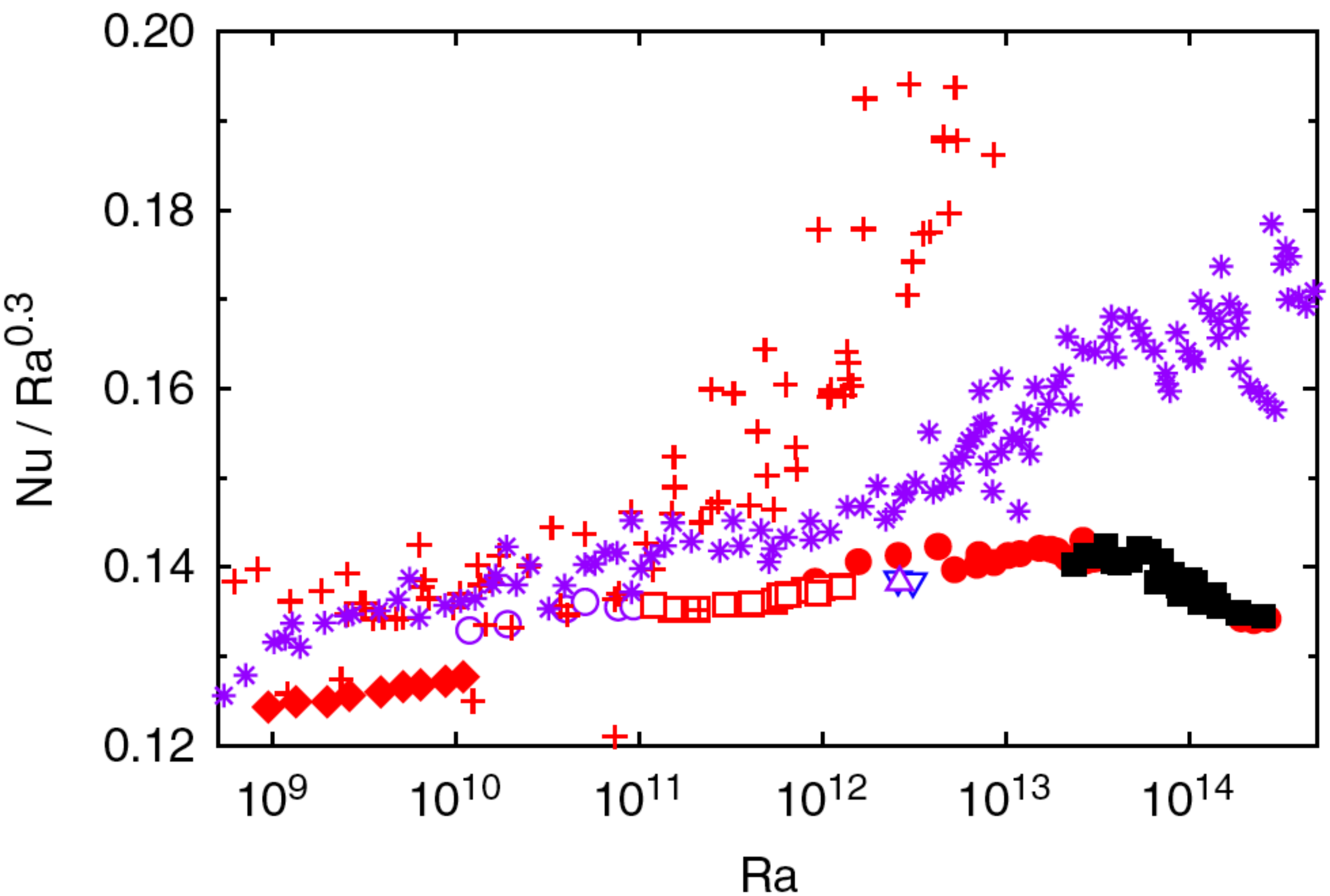
[2]Max Planck Institute for Dynamics and Self-Organization, Am Fassberg 17, D-37077 Goettingen, Germany
[3]Department of Physics, University of California, Santa Barbara, California 93106, USA
(Received 16 April 2009; published 1 July 2009)



Efficiency of Heat Transfer in Turbulent Rayleigh-Bénard Convection

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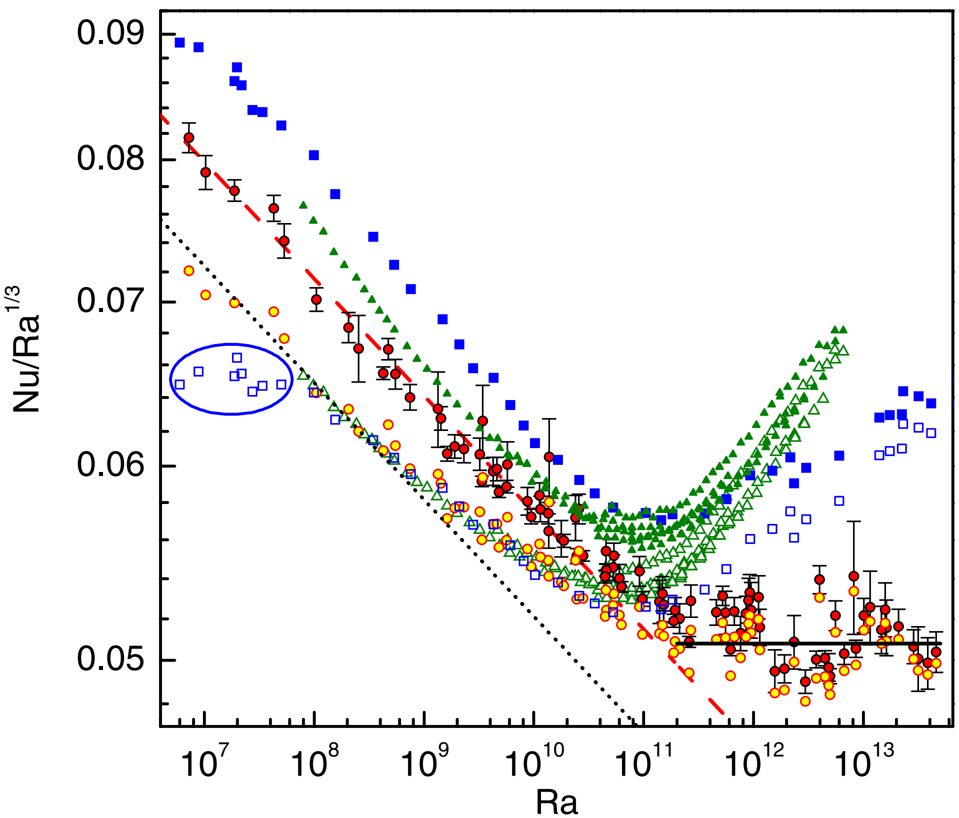

FIG. 2 (color online). The compensated $NuRa^{-1/3}$ plot versus Ra: our measured data (without wall correction) are shown as (red filled) circles with error bars representing the total uncertainty in $NuRa^{-1/3}$ caused by uncertainties in the determination of T_m (4 mK), p (0.1%), ΔT (2 mK) and heat power to the bottom plate (0.5%); (red, yellow filled) circles are our data with the wall corrections applied as described in the text; (olive) triangles and open (olive) triangles represent the uncorrected and corrected ($\Gamma = 1.14$) Grenoble data set [7]; solid (blue) squares and open (blue) squares are the uncorrected and corrected ($\Gamma = 1$) data sets from Trieste ($T_m = 5.34 \pm 0.02$ K) [9]. The dashed (red) line is functional dependence $Nu = 0.172Ra^{2/7}$, the dotted line $Nu = 0.156Ra^{2/7}$, and the solid line $Nu = 0.0508Ra^{1/3}$.

Selected for a Viewpoint in *Physics*

PRL **108,** 024502 (2012) PHYSICAL REVIEW LETTERS week ending 13 JANUARY 2012

Transition to the Ultimate State of Turbulent Rayleigh-Bénard Convection

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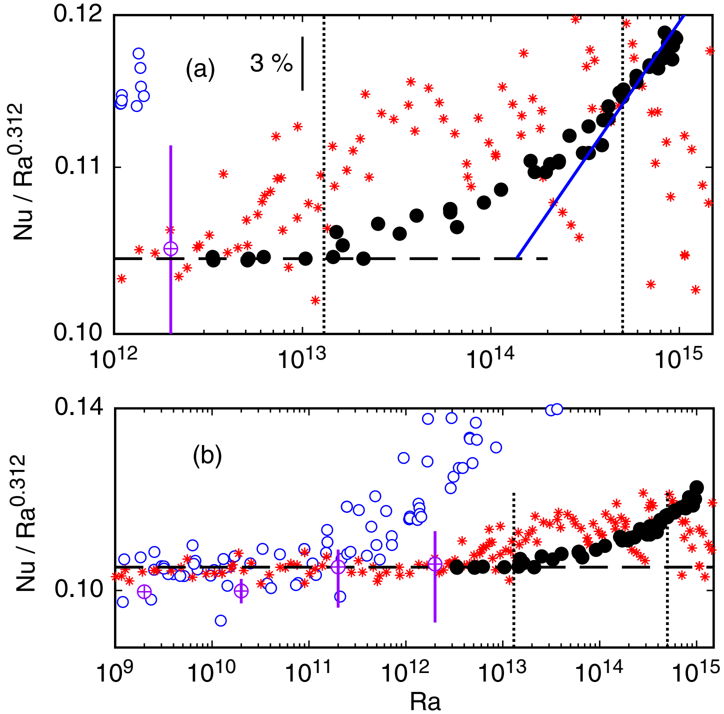
[4]*Laboratory of Atomic and Solid-State Physics and Sibley School of Mechanical and Aerospace Engineering, Cornell University, Ithaca, New York 14853*
[5]*Department of Physics, University of California, Santa Barbara, California 93106, USA*



Turbulent transport bounds:

J. Fluid Mech. 17 (1963) 405–432.

Heat transport by turbulent convecti

By LOUIS N. HOWARD

$\mathrm{Nu} < C\,\mathrm{Ra}^{1/2}$

uniformly in Pr

J. Fluid Mech. 37 (1969) 457–477.

On Howard's upper bound for heat transport by turbulent convection

By F. H. BUSSE

$\mathrm{Nu} < c\,\mathrm{Ra}^{1/2}$

uniformly in Pr

no-slip boundaries, 2-d and 3-d

VOLUME 69, NUMBER 11 PHYSICAL REVIEW LETTERS 14 SEPTEMBER 1992

Energy Dissipation in Shear Driven Turbulence

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Peter Constantin[b]

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(Received 21 February 1992; revised manuscript received 13 August 1992)

PHYSICAL REVIEW E VOLUME 49, NUMBER 5 MAY 1994

Variational bounds on energy dissipation in incompressible flows: Shear flow

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Peter Constantin†

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(Received 18 October 1993)

PHYSICAL REVIEW E VOLUME 53, NUMBER 6 JUNE 1996

Variational bounds on energy dissipation in incompressible flows. III. Convection

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(Received 8 May 1995; revised manuscript received 16 January 1996)

ELSEVIER Physica D 101 (1997) 178–190 PHYSICA D

Improved variational principle for bounds on energy dissipation in turbulent shear flow

Rolf Nicodemus [1], Siegfried Grossmann [2], Martin Holthaus *

Fachbereich Physik der Philipps-Universität, Renthof 6, D-35032 Marburg, Germany

Received 18 July 1996; revised 2 September 1996; accepted 3 September 1996

Communicated by F.H. Busse

ELSEVIER Physica D 121 (1998) 175–192 PHYSICA D

Unification of variational principles for turbulent shear flows: the background method of Doering–Constantin and the mean-fluctuation formulation of Howard–Busse

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Turbulent transport bounds:

J. Fluid Mech. (2003), *vol.* 477, *pp.* 363–379.

Improved upper bound on the energy dissipation rate in plane Couette flow: the full solution to Busse's problem and the Constantin–Doering–Hopf problem with one-dimensional background field

By S. C. PLASTING AND R. R. KERSWELL
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$$\mathrm{Nu} \leq .02634\,\mathrm{Ra}^{1/2}$$

uniformly in Pr

in 2-d & 3-d

J. Fluid Mech. (2006), *vol.* 560, *pp.* 159–227. © 2006 Cambridge University Press 159
doi:10.1017/S0022112006000450 Printed in the United Kingdom

Infinite-Prandtl-number convection. Part 2. A singular limit of upper bound theory

By G. R. IERLEY[1], R. R. KERSWELL[2] AND S. C. PLASTING[3]

Numerical evidence:

$\mathrm{Nu} \leq c\,\mathrm{Ra}^{1/3}$

at $\mathrm{Pr} = \infty$

J. Fluid Mech. (2006), *vol.* 560, *pp.* 229–241. © 2006 Cambridge University Press 229
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Bounds on vertical heat transport for infinite-Prandtl-number Rayleigh–Bénard convection

By CHARLES R. DOERING[1], FELIX OTTO[2] AND MARIA G. REZNIKOFF[2,3]

Theorem:

$\mathrm{Nu} \leq .644 \times \mathrm{Ra}^{1/3}[\ln(\mathrm{Ra})]^{1/3}$

at $\mathrm{Pr} = \infty$

JOURNAL OF MATHEMATICAL PHYSICS **52**, 083702 (2011)

Rayleigh–Bénard convection: Improved bounds on the Nusselt number

Felix Otto and Christian Seis[a)]
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(Received 1 March 2011; accepted 8 July 2011; published online 18 August 2011)

Theorem:

$\mathrm{Nu} \leq C\,\mathrm{Ra}^{1/3}\ln[\ln(\mathrm{Ra})]^{1/3}$

at $\mathrm{Pr} = \infty$

NO-SLIP BOUNDARIES

J. Fluid Mech. (2006), *vol.* 560, *pp.* 159–227.
doi:10.1017/S0022112006000450 Printed in the United Kingdom

Infinite-Prandtl-number convection. Part 2. A singular limit of upper bound theory

By G. R. IERLEY[1], R. R. KERSWELL[2] AND S. C. PLASTING[3]

Numerical evidence: "In the case of free-slip, we find an asymptotic scaling [bound] of $\mathrm{Nu} \leq c\,\mathrm{Ra}^{5/12}$ "

Turbulent transport bounds:

BOUNDS FOR THE HEAT TRANSPORT IN TURBULENT CONVECTION

Arbitrary Pr

2-d free-slip boundaries

with an enstrophy constraint

by

Jesse Otero

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2002

Turbulent transport bounds:

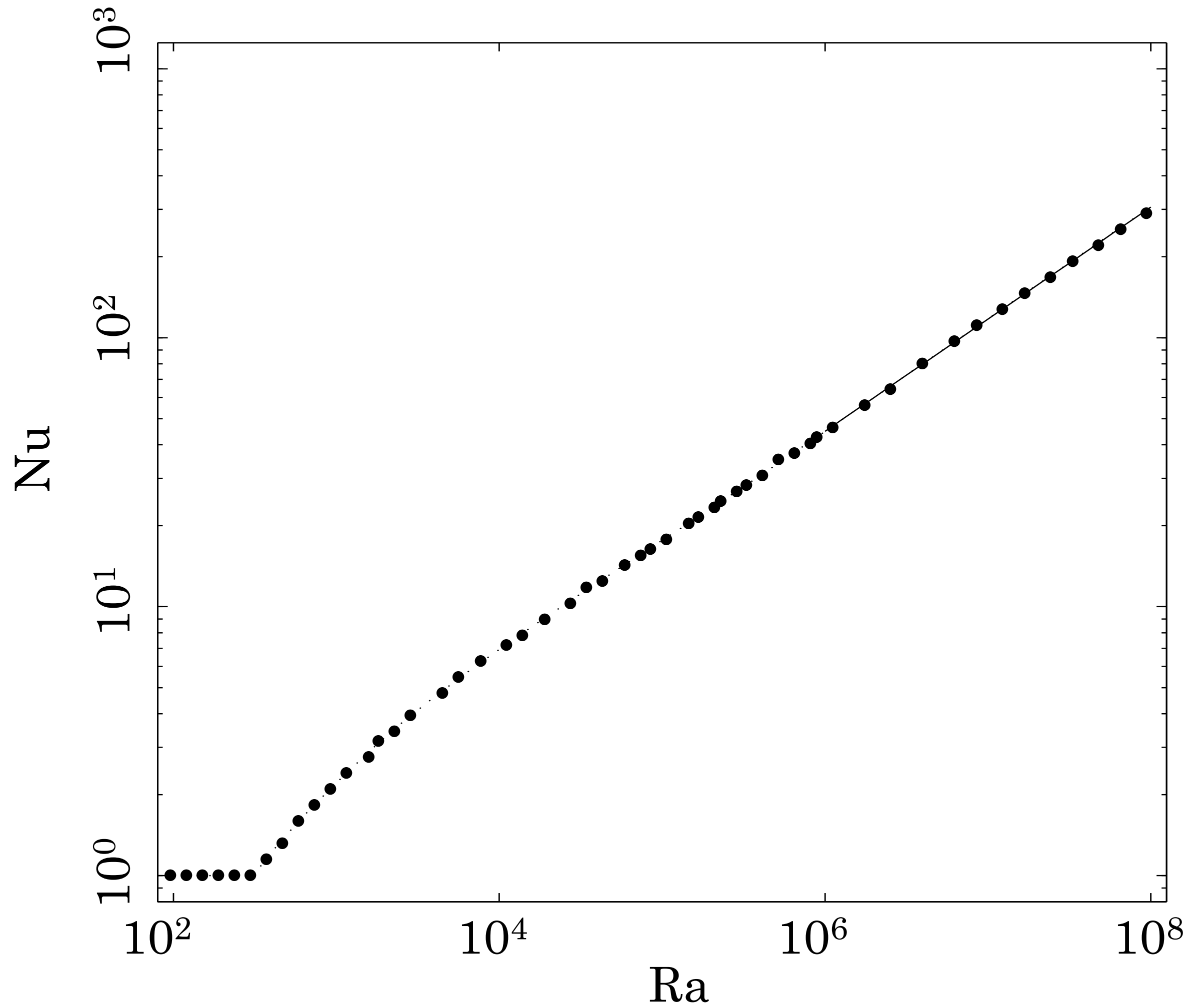

Figure 6.2: Semi-optimal bound. The dotted line shows the numerical bound. The solid line is a graph of $\mathrm{Nu} = .142\mathrm{Ra}^{5/12}$.

PRL 106, 244501 (2011) PHYSICAL REVIEW LETTERS week ending 17 JUNE 2011

Ultimate State of Two-Dimensional Rayleigh-Bénard Convection between Free-Slip Fixed-Temperature Boundaries

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Computational Physics Group (CCS-2),
Los Alamos National Laboratory

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National Science Foundation
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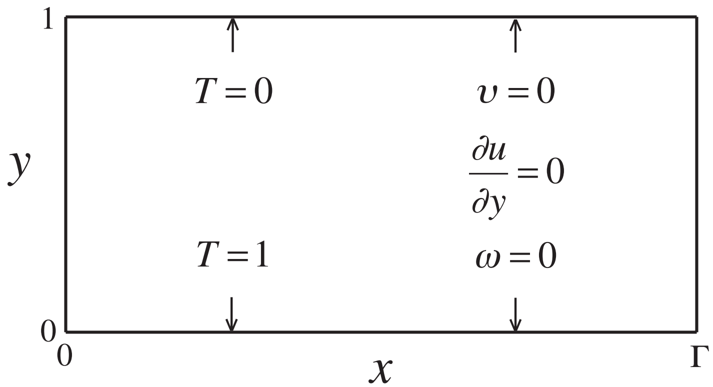
Background method:



$$\omega = \partial v/\partial x - \partial u/\partial y,$$

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + \nabla p = \nabla^2\boldsymbol{u} + \mathrm{Ra}\,\hat{\boldsymbol{j}}T, \qquad (1)$$

$$\nabla\cdot\boldsymbol{u} = 0, \qquad (2)$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T = \nabla^2 T, \qquad (3)$$

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \omega}{\partial t} + \boldsymbol{u}\cdot\nabla\omega\right) = \nabla^2\omega + \mathrm{Ra}\frac{\partial T}{\partial x}. \qquad (4)$$

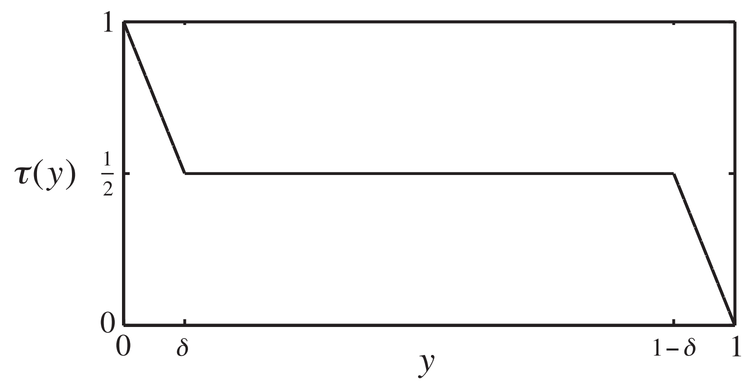
$$\frac{\partial \theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta = \nabla^2\theta + \tau''(y) - v\tau'(y). \qquad (5)$$

Background decomposition:

$$T(x,y,t) = \tau(y) + \theta(x,y,t)$$

$$\tau(0) = 1 \text{ and } \tau(1) = 0$$

$$\theta(x,0,t) = 0 = \theta(x,1,t)$$



Background method:

Then the equations of motion together with the boundary conditions and the background decomposition imply

$$\frac{1}{2\,\mathrm{Pr}}\frac{d}{dt}\|\boldsymbol{u}\|_2^2 = -\|\omega\|_2^2 + \mathrm{Ra}\int v\theta dxdy, \qquad (6)$$

$$\frac{1}{2\,\mathrm{Pr}}\frac{d}{dt}\|\omega\|_2^2 = -\|\nabla\omega\|_2^2 + \mathrm{Ra}\int \omega\frac{\partial\theta}{\partial x}dxdy, \qquad (7)$$

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_2^2 = -\|\nabla\theta\|_2^2 - \int\left[\tau'\frac{\partial\theta}{\partial y} + \tau' v\theta\right]dxdy, \qquad (8)$$

$$\|\nabla T\|_2^2 = \|\nabla\theta\|_2^2 + 2\int \tau'\frac{\partial\theta}{\partial y}dxdy + \|\tau'\|_2^2, \qquad (9)$$

where $\|\cdot\|_2$ is the L^2 norm on the spatial domain and the elementary identity $\|\nabla\boldsymbol{u}\|_2^2 = \|\omega\|_2^2$ was used in (6).

$$\frac{b}{\mathrm{Ra}}\times(6) + \frac{a}{\mathrm{Ra}^{3/2}}\times(7) + 2\times(8) + (9), \qquad (10)$$

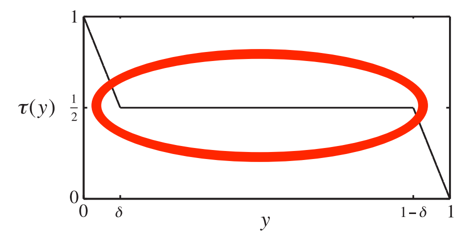
Background method:

$$\mathrm{Nu} = \frac{1}{1-b}\left(\int_0^1 \tau'(y)^2 dy - b\right) - \frac{1}{1-b}\mathcal{Q}$$

$$\mathcal{Q} = \left\langle |\nabla\theta|^2 + \frac{a}{\mathrm{Ra}^{3/2}}|\nabla\omega|^2 + \frac{b}{\mathrm{Ra}}|\omega|^2 + 2\tau' v\theta - \frac{a}{\mathrm{Ra}^{1/2}}\,\omega\,\frac{\partial\theta}{\partial x}\right\rangle$$

Hence if we can choose the background profile $\tau(y)$ and coefficients $a > 0$ and $0 < b < 1$ so that $\mathcal{Q} \geq 0$ for all relevant θ, ω and v, then the first term on the right hand side of (11) is an upper bound on Nu. For the problem at hand we may use the piecewise linear profile shown in Fig. 2 where the thickness δ of the "boundary layers" is to be determined as a function of Ra to satisfy $\mathcal{Q} \geq 0$. With this choice of $\tau(y)$ the bound will be

$$\mathrm{Nu} \leq \frac{1}{2\delta(1-b)} - \frac{b}{1-b}.$$



Background method:

Applying the horizontal Fourier transform and introducing the shorthand $D = \frac{d}{dy}$, it is evident that positivity of $\mathcal{Q}$ is equivalent to the positivity of

$$\mathcal{Q}_k = \|D\hat{\theta}_k\|^2 + k^2\|\hat{\theta}_k\|^2 + \frac{a}{\mathrm{Ra}^{3/2}}\|D\hat{\omega}_k\|^2 + \frac{a}{\mathrm{Ra}^{3/2}}k^2\|\hat{\omega}_k\|^2 + \frac{b}{\mathrm{Ra}}\|\hat{\omega}_k\|^2 + \mathrm{Re}\left\{2\int_0^1 \tau'\hat{v}_k\hat{\theta}_k^* dy - \frac{aik}{\mathrm{Ra}^{1/2}}\int_0^1 \hat{\omega}_k\hat{\theta}_k^* dy\right\}$$

for each horizontal wave number k where $\|\cdot\|$ is now the L^2 norm on complex valued functions of $y \in [0,1]$ and $\mathrm{Re}\{\cdot\}$ indicates the real part of a complex quantity.

Cauchy-Schwarz and Young inequalities imply

$$\left|\frac{aik}{\mathrm{Ra}^{1/2}}\int_0^1 \hat{\omega}_k\hat{\theta}_k^* dy\right| \le \frac{a^2}{4\mathrm{Ra}}\|\hat{\omega}_k\|^2 + k^2\|\hat{\theta}_k\|^2$$

so dropping the manifestly non-negative term $\|D\hat{\omega}_k\|^2$,

$$\mathcal{Q}_k \ge \|D\hat{\theta}_k\|^2 + \left[\frac{ak^2}{\mathrm{Ra}^{3/2}} + \frac{1}{\mathrm{Ra}}\left(b - \frac{a^2}{4}\right)\right]\|\hat{\omega}_k\|^2 - \frac{1}{\delta}\mathrm{Re}\left\{\int_0^\delta \hat{v}_k(y)\hat{\theta}_k^*(y)dy + \int_{1-\delta}^1 \hat{v}_k(y)\hat{\theta}_k^*(y)dy\right\}.$$

Background method:

Because $\hat{\theta}_k(y)$ vanishes at $y = 0$ and 1, applications of the fundamental theorem of calculus and Cauchy-Schwarz inequality yield the pointwise bounds

$$|\hat{\theta}_k(y)| \leq y^{1/2} \left(\int_0^{1/2} |D\hat{\theta}_k(y')|^2 dy' \right)^{1/2}$$

for $0 \leq y \leq 1/2$ and, for $1/2 \leq y \leq 1$,

$$|\hat{\theta}_k(y)| \leq (1-y)^{1/2} \left(\int_{1/2}^{1} |D\hat{\theta}_k(y')|^2 dy' \right)^{1/2}$$

The Fourier coefficients of the vertical velocity and vorticity (suppressing the time dependence) are related by

$$ik\hat{\omega}_k(y) = D^2\hat{v}_k(y) - k^2\hat{v}_k(y)$$

$$\vdots$$

$$|\hat{v}_k(y)| \leq \frac{3^{3/4}}{2^{3/2}} k^{1/2} \min\{y, 1-y\} \|\hat{\omega}_k\|$$

Background method:

Hence $\mathcal{Q}_k \geq 0$ is guaranteed by a δ small enough that

$$\frac{ak^2}{\mathrm{Ra}^{3/2}} + \frac{1}{\mathrm{Ra}}\left(b - \frac{a^2}{4}\right) - \frac{3^{3/2}k}{5^2 \times 2^2}\,\delta^3 \geq 0$$

Inserting $a = \frac{2}{\sqrt{15}}$ and $b = \frac{1}{5}$ into (28)—chosen to minimize the prefactor in the bound—and minimizing the suitable δ over k, this is satisfied by choosing $\delta = \frac{2^{4/3}\cdot 5^{5/12}}{3^{3/4}}\,\mathrm{Ra}^{-5/12}$ where $k = \frac{1}{3^{1/4}\cdot 5^{1/4}}\,\mathrm{Ra}^{1/4}$ is the minimizing wave number. Inserting these δ and b into (13) we see that for $\mathrm{Ra} > 33.57$ (actually for $\mathrm{Ra} > \frac{27}{4}\pi^4$)

$$\mathrm{Nu} \leq \frac{5^{7/12} \times 3^{3/4}}{2^{13/3}}\,\mathrm{Ra}^{5/12} - \frac{1}{4} \lesssim 0.2891\mathrm{Ra}^{5/12}$$

Turbulent transport bounds:

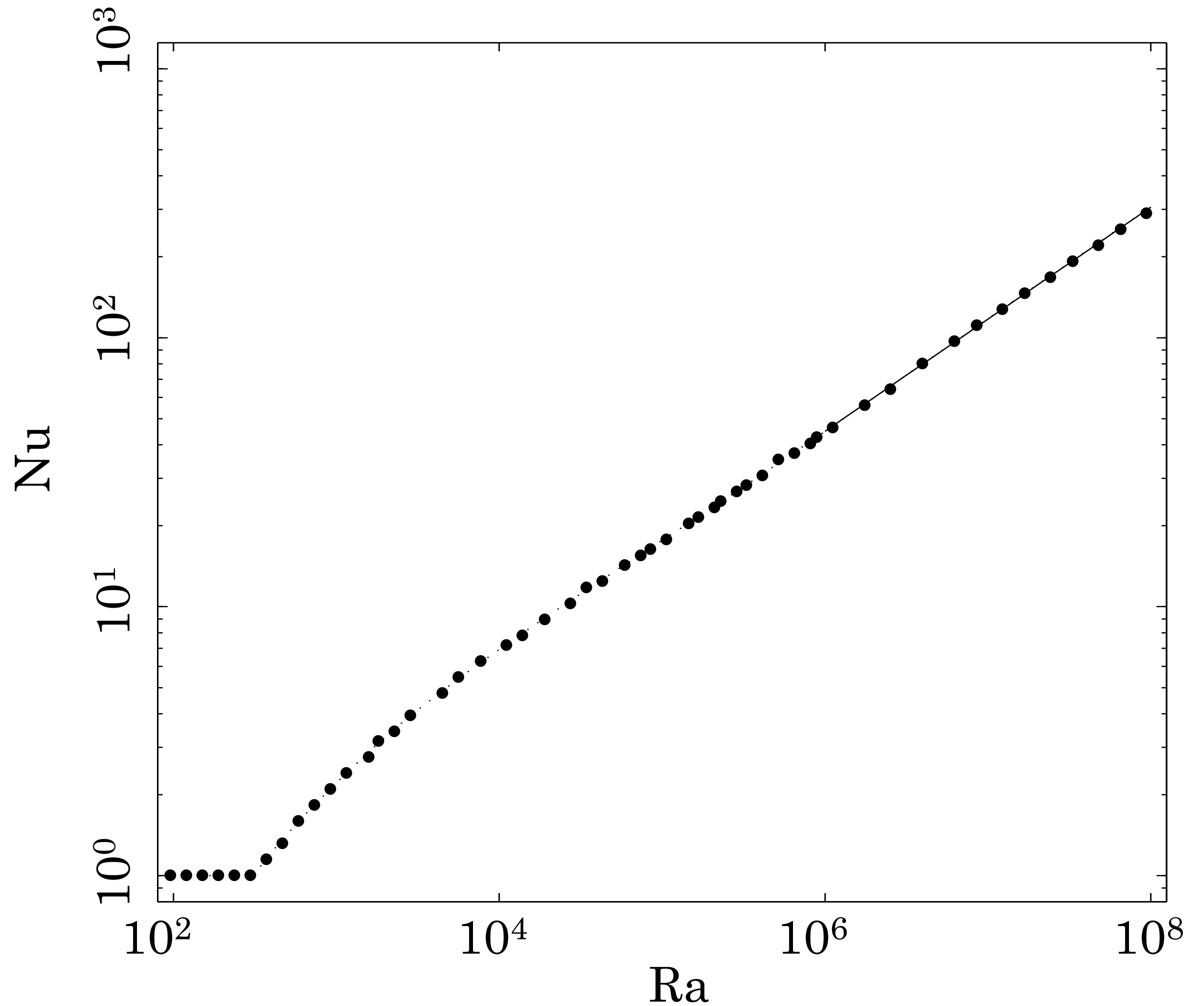


Figure 6.2: Semi-optimal bound. The dotted line shows the numerical bound. The solid line is a graph of $\mathrm{Nu} = .142\mathrm{Ra}^{5/12}$.

Background method: Pr = ∞

J. Fluid Mech., page 1 of 19.
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Rigid bounds on heat transport by a fluid between slippery boundaries

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Rigorous bounds on heat transport are derived for thermal convection between stress-free horizontal plates. For three-dimensional Rayleigh–Bénard convection at infinite Prandtl number (Pr), the Nusselt number (Nu) is bounded according to $Nu \leqslant 0.28764Ra^{5/12}$ where Ra is the standard Rayleigh number. For convection driven by a uniform steady internal heat source between isothermal boundaries, the spatially and temporally averaged (non-dimensional) temperature is bounded from below by $\langle T \rangle \geqslant 0.6910R^{-5/17}$ in three dimensions at infinite Pr and by $\langle T \rangle \geqslant 0.8473R^{-5/17}$ in two dimensions at arbitrary Pr, where R is the heat Rayleigh number proportional to the injected flux.

The $\mathrm{Nu} \lesssim \mathrm{Ra}^{5/12}$ bound derived here raises questions of precisely how the spatial dimension and the nature of even very thin boundary layers enter into the problem at high Rayleigh numbers. At least in two dimensions with free-slip boundaries, no matter how high the Rayleigh number is it is apparent that boundary layers continue to play a limiting role in the turbulent heat transport.

$$\text{Take away:} \qquad \frac{1}{3}\,? \qquad \frac{1}{2}\,?$$

$$\text{Split the difference:} \left(\frac{1}{3} + \frac{1}{2} \right) \div 2 = \frac{5}{12}\;!$$

Ultimate State of Two-Dimensional Rayleigh-Bénard Convection between Free-Slip Fixed-Temperature Boundaries

Jared P. Whitehead[1,*] and Charles R. Doering[1,2,3]

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[2]*Department of Physics, University of Michigan, Ann Arbor, Michigan 48109-1040, USA*
[3]*Center for the Study of Complex Systems, University of Michigan, Ann Arbor, Michigan 48109-1107, USA*

Rigorous upper limits on the vertical heat transport in two-dimensional Rayleigh-Bénard convection between stress-free isothermal boundaries are derived from the Boussinesq approximation of the Navier-Stokes equations. The Nusselt number Nu is bounded in terms of the Rayleigh number Ra according to $\mathrm{Nu} \leq 0.2891\mathrm{Ra}^{5/12}$ uniformly in the Prandtl number Pr. This scaling challenges some theoretical arguments regarding asymptotic high Rayleigh number heat transport by turbulent convection.

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Comparison of Turbulent Thermal Convection between Conditions of Constant Temperature and Constant Flux

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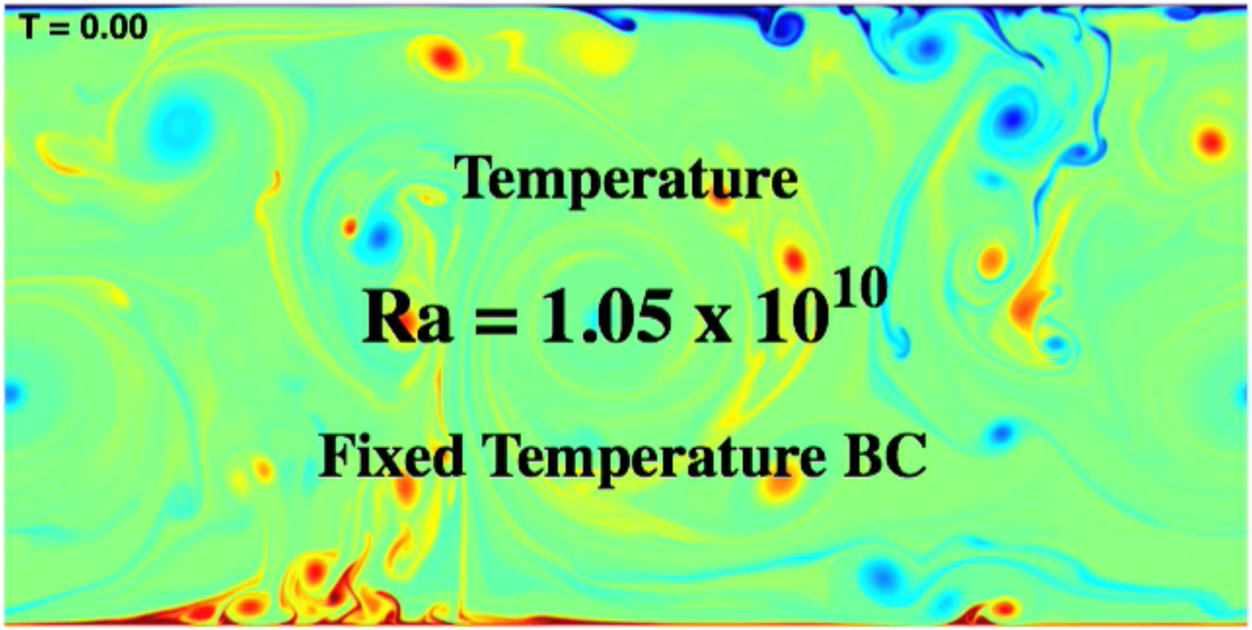
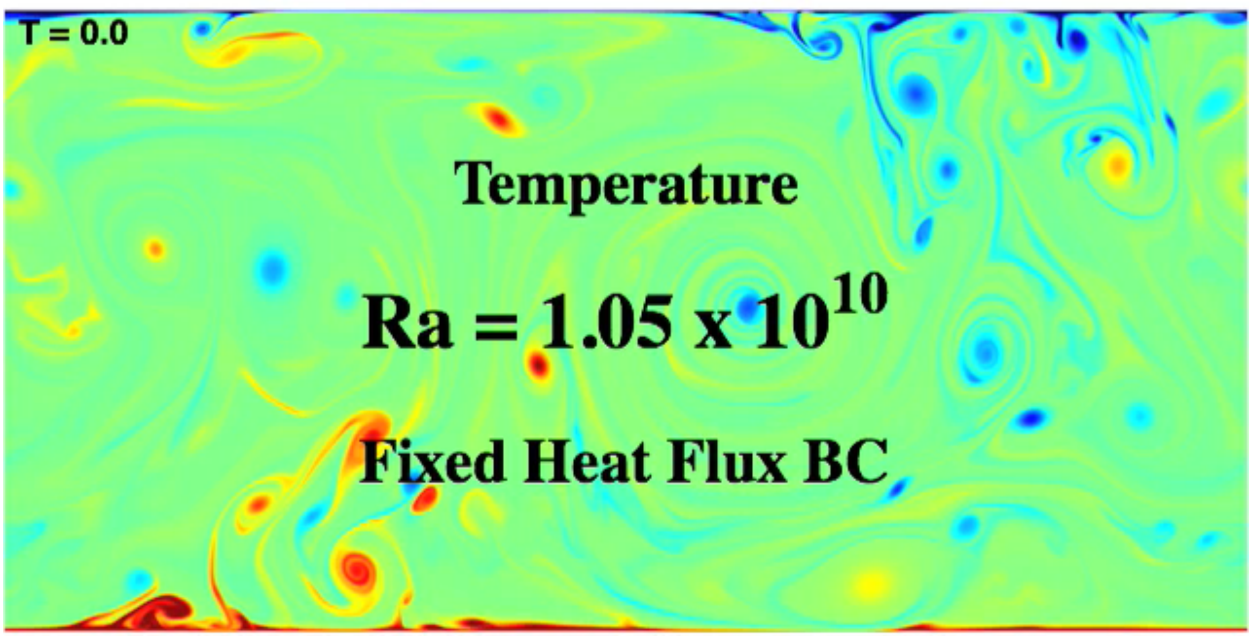
(Received 26 October 2008; published 10 February 2009)

We report the results of high-resolution direct numerical simulations of two-dimensional Rayleigh-Bénard convection for Rayleigh numbers up to $\mathrm{Ra} = 10^{10}$ in order to study the influence of temperature boundary conditions on turbulent heat transport. Specifically, we considered the extreme cases of fixed heat flux (where the top and bottom boundaries are poor thermal conductors) and fixed temperature (perfectly conducting boundaries). Both cases display identical heat transport at high Rayleigh numbers fitting a power law $\mathrm{Nu} \approx 0.138 \times \mathrm{Ra}^{0.285}$ with a scaling exponent indistinguishable from $2/7 = 0.2857\ldots$ above $\mathrm{Ra} = 10^7$. The overall flow dynamics for both scenarios, in particular, the time averaged temperature profiles, are also indistinguishable at the highest Rayleigh numbers.

DOI: 10.1103/PhysRevLett.102.064501 PACS numbers: 47.27.te, 02.70.Hm, 44.25.+f

Both cases display identical heat transport at high Rayleigh numbers
fitting a power law $\mathrm{Nu} \approx 0.138 \times \mathrm{Ra}^{0.285}$
with a scaling exponent indistinguishable from $2/7$


T = 0.00
Temperature
Ra = 1.05 x 10^10
Fixed Temperature BC


T = 0.0
Temperature
Ra = 1.05 x 10^10
Fixed Heat Flux BC