Modeling and analysis of turbulence in rotating flow

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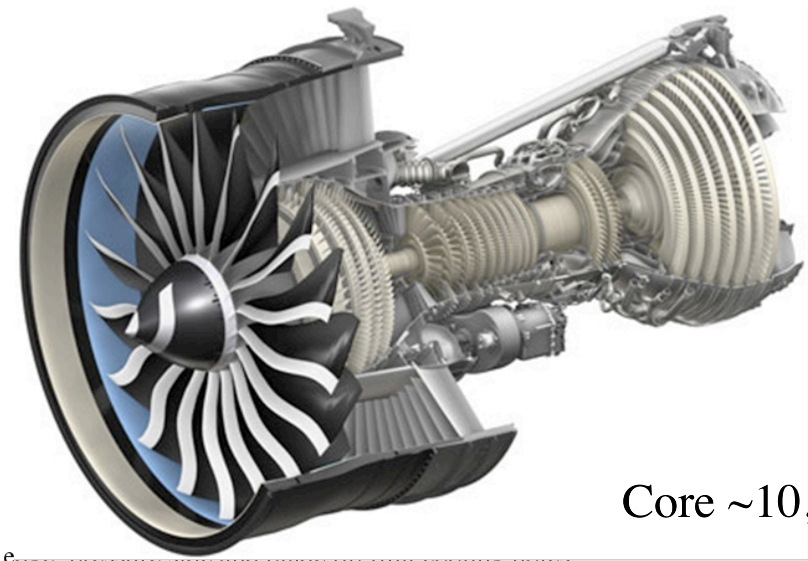
EPTT 2012
SÃO PAULO

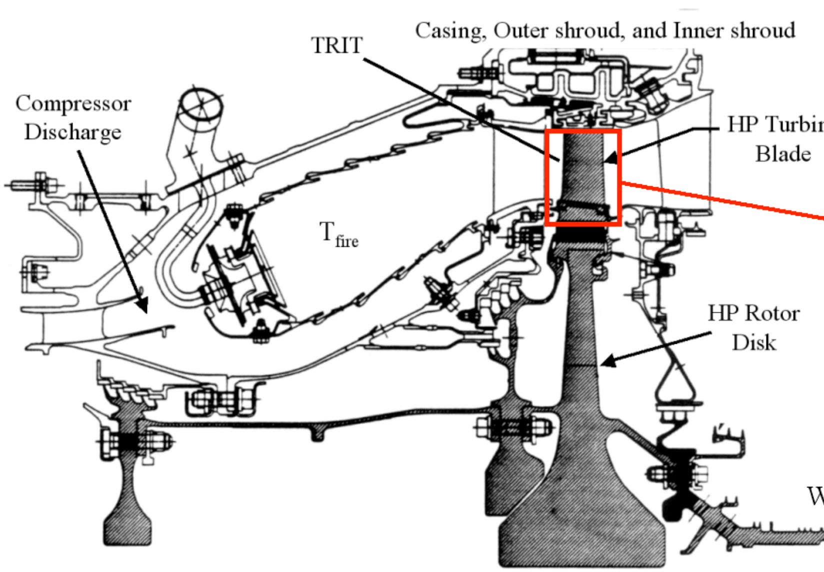
25 sept 2012 4:00-5:45 PM

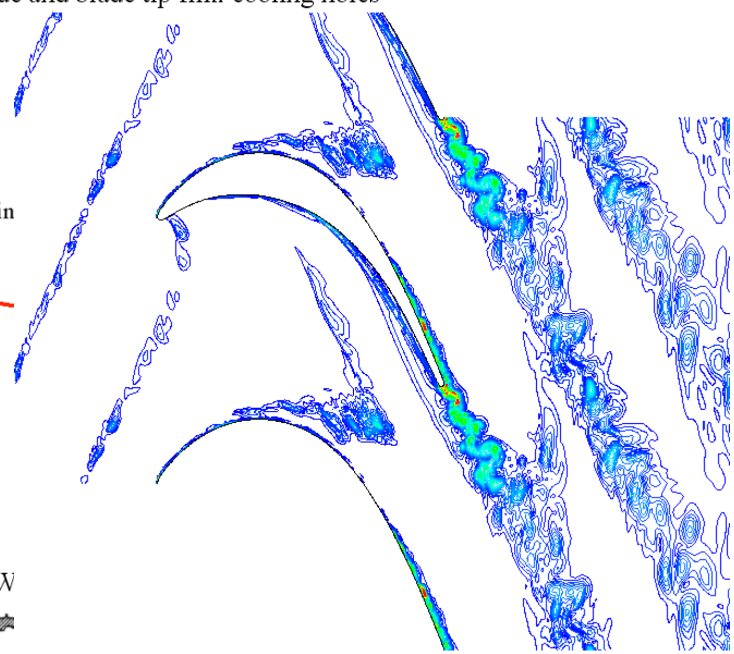
Introduction

Rotation

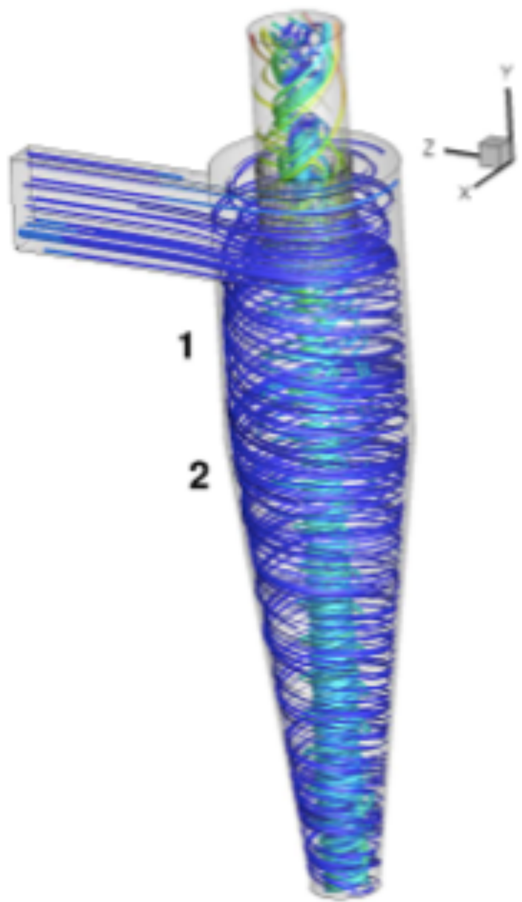


Core ~10,000 RPM

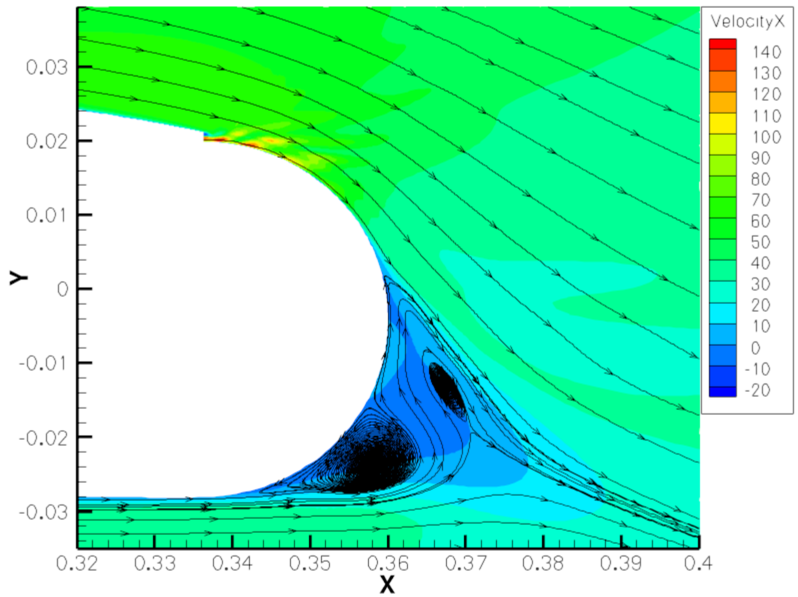


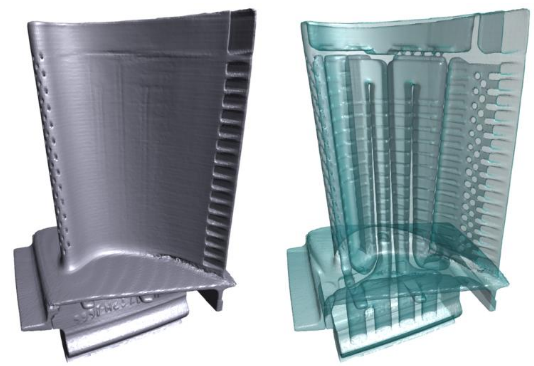


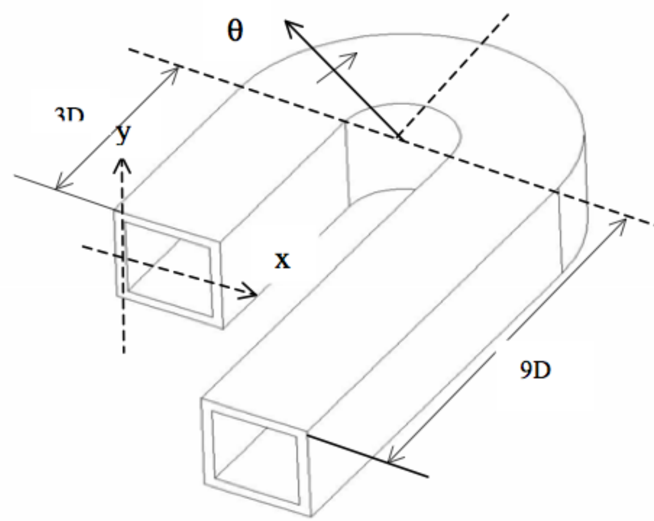
Swirl



Curvature


VelocityX
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
-10
-20
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
Y
0.32
0.33
0.34
0.35
0.36
0.37
0.38
0.39
0.4
X




θ
3D
y
x
9D

Plan

Part I

- Elements of rotating turbulence
  - Illustration via eddy resolving simulations
  - Phenomenology
  - Analytical concepts
- Linear stability of uniform gradient, rotating flow

Part II

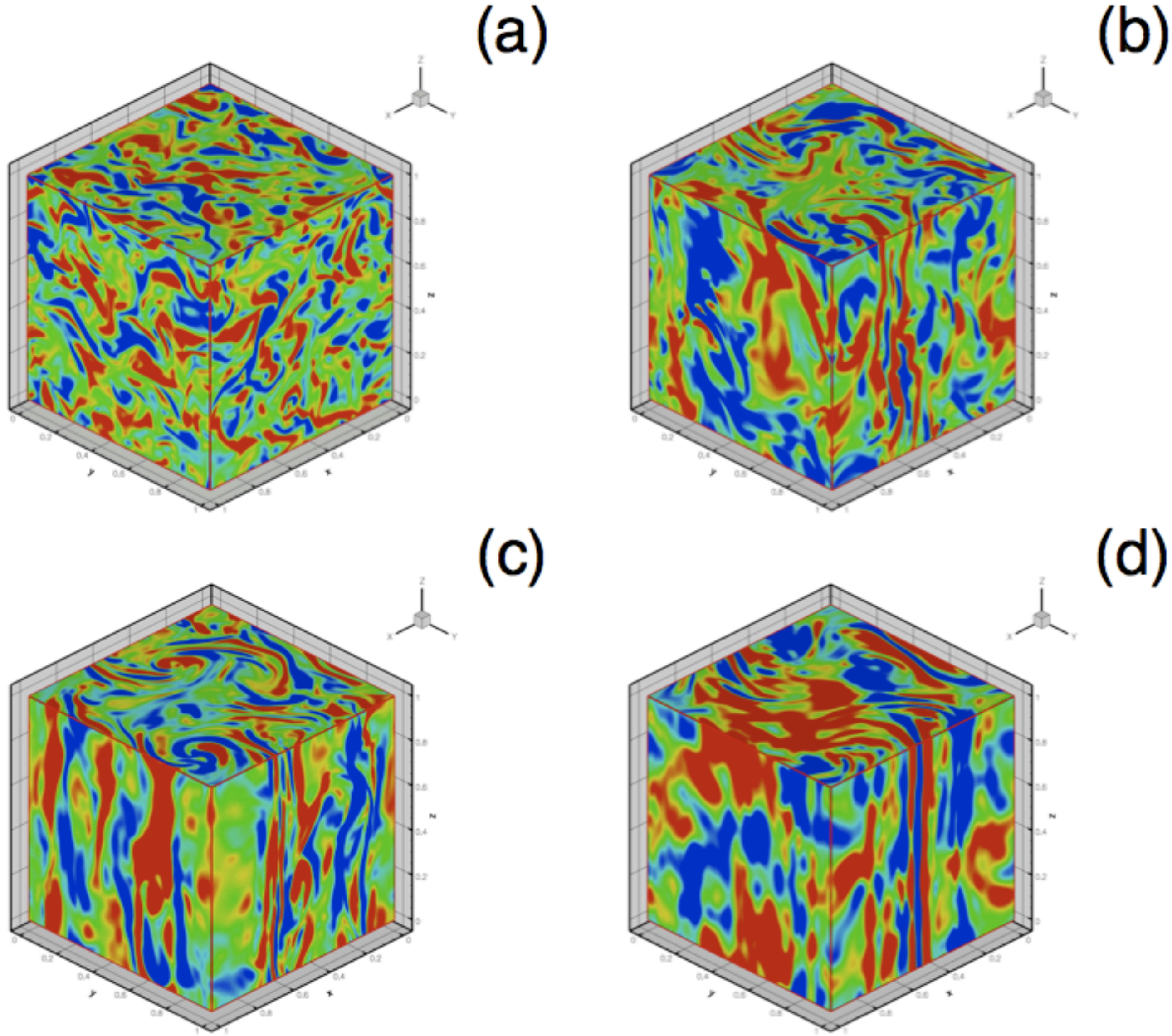
- Reynolds stress transport equations
  - Transformation to rotating coordinates
  - Absolute and relative rotation
  - Transformation and closure models
- Equilibrium analysis
  - Second moment closure and bifurcation
  - Modified coefficient, two equation models

Part I. Eddy Simulations

Basic Concepts

Phenomenology

Rotating grid turbulence



Length scale Anisotropy at intermediate Rossby numbers

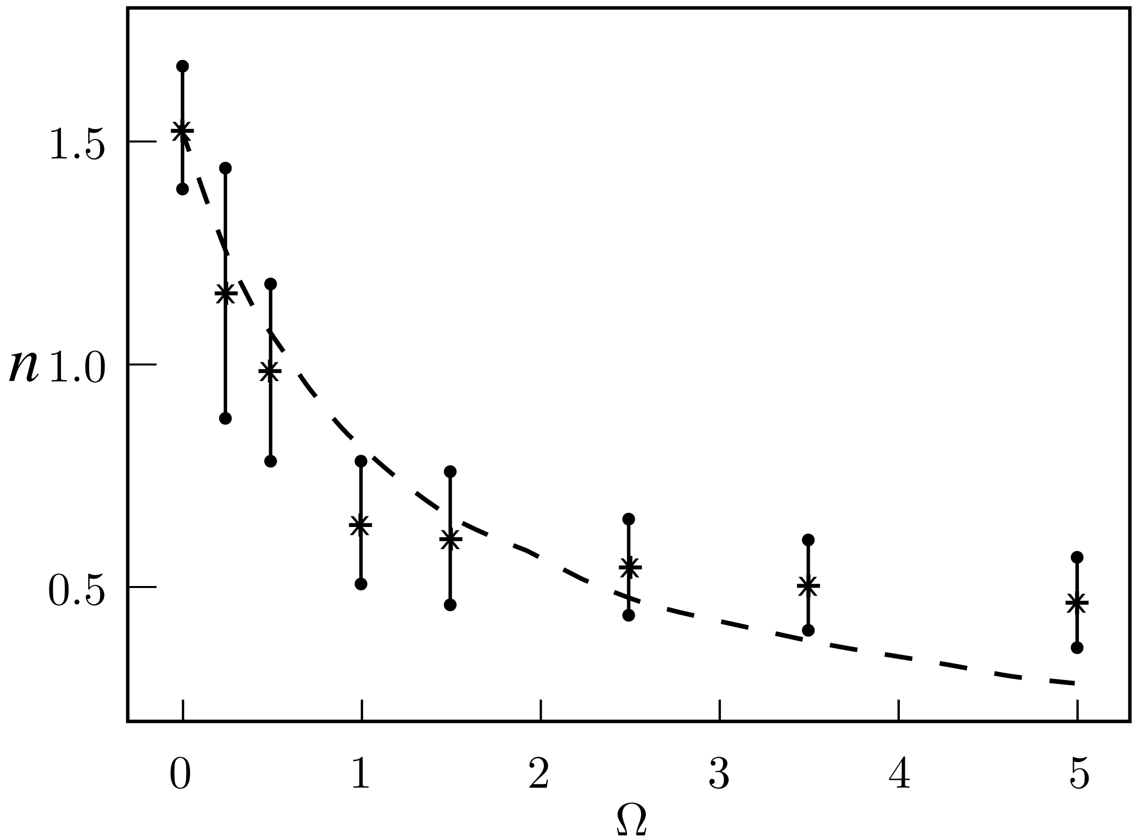
Stress anisotopy???

Geurts *et al* (2007) DNS

Decay exponent

$k \propto t^{-n}$ $\qquad\qquad C_{\epsilon 2} = 1 + 1/n$

Low Re DNS



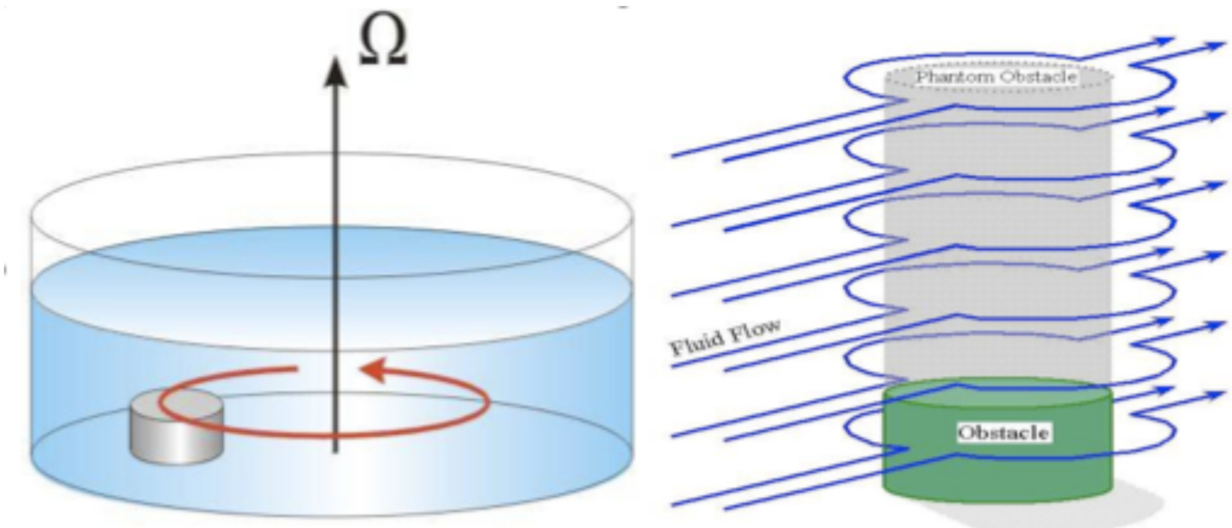
But dk/dt =-ε ; no explicit effect of rotation

And standard return to isotropy says b_{ij}=0

Taylor columns

Taylor-Proudman reorganization?

$$v = \frac{\partial_x p}{2\rho\Omega^F} = \partial_x \Psi \; ; \quad u = -\frac{\partial_y p}{2\rho\Omega^F} = -\partial_y \Psi$$



$$\text{Rossby number} = U/\Omega^F L \text{ or } \varepsilon/\Omega^F k$$

Momentum equation in rotating frame

$$\frac{\partial u_i}{\partial t} + u_j \partial_j u_i + 2\epsilon_{ijk}\Omega_j^F u_k = -\frac{1}{\rho}\partial_i p + \nu \nabla^2 u_i.$$

The Coriolis acceleration (2**Ω** × **u**) conserves energy

Arises by expressing equations in non-inertial frame

In *x*,*y*-component form, rotation about *z*

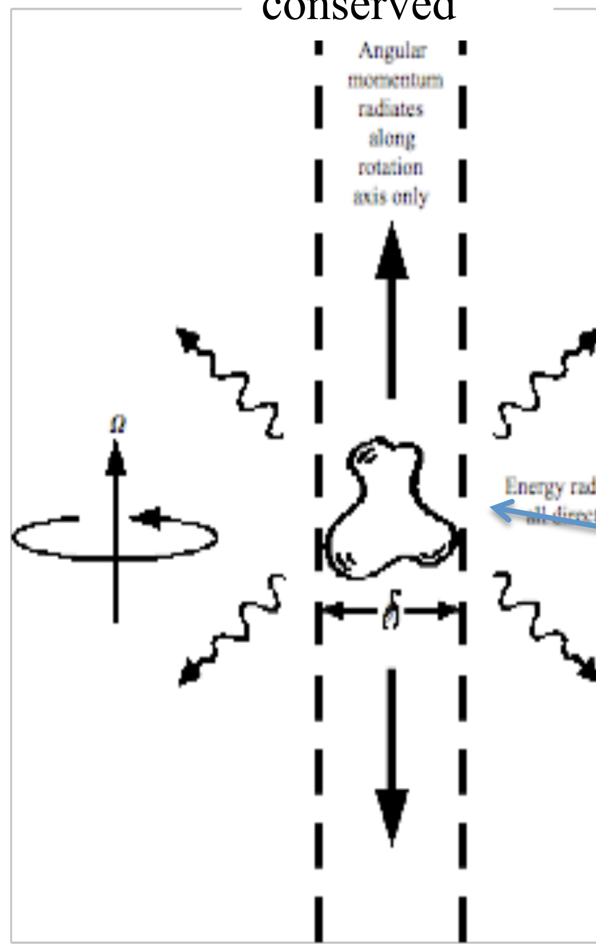
$$\begin{aligned}\frac{\partial u}{\partial t} + u_j \partial_j u - 2\Omega^F v &= -\frac{1}{\rho}\partial_x p + \nu \nabla^2 u \\ \frac{\partial v}{\partial t} + u_j \partial_j v + 2\Omega^F u &= -\frac{1}{\rho}\partial_y p + \nu \nabla^2 v\end{aligned}$$

How might Taylor-Proudman reorganization occur in turbulence?

P. Davidson (J. Fluid Mech. 557, 2006): turbulent eddies may generate transient Taylor Columns

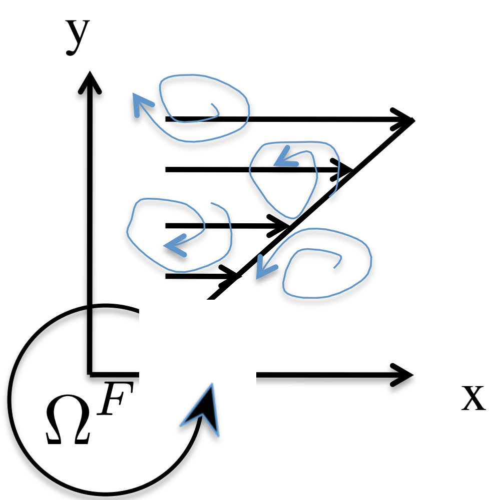
But only seen in DNS at intermediate Rossby numbers

Ros= U/ΩL



Relevance to engineering modeling unknown

Homogeneous shear



U = S y

Figure defines positive rotation and positive shear

N.B mean rotation is opposite to frame rotation

Linearized equations

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \boxed{vS - 2\Omega^F v} = -\frac{1}{\rho}\partial_x p$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} \boxed{+ 2\Omega^F u} = -\frac{1}{\rho}\partial_y p$$

Displacement produces 'streaks' or 'jets'

For x-independent perturbation (a.k.a 'streak'):

$$\frac{\partial u}{\partial t} \sim (2\Omega^F - s)v$$

'displacement effect' is

$$\Delta u \sim (2\Omega^F - S)\int v dt = (2\Omega^F - S)\Delta y$$

If $S > 2\Omega^F$ $(R > -1)$, $\Delta y > 0 \rightarrow \Delta u < 0$

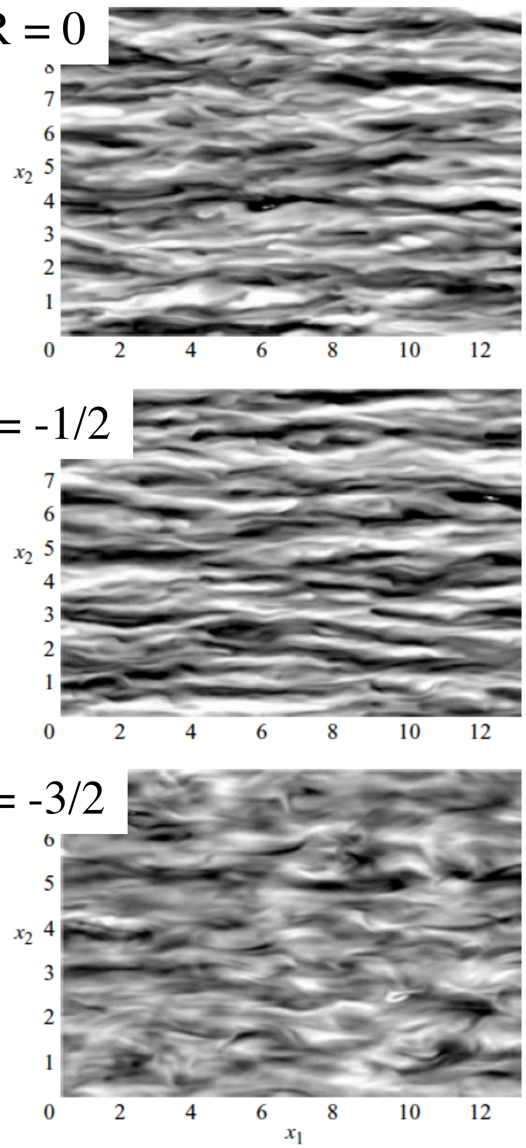
Brethouwer
(JFM 542, 2005)

u-contours

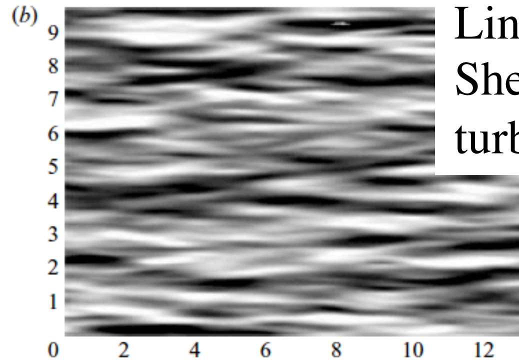
DNS R = 0

DNS R = -1/2

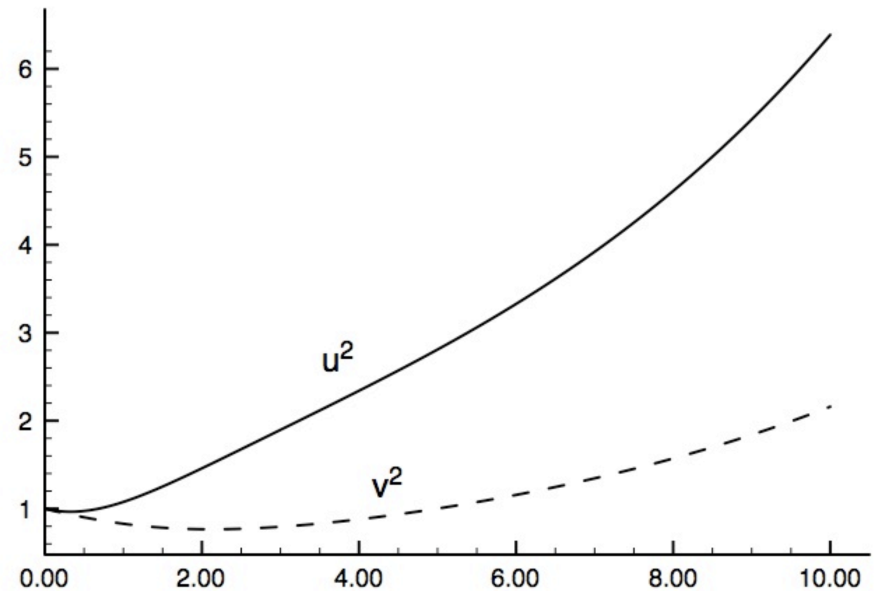
DNS R = -3/2



Linear solution for
Sheared isotropic
turbulence

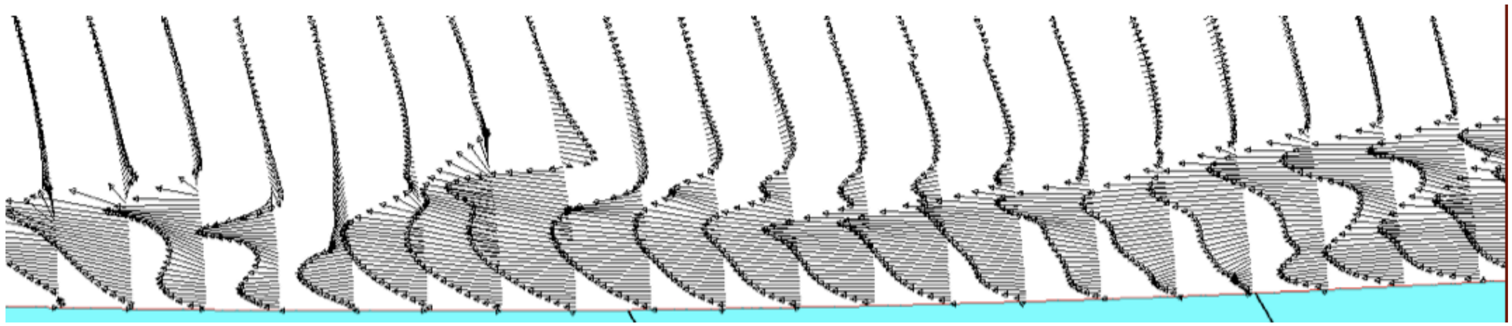


DNS R = -1

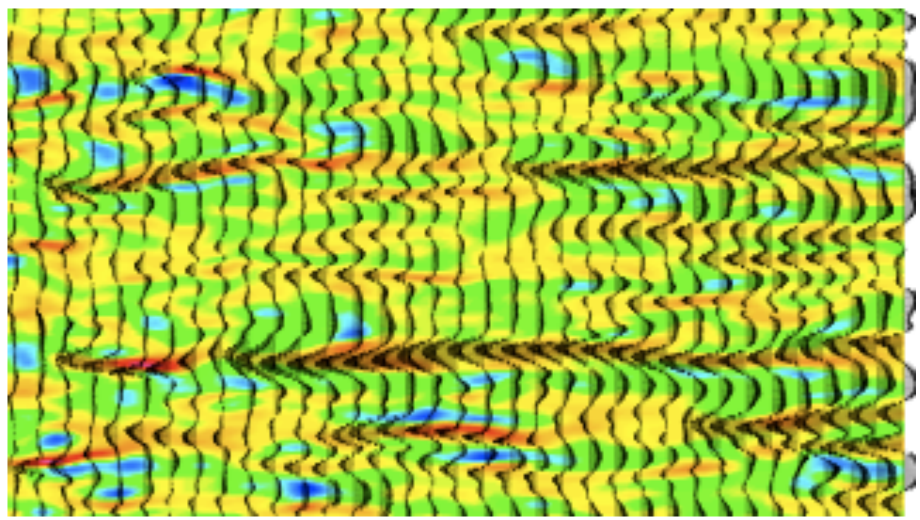


streaks
weaken,
R<-1, R>0

Streaky contours are jet like velocity perturbations: u'(y,z)



Side view



Top view

Admission: there are jets in transitional flow

Simple stability analysis

Consider an elongated disturbance:

$$\partial/\partial x(\bullet) = 0 \quad \rightarrow \quad p = 0$$

$$\frac{\partial u}{\partial t} + vS - 2\Omega^F v \quad = \quad 0$$

$$\frac{\partial v}{\partial t} \qquad\qquad + 2\Omega^F u \quad = \quad 0$$

Production from shear

Coriolis acceleration

Rotation number

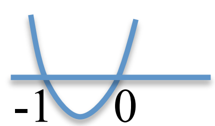
$$\Rightarrow \quad \frac{d^2u}{dt^2} = -R(R+1)S^2u \quad ; \quad R \equiv -2\Omega^F/S$$

$R = \dfrac{\text{coordinate rotation}}{\text{flow rotation}}$

$$u \propto e^{\pm\sqrt{-R(R+1)}\,St}$$

unstable $-1 < R < 0$

Comment: max @ R=-1/2
symmetric about -1/2

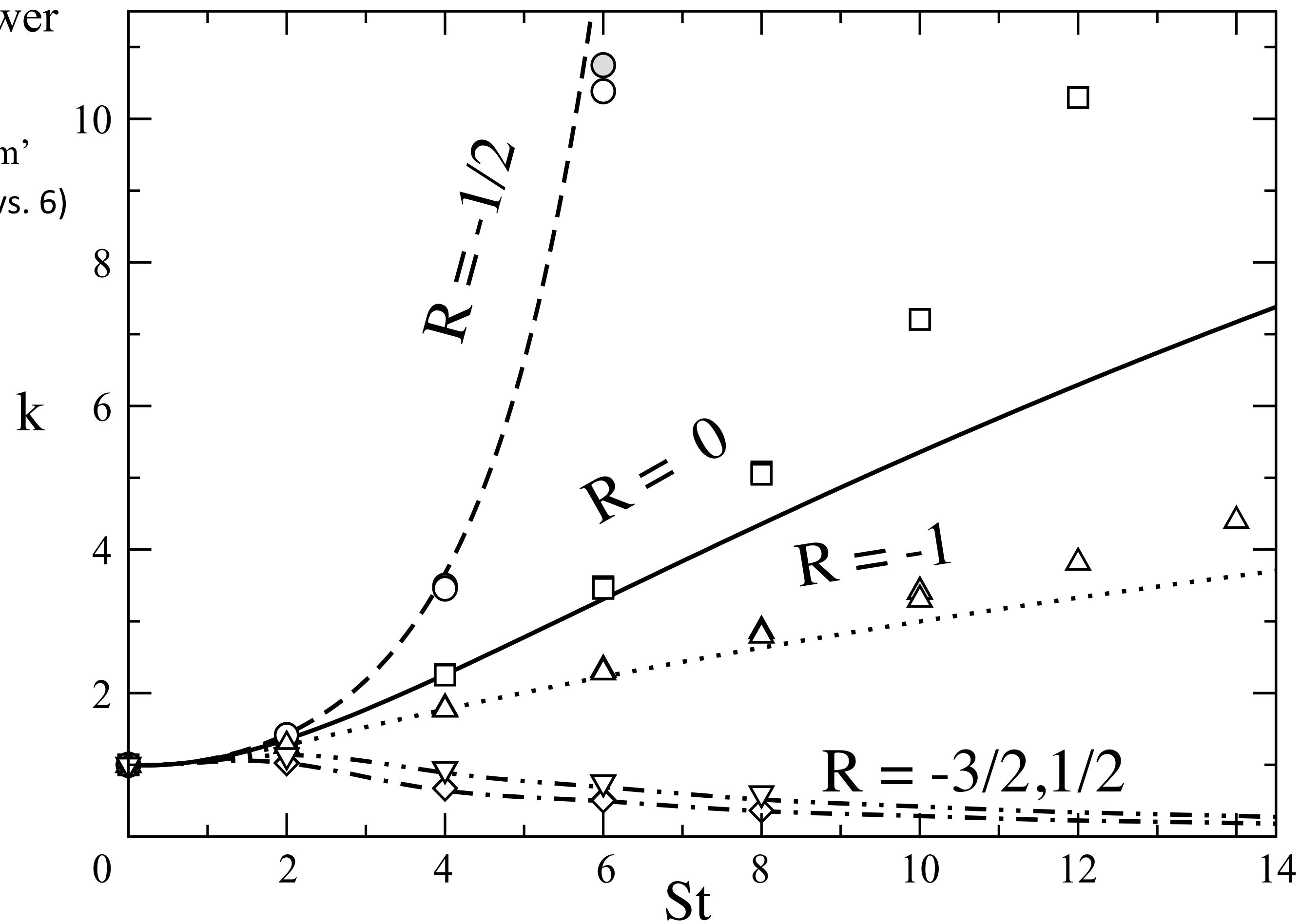


R< 0 for rotation against shear
(mean vorticity is $\omega_z = -S$)

t.k.e. in rotating homogeneous shear

Brethouwer

NB: Not
`equilibrium'
Sk/ε ~12 (vs. 6)

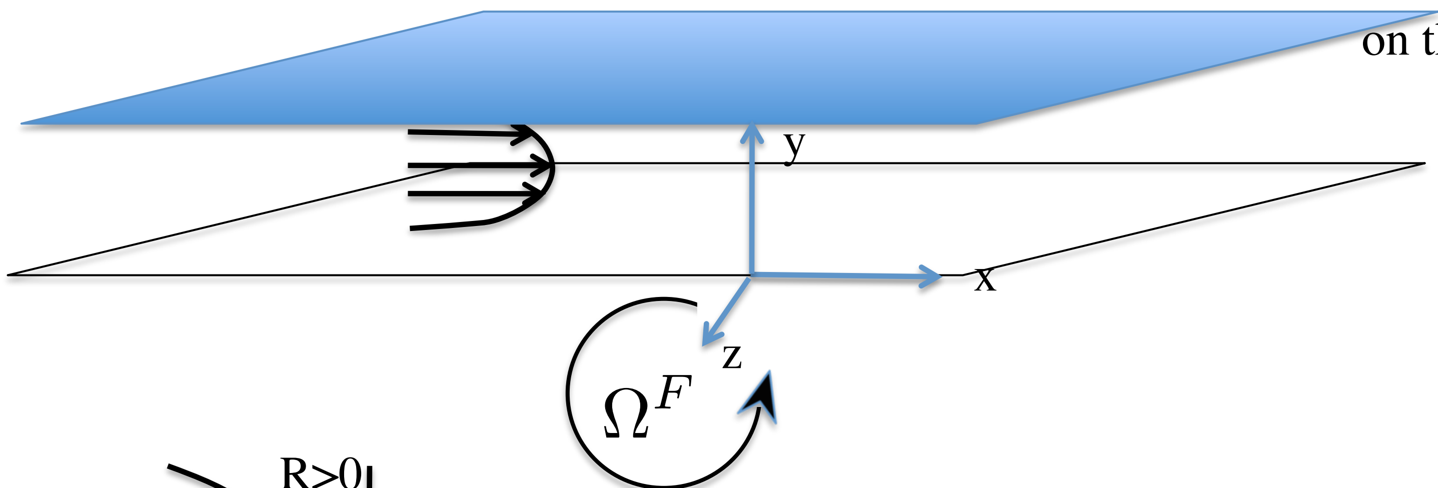


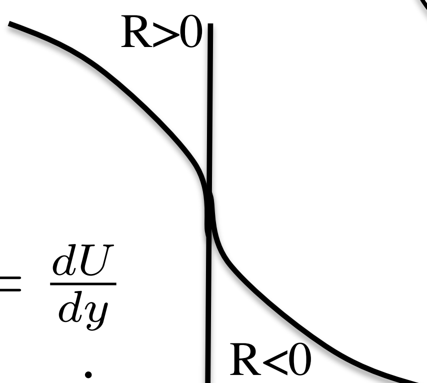
Note R=0 not same as R=-1

Bradshaw parameter, R(R+1) is not controlling

Symbols are DNS
Lines are linear, RDT
solution

Rotating channel





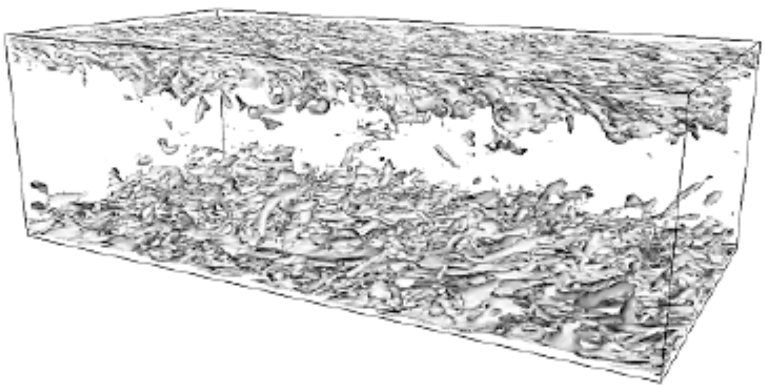
$$S = \frac{dU}{dy}$$

changes sign

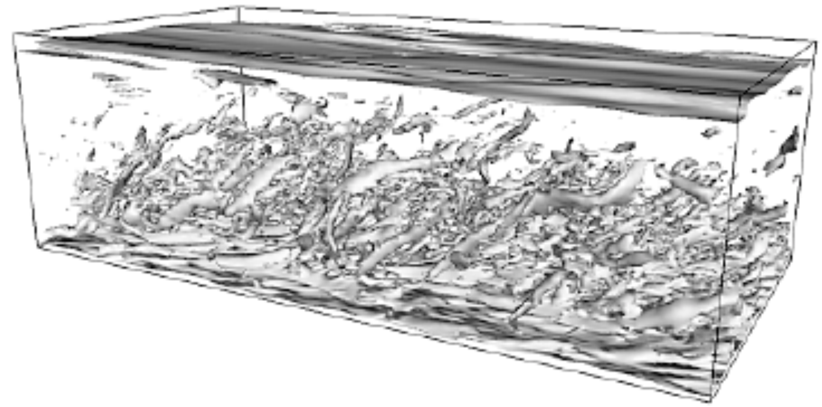
$$R = -2\Omega^F / S$$

changes sign

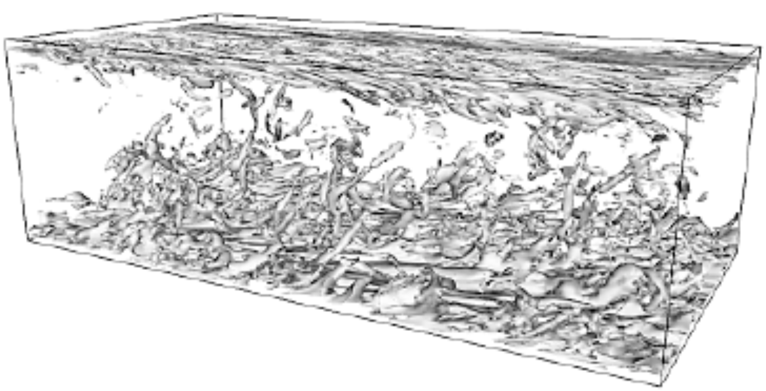
Vorticity



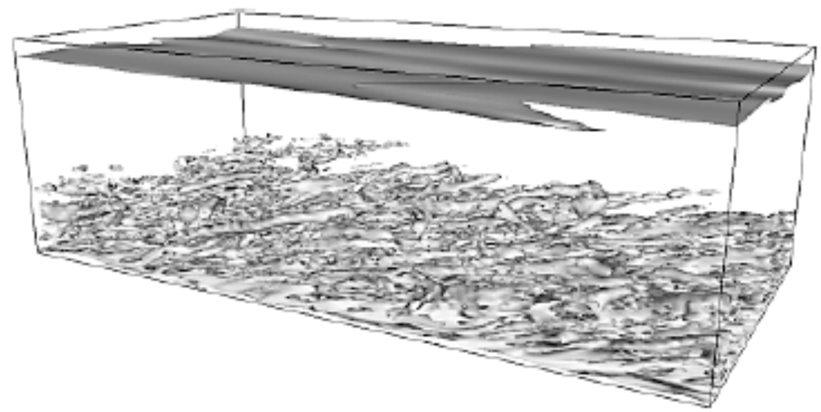
$Ro_g = \infty$, $Re = 14000$ (LES)



$Ro_g = 6$, $Re = 14000$ (LES)



$Ro_g = 18$, $Re = 14000$ (LES)

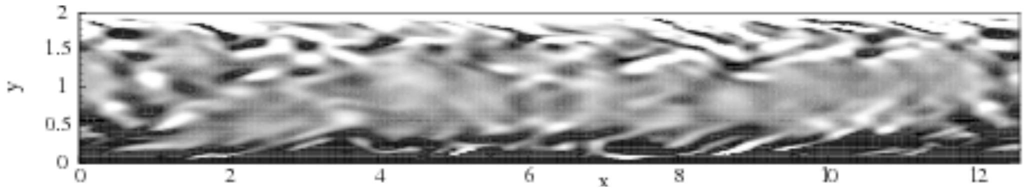


$Ro_g = 2$, $Re = 14000$ (LES)

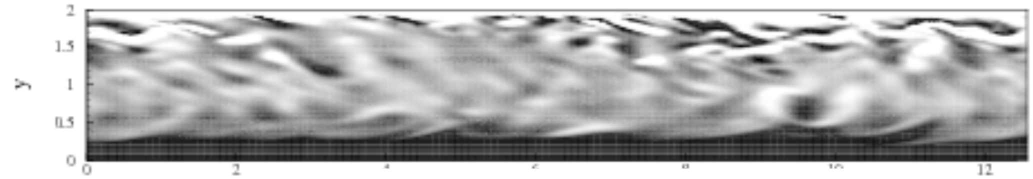
Anticlockwise rotation

Lambalais *(Theoretical and Comput. Fluid Dynam., 12, 1998)*

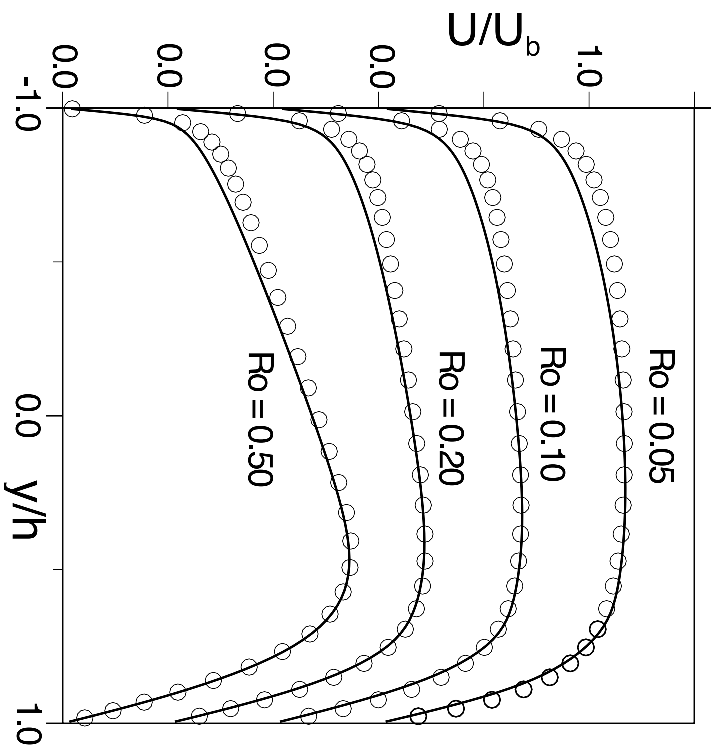
Velocity, stress

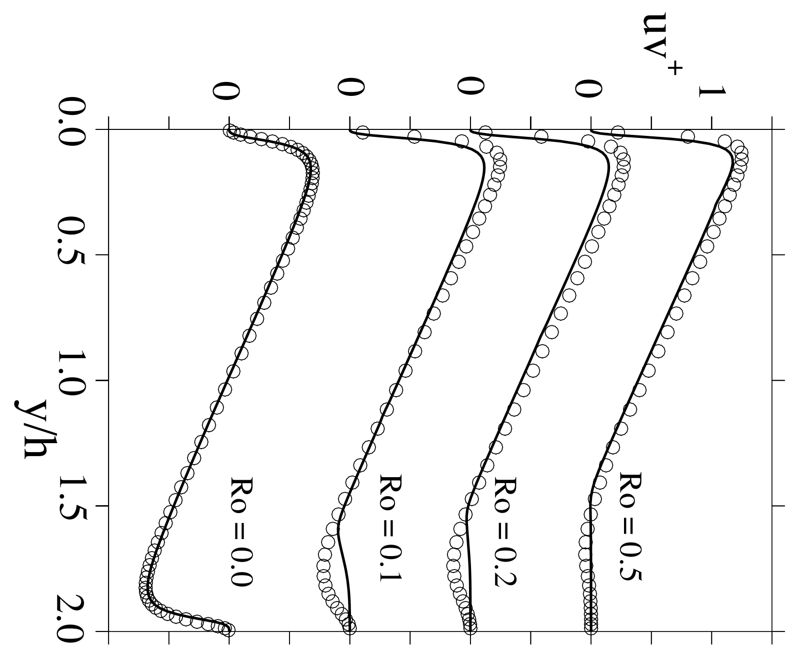


stationary

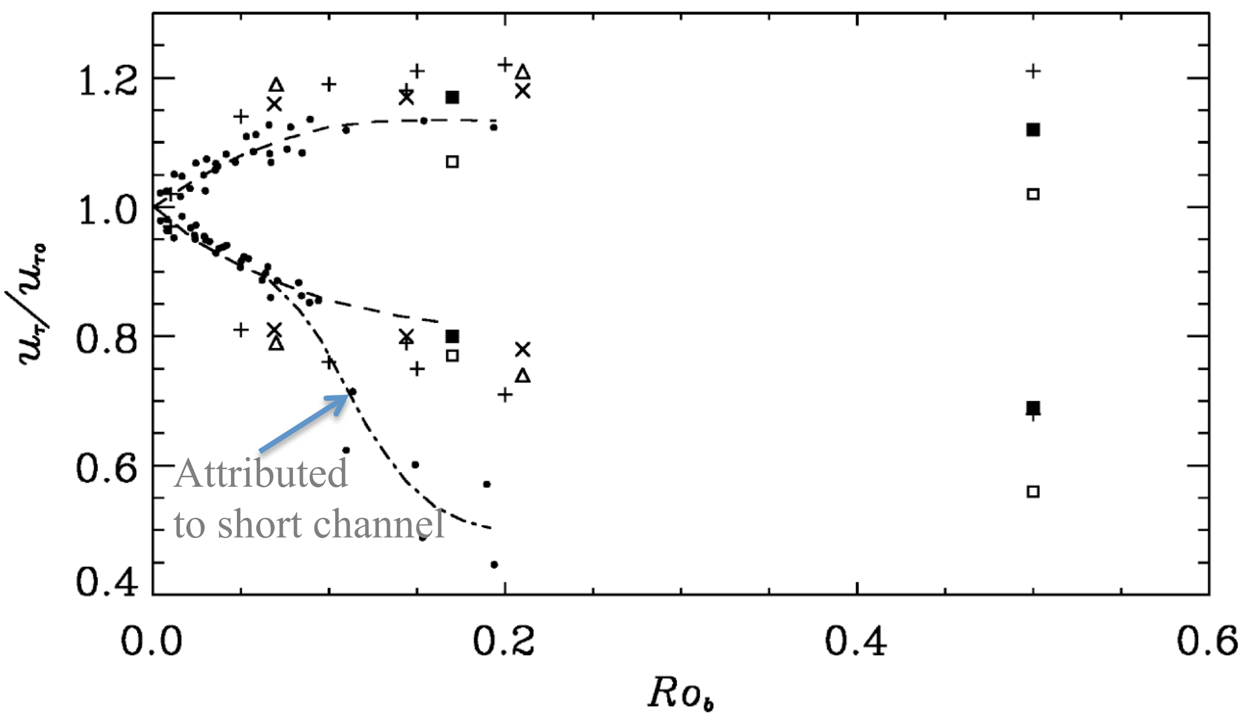


Clockwise rotation

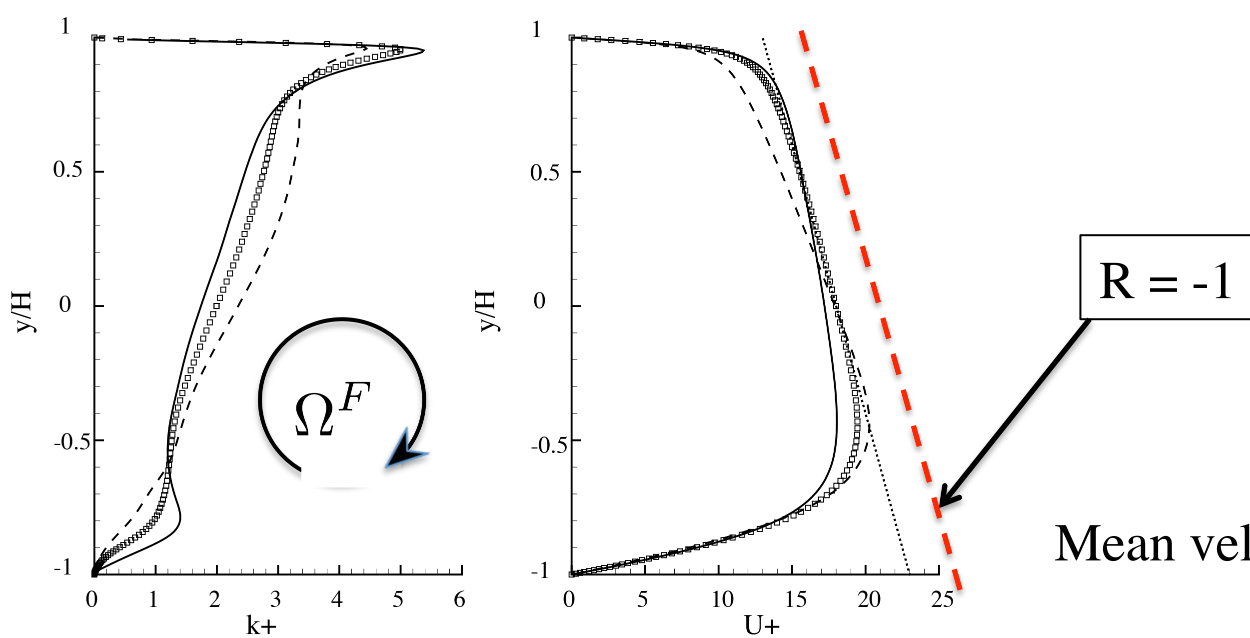




Wall shear stress



t. k. e



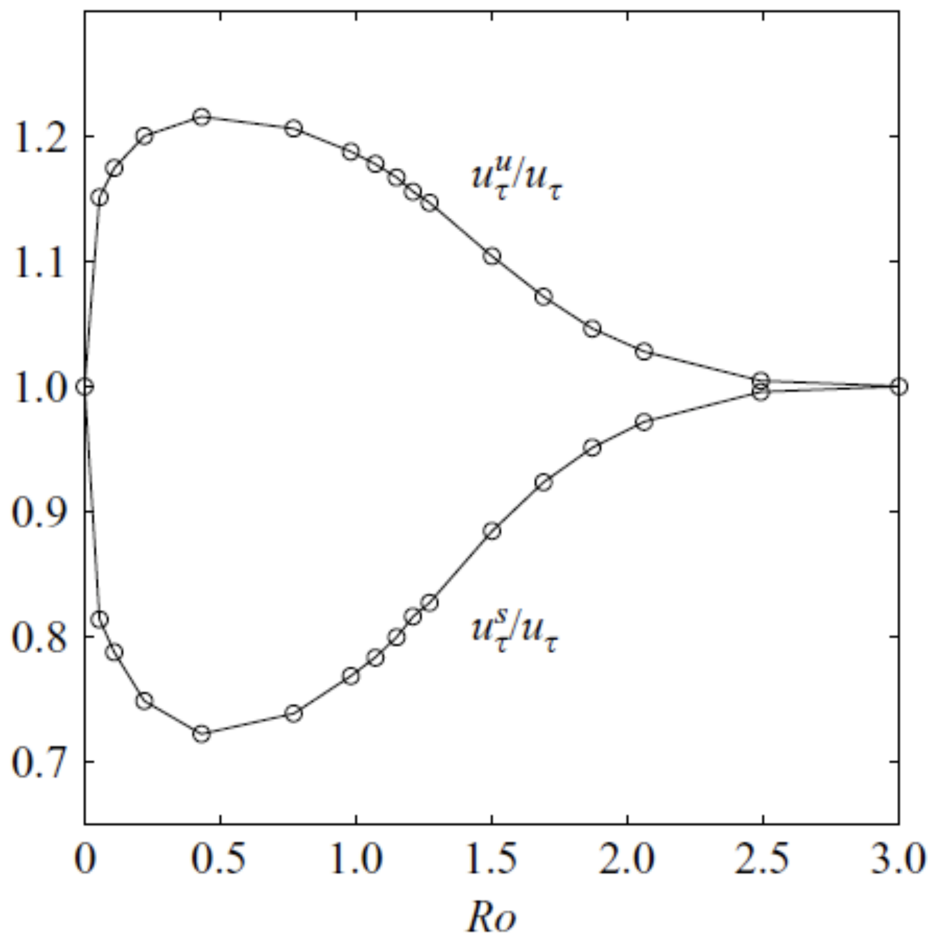
Mean velocity

Relaminarization at high rotation

Grundestam etal *(JFM 598,2008)* DNS

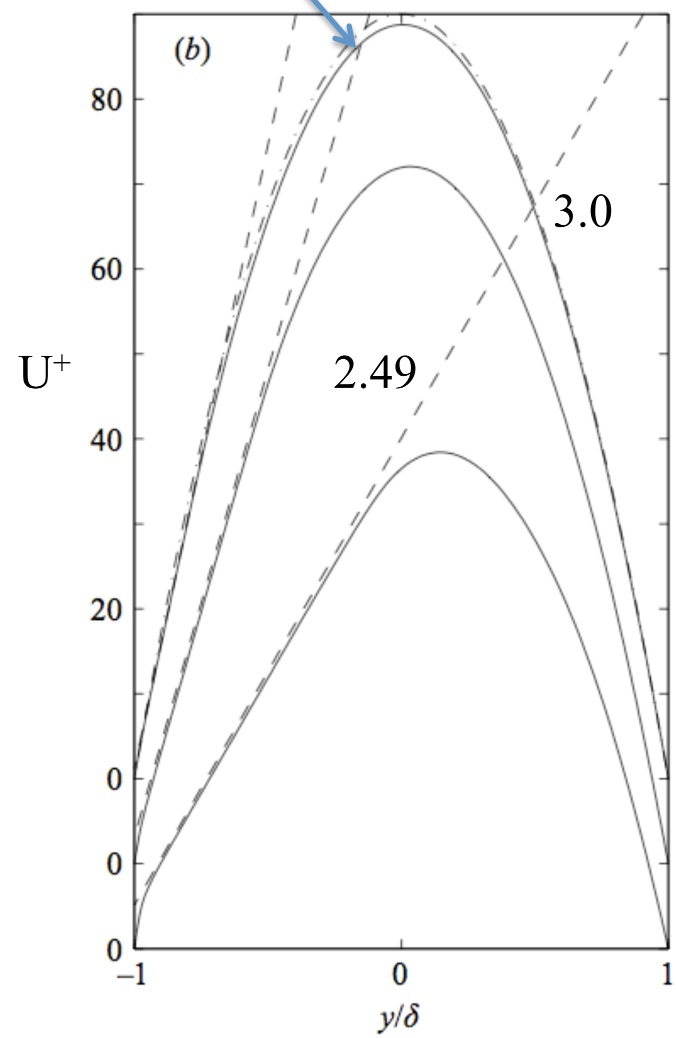
$Re_\tau = 180$

$Ro = 2\Omega^F H/U_{bulk}$



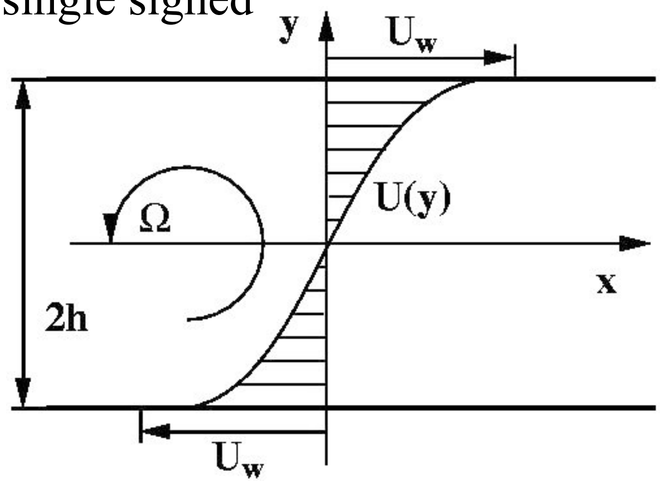
for laminar Poiseuille flow

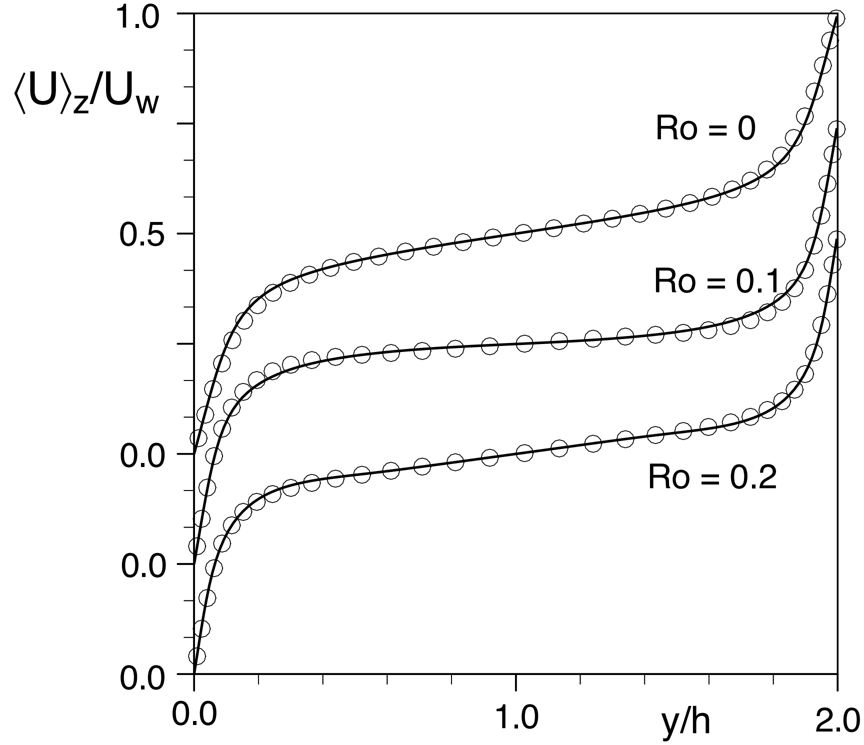
$U_{cl}^+ = Re_\tau/2 = 90$



Couette flow

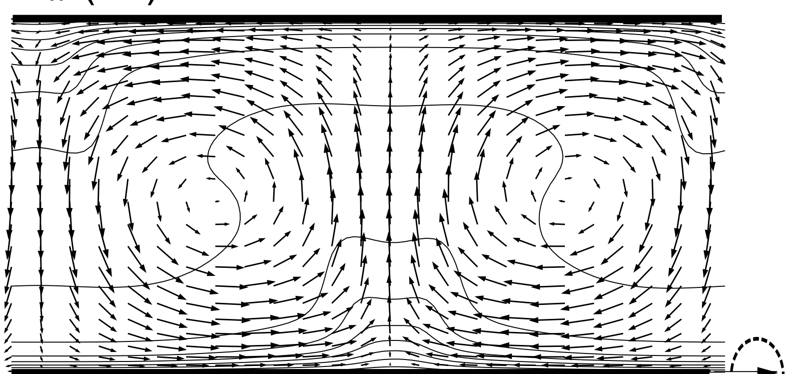
R is single signed





Streamwise rolls (Taylor-Goertler)

U_w (2h) = U



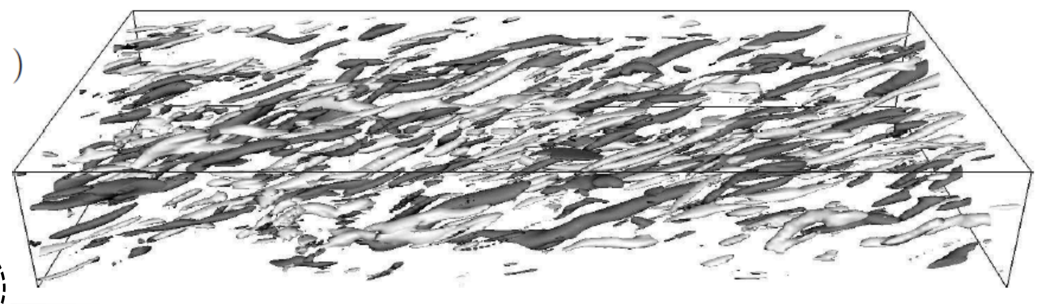
U(0) = 0

y

z

Ω

Average (RANS)



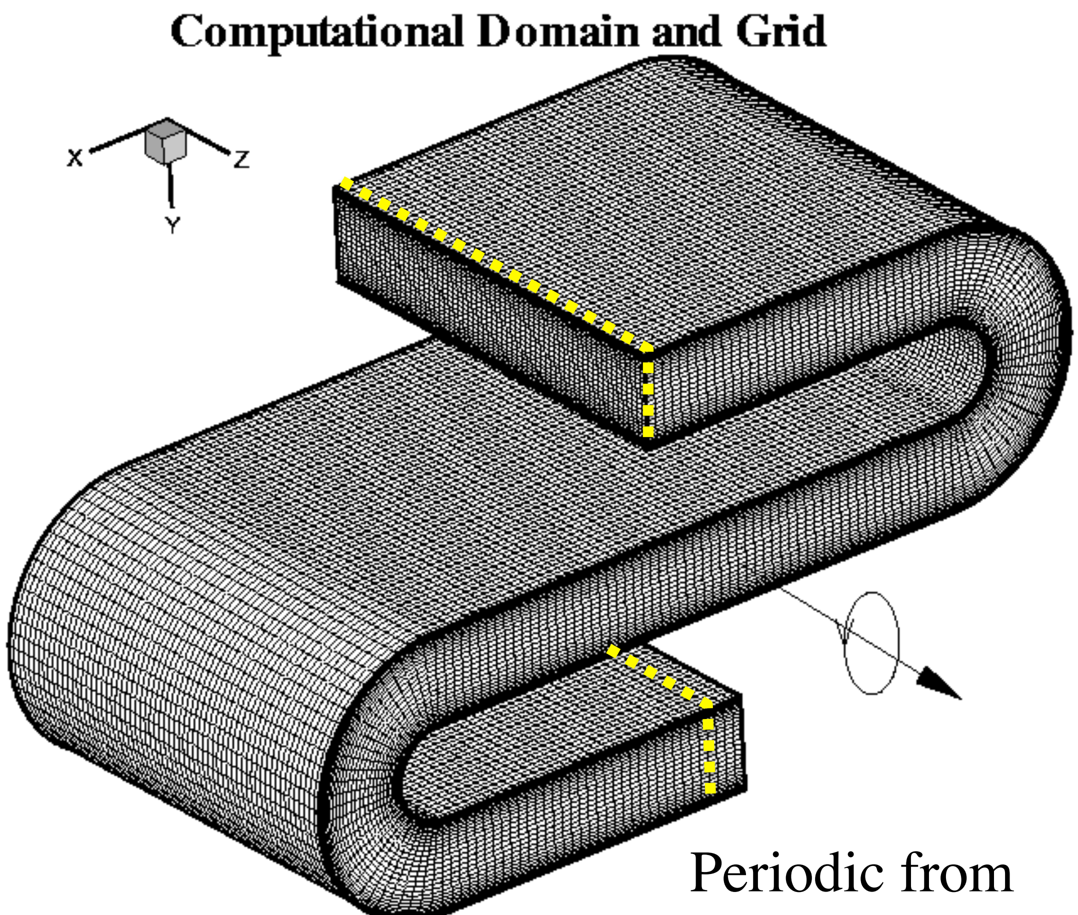
instantaneous

Andersson, Petersson, Bech

Bari & Andersson

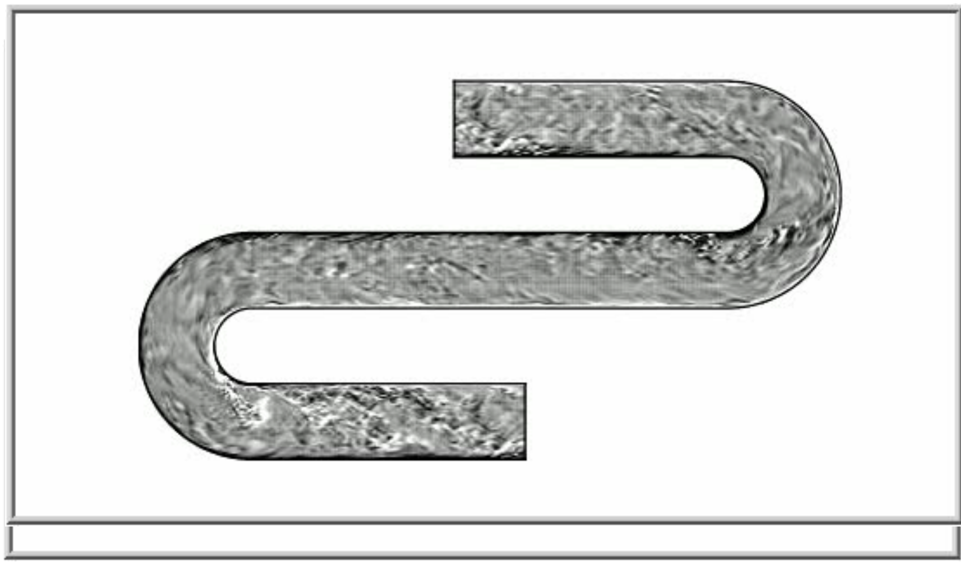
Serpentine channel

Laskowski and
Durbin *(Phys Fl. 19 2007)*

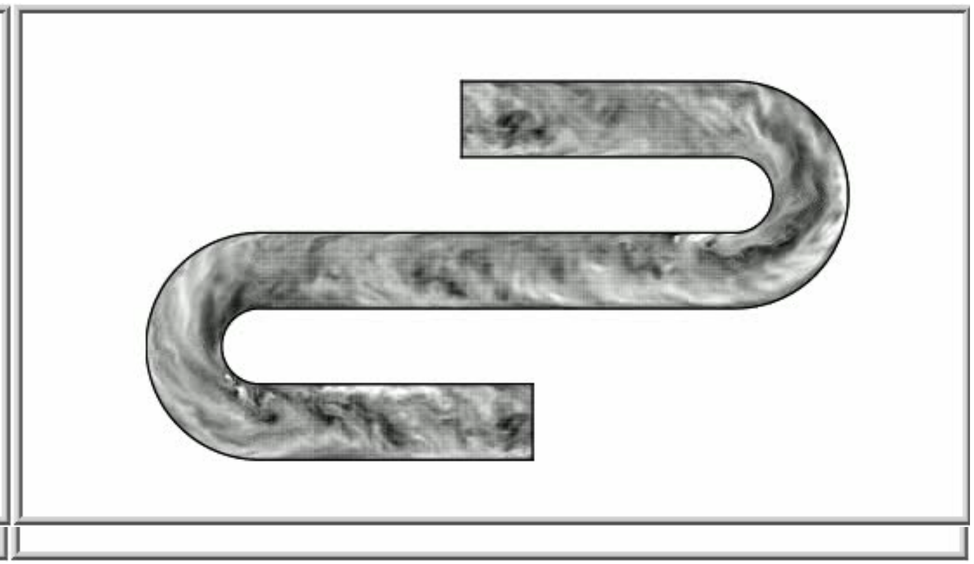


Periodic from
inlet to exit

Serpentine channel



Vorticity

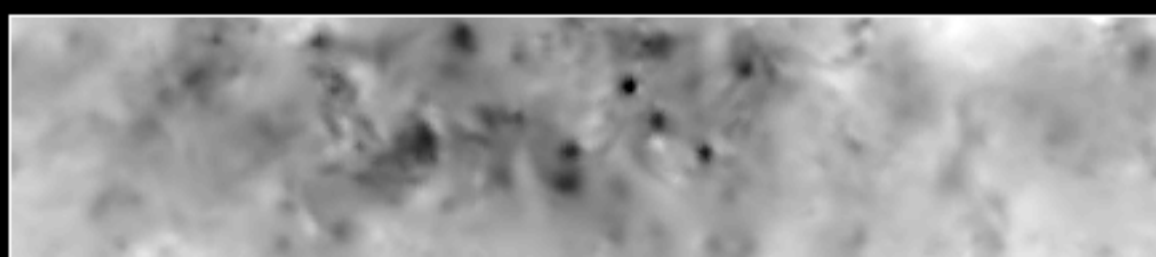


u-velocity

Cross-section

w

p



Analogy between rotation and curvature

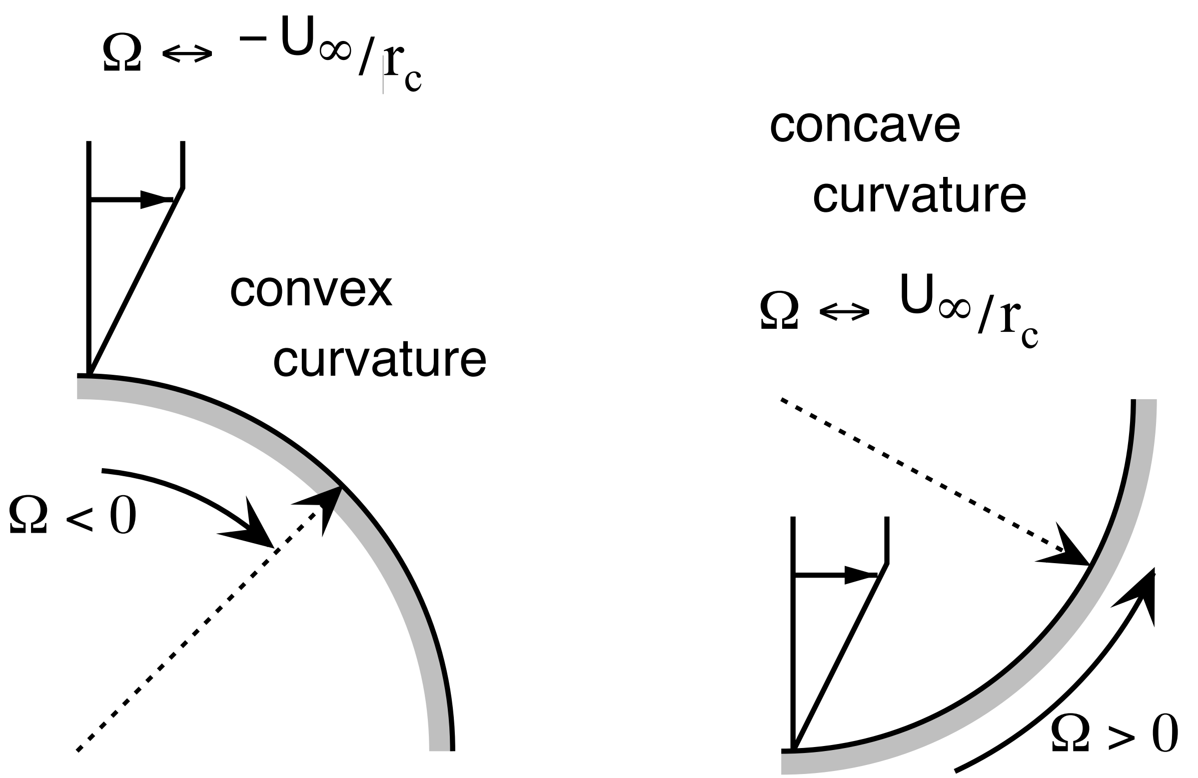
Analogy: $U/r_c \sim \Omega^F$ N.B. convex and concave curvature have opposite r_c

$$C = 2U/r_c S$$

The linearized equation for v^2 is $${}^1\!/_2 \frac{d\overline{v^2}}{d\tau} = \overline{uv}(R + C)$$

Curvature can enhance or counter rotation. In isolation it acts either with or against the shear.

Convex and concave curvature



Curvature with (R>0) and against (R<0) shear

Goertler vortices

Convex curvature

Are they significant
in turbulent flow?

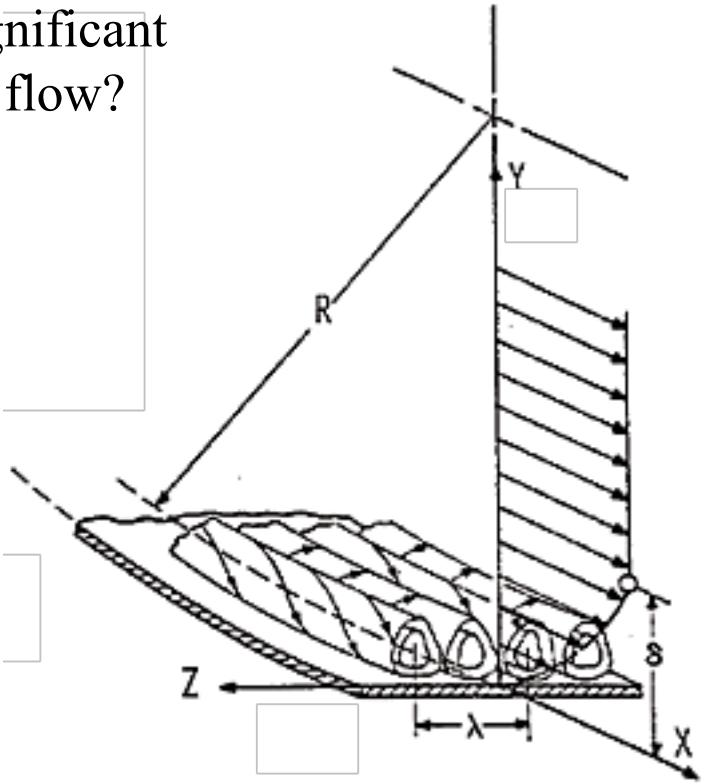
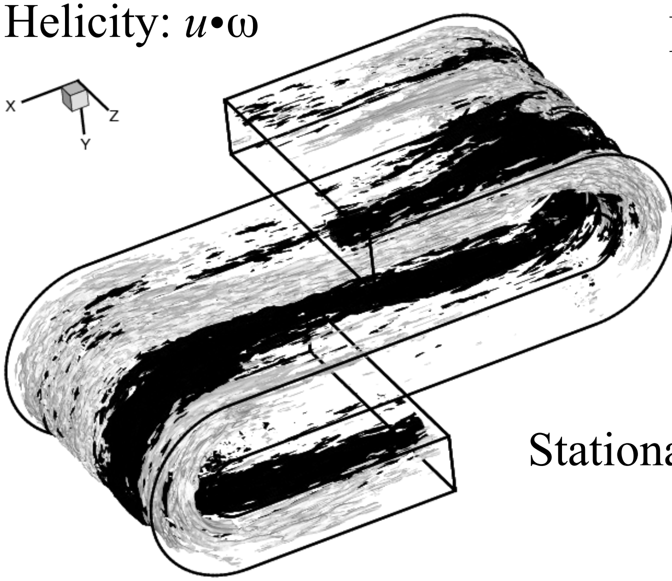


Figure 1. Görtler vortices over a concave wall.

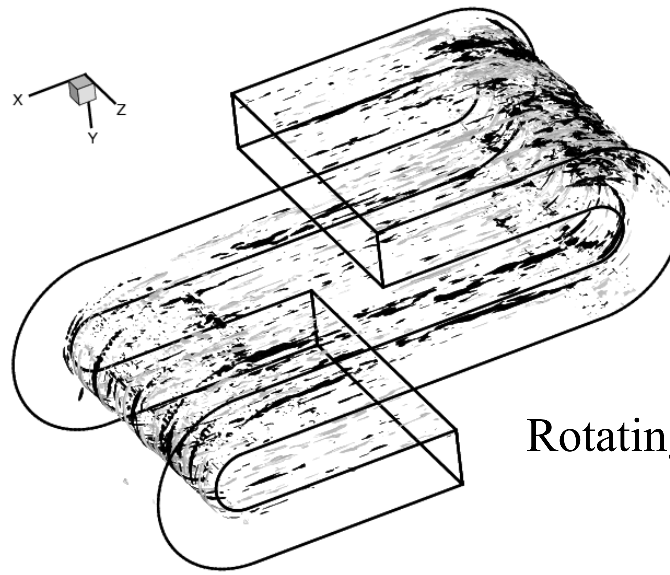
Persistent streamwise vortices?

Helicity: $u \cdot \omega$

Low pass filtered

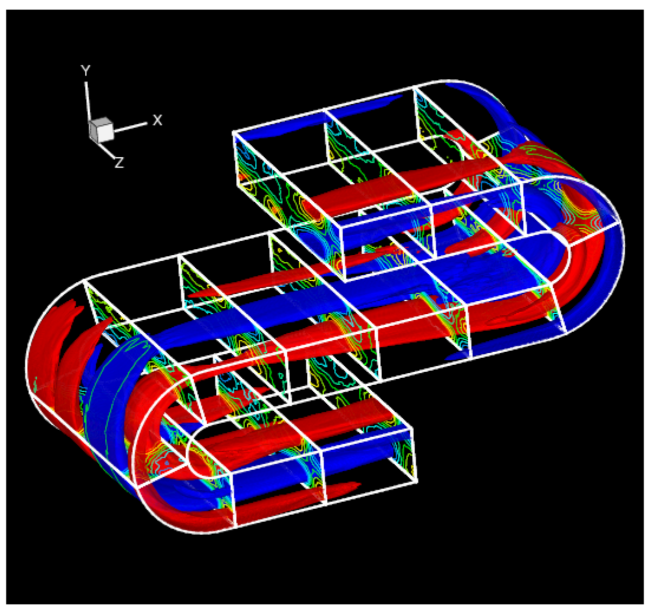


Stationary



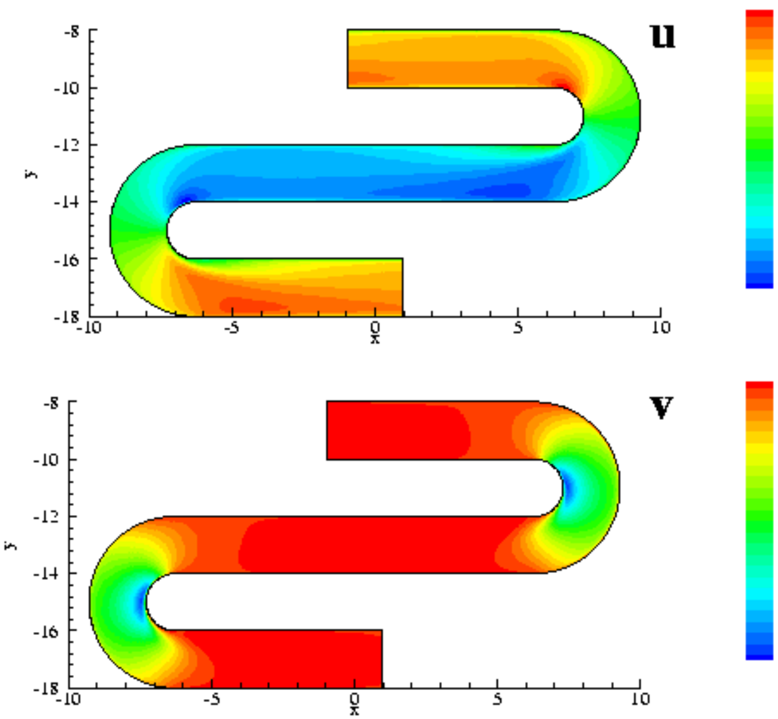
Rotating

w

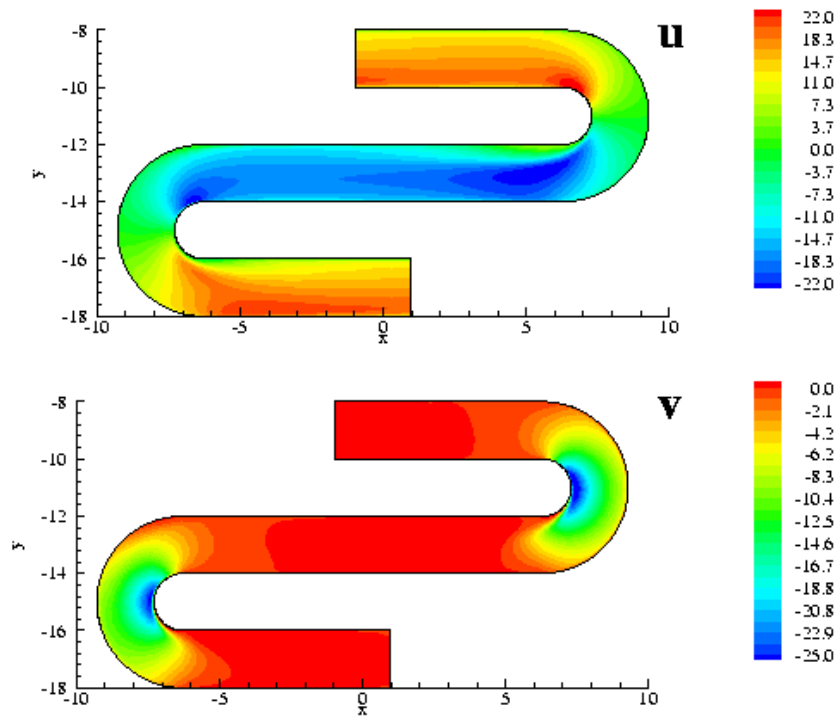


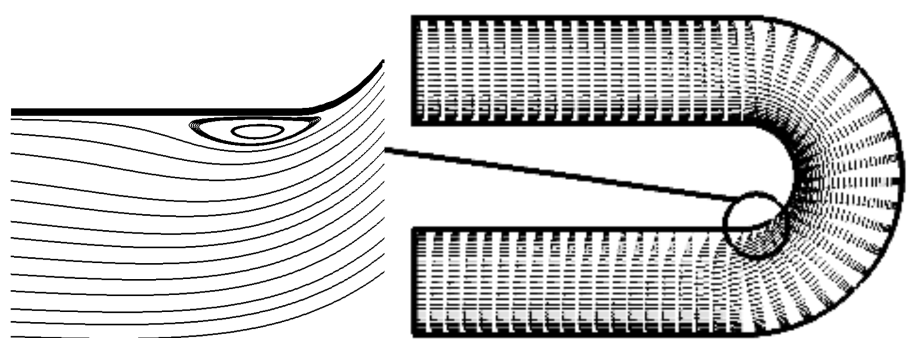
Overview serpentine passage

Average Velocity, Re_τ = 180, Ro = 0

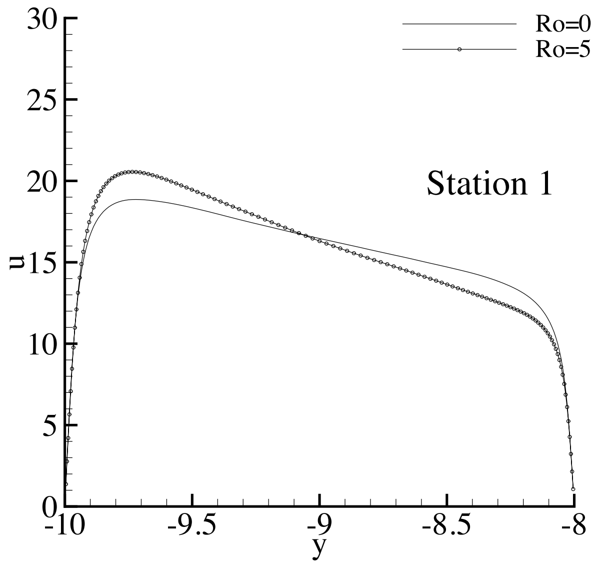


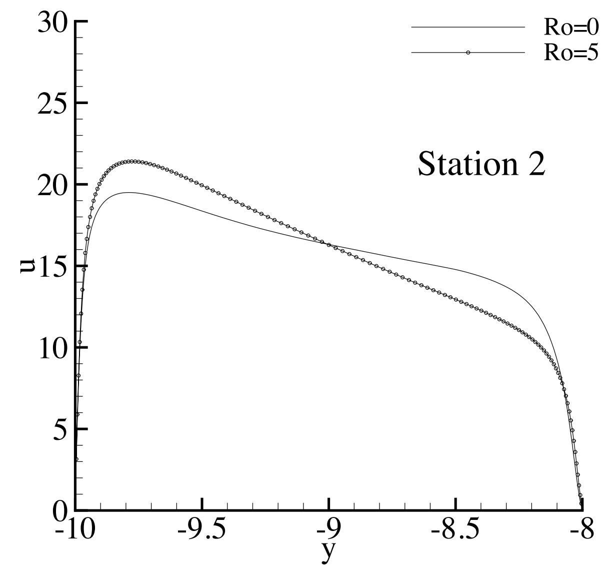
Average Velocity, Re_τ = 180, Ro = 5

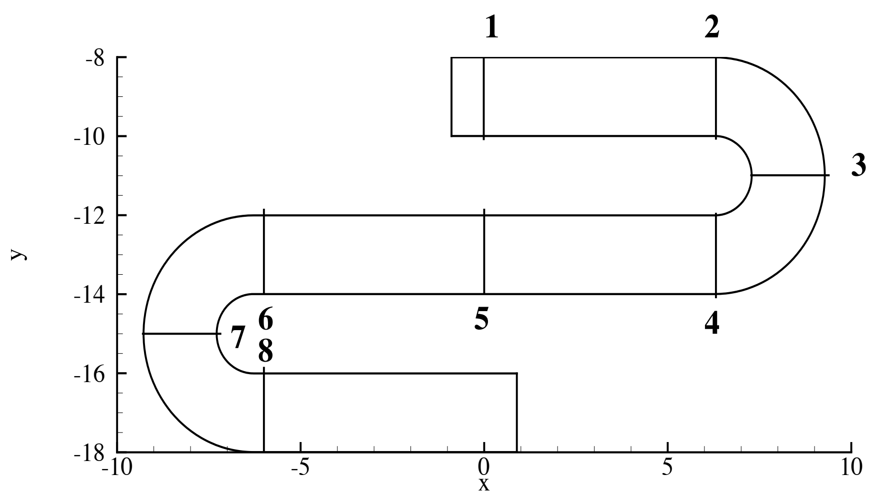


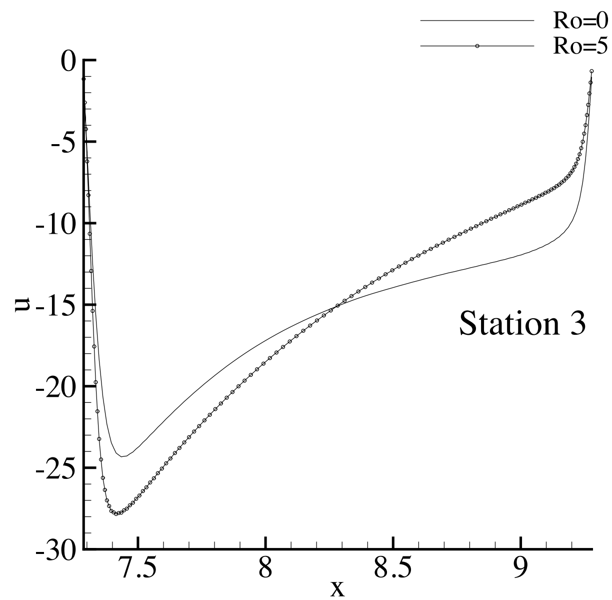


Mean profiles



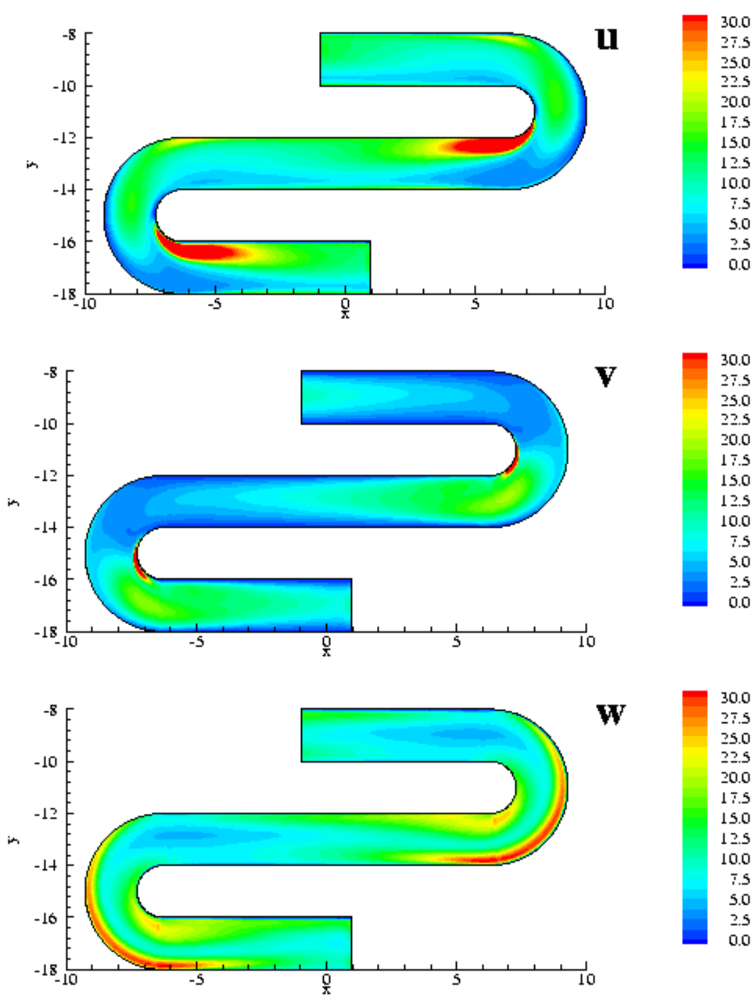




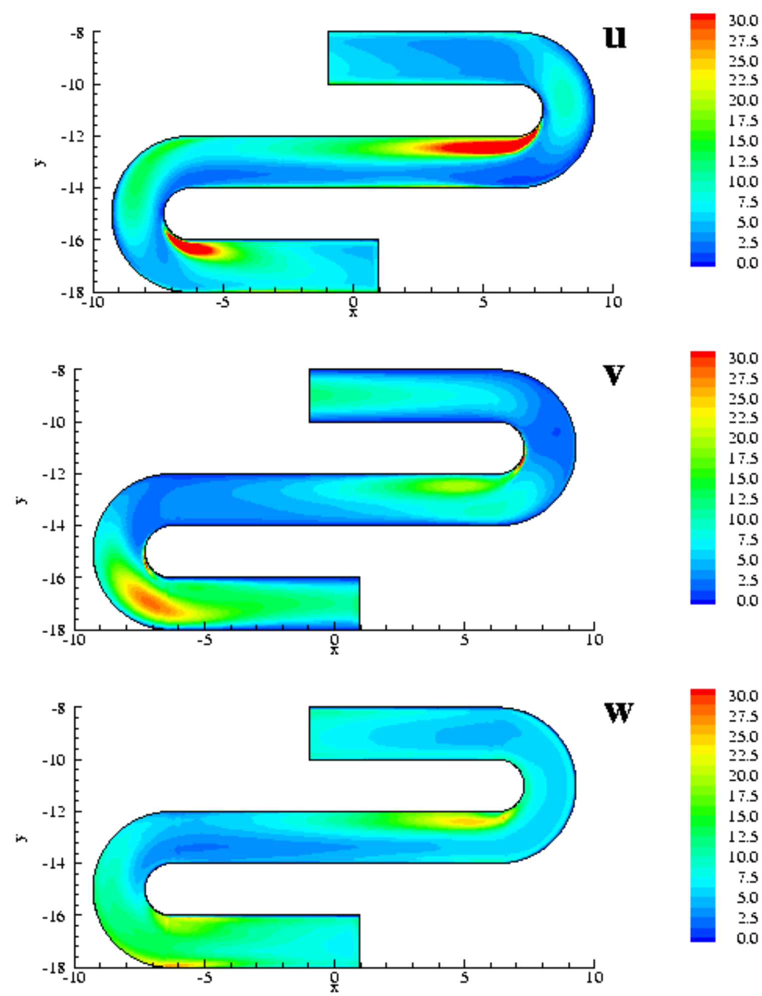


Overview fluctuations

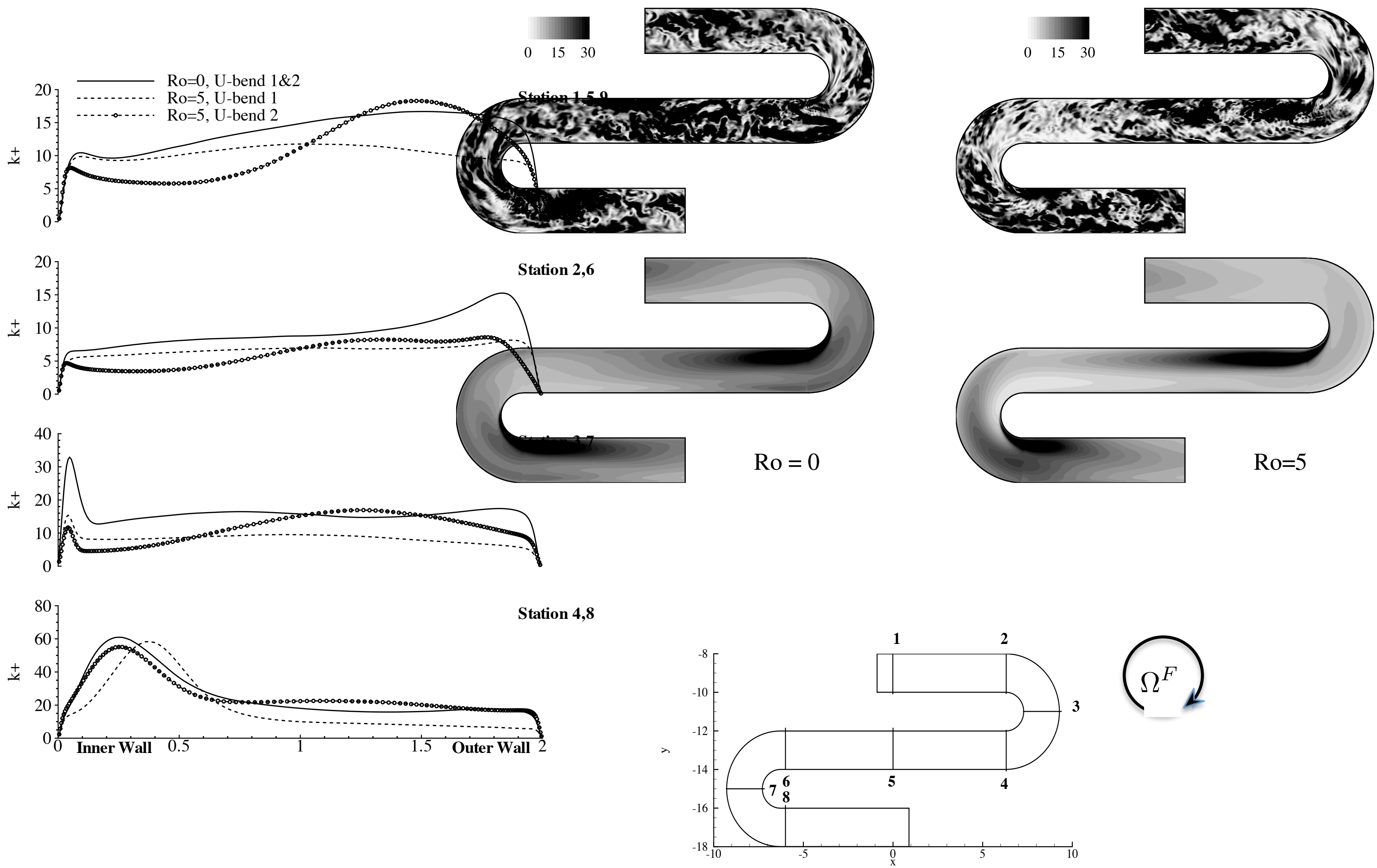
Variance of Velocity, Re_τ = 180, Ro = 0



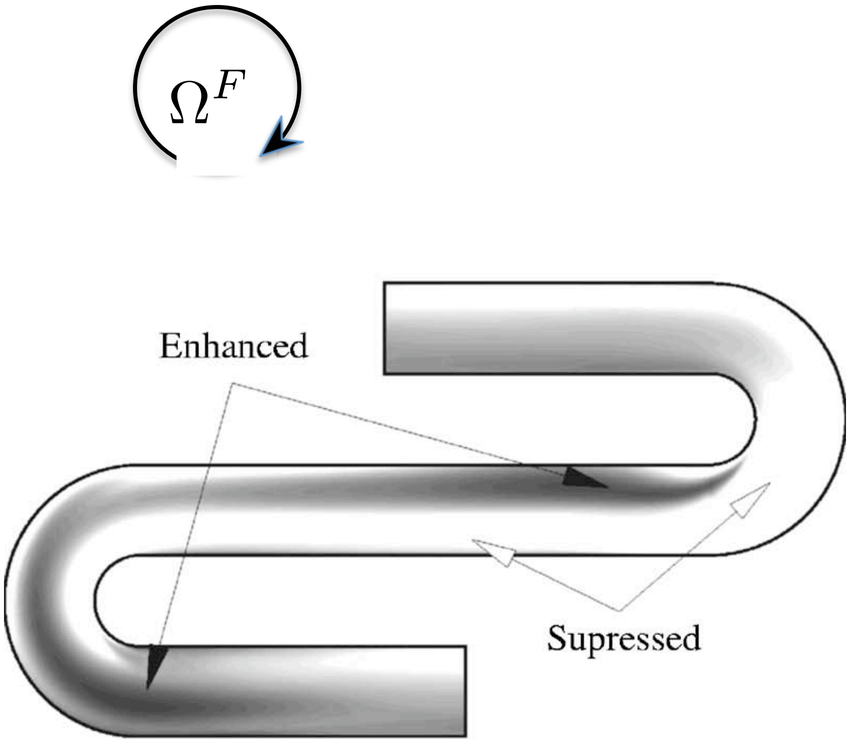
Variance of Velocity, Re_τ = 180, Ro = 5



Kinetic energy

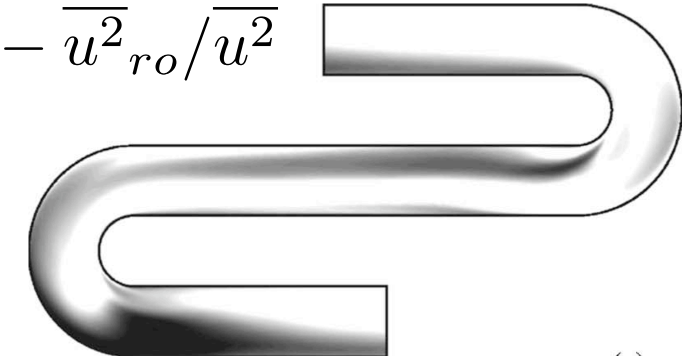


Enhancement and suppression of Reynolds stresses



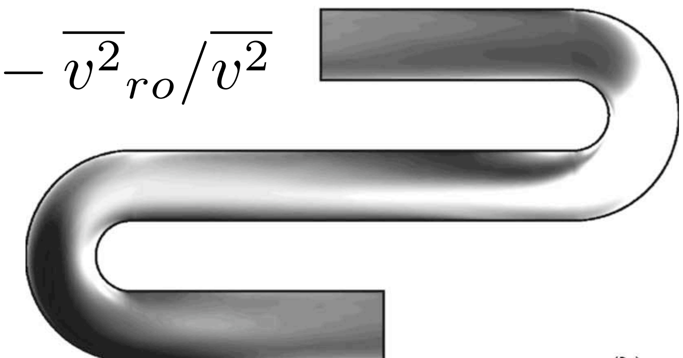
$$1 - k_{ro}/k$$

$$1 - \overline{u^2}_{ro}/\overline{u^2}$$



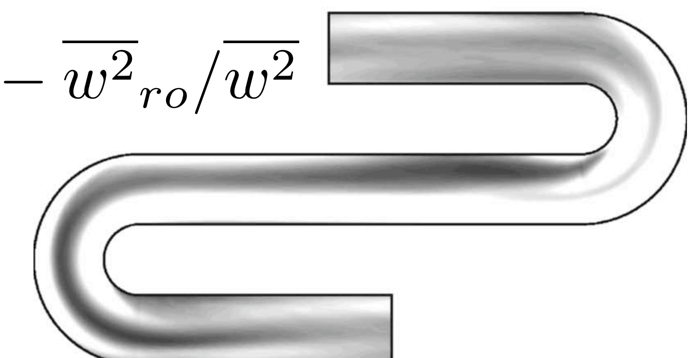
(a)

$$1 - \overline{v^2}_{ro}/\overline{v^2}$$



(b)

$$1 - \overline{w^2}_{ro}/\overline{w^2}$$



(c)

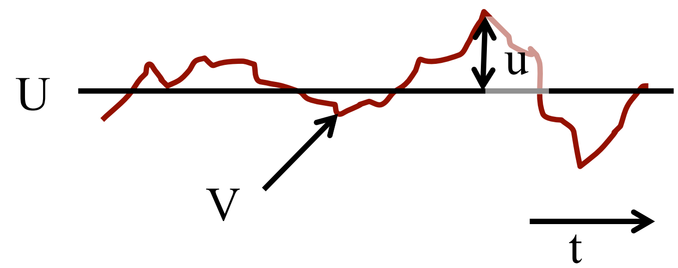
Summary, part I

- Rotation reduces the rate of decay of grid turbulence
- Rotation in the direction of shear is stabilizing
- Moderate rotation against the shear is destabilizing; larger rotation is stabilizing
- Curvature is analogous to rotation – to a large extent
- In non-homogeneous flow the *rotation number* varies with position and can change sign. The net effect is not entirely obvious

Part II. Single point closures

Reynolds decomposition

Total velocity (V) = Average (U) + Fluctuation(u)



U is the mean flow
u is the turbulence

Navier-Stokes
$$\frac{\partial V}{\partial t} + V \cdot \nabla V = -\nabla P + \nu \nabla^2 V$$

Let V=U+u, substitute and average: Reynolds Averaged N-S (RANS)

Equation of the mean flow
$$\partial_t U_i + U_j \partial_j U_i = -\frac{1}{\rho}\partial_i P + \nu \nabla^2 U_i \underbrace{-\boldsymbol{\partial_j \overline{u_j u_i}}}$$

Reynolds stress

Comment: eddy viscosity closure

Constitutive formula (mean flow closure):

$$-\overline{u_i u_j} = \nu_T S_{ij} - \frac{2}{3}\delta_{ij}$$

Model:

$$\nu_T = C_\mu k T; \quad T = 1/\omega,\ or, T = k/\varepsilon$$

Reynolds stress transport equation

Equation of the turbulent stress

$$\partial_t \overline{u_i u_j} + U_k \partial_k \overline{u_i u_j} = -\frac{1}{\rho} \underbrace{\left(\overline{u_j \partial_i p} + \overline{u_i \partial_j p}\right)}_{\text{redistribution}} - \underbrace{2\nu \overline{\partial_k u_i \partial_k u_j}}_{\text{dissipation}}$$
$$\underbrace{-\partial_k \overline{u_k u_i u_j}}_{\text{turbulent transport}} \underbrace{-\overline{u_j u_k} \partial_k U_i - \overline{u_i u_k} \partial_k U_j}_{\text{production}} + \nu \nabla^2 \overline{u_i u_j}.$$

These are unclosed equations: models are needed

The focus of second moment closure modeling is the redistribution tensor: make it a function of the Reynolds stress tensor

Rotation effects enter through production

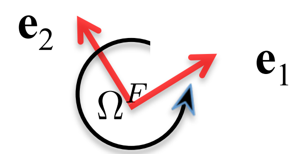
$$\mathcal{P}_{ij} = -\overline{u_j u_k} \partial_k U_i - \overline{u_i u_k} \partial_k U_j.$$

and convection

$$\partial_t \overline{u_i u_j} + U_k \partial_k \overline{u_i u_j}$$

Reynolds stress equations in rotating frame

If the unit directions rotate as



$$\boldsymbol{e}_1 = (\cos\Omega^F t, \sin\Omega^F t, 0), \quad \boldsymbol{e}_2 = (-\sin\Omega^F t, \cos\Omega^F t, 0)$$

then

$$d_t\boldsymbol{e}_1 = \Omega^F(-\sin\Omega^F t, \cos\Omega^F t, 0) = \Omega^F\boldsymbol{e}_2, \quad d_t\boldsymbol{e}_2 = -\Omega^F\boldsymbol{e}_1$$

and

$$d_t(u_i\boldsymbol{e}_i) = \boldsymbol{e}_i d_t u_i + \boldsymbol{e}_i\epsilon_{ijk}\Omega^F_j u_k$$

Reynolds stress equations are

$$d_t\overline{u_iu_j} + \overline{u_iu_l}\varepsilon_{jkl}\Omega^F_k + \overline{u_ju_l}\varepsilon_{ikl}\Omega^F_k = P_{ij} - {}^2\!/_3\,\delta_{ij}\varepsilon + \wp_{ij}$$

Where is the 2 Ω?

Unclosed pressure-strain

Production tensor

$$P_{ij} = -\overline{u_i u_k}\partial_j U_k - \overline{u_j u_k}\partial_i U_k$$

$$\partial_j U_k = \underbrace{\frac{1}{2}[\partial_j U_k + \partial_k U_j]}_{S_{jk}} + \underbrace{\frac{1}{2}[\partial_j U_k - \partial_k U_j]}_{\Omega_{jk}}$$

In terms of rate of strain and rate of rotation

$$P_{ij} = -\overline{u_i u_k}(S_{kj} + \Omega_{kj}) - \overline{u_j u_k}(S_{ki} + \Omega_{ki})$$

The apparently missing factor of 2: $\underbrace{\partial_k U_j^A}_{absolute} = \underbrace{\partial_k U_j^F}_{relative} + \varepsilon_{jkl}\Omega_l^F$

Hence $P_{ij}^A = P_{ij}^F - \overline{u_i u_l}\varepsilon_{jkl}\Omega_k^F - \overline{u_j u_l}\varepsilon_{ikl}\Omega_k^F$

In closure modeling it is necessary to distinguish the production tensor. Production is frame independent:

$$P_{ij} = -\overline{u_i u_k}(S_{kj} + \Omega_{kj}^A) - \overline{u_j u_k}(S_{ki} + \Omega_{ki}^A)$$

$$\underbrace{\Omega_{ij}^A}_{absolute\ rotation} = \underbrace{\Omega_{ij}}_{vorticity} + \underbrace{\Omega_k^F \varepsilon_{ijk}}_{frame\ rotation}$$

Reynolds stress depends on both Ω^F and Ω^A. The former comes from evolution; the latter from production.

The notion that constitutive formulas depend only on absolute rotation is not right for turbulence.

Rotation effect via SMC

For IP model:

Homogeneous shear

$$d_t\overline{u^2} - 2\overline{uv}\Omega^F = {}^4\!/_5\,\Omega^F\overline{uv} - {}^6\!/_5\,\overline{uv}\mathcal{S}\ \ldots$$

$$d_t\overline{v^2} + 2\overline{uv}\Omega^F = -\,{}^4\!/_5\,\Omega^F\overline{uv} - {}^2\!/_5\,\overline{uv}\mathcal{S}\ldots$$

$$d_t\overline{uv} = {}^2\!/_5\,\Omega^F(\overline{v^2} - \overline{u^2}) - {}^2\!/_5\,\overline{v^2}\mathcal{S}\ldots$$

With $R = -2\Omega^F/\mathcal{S}$ and $\tau = \mathcal{S}t$:

$$d_\tau\overline{u^2} = -({}^7\!/_5\,R + {}^6\!/_5)\overline{uv}\ \ldots$$

$$d_\tau\overline{v^2} = ({}^7\!/_5\,R - {}^2\!/_5)\overline{uv}\ldots$$

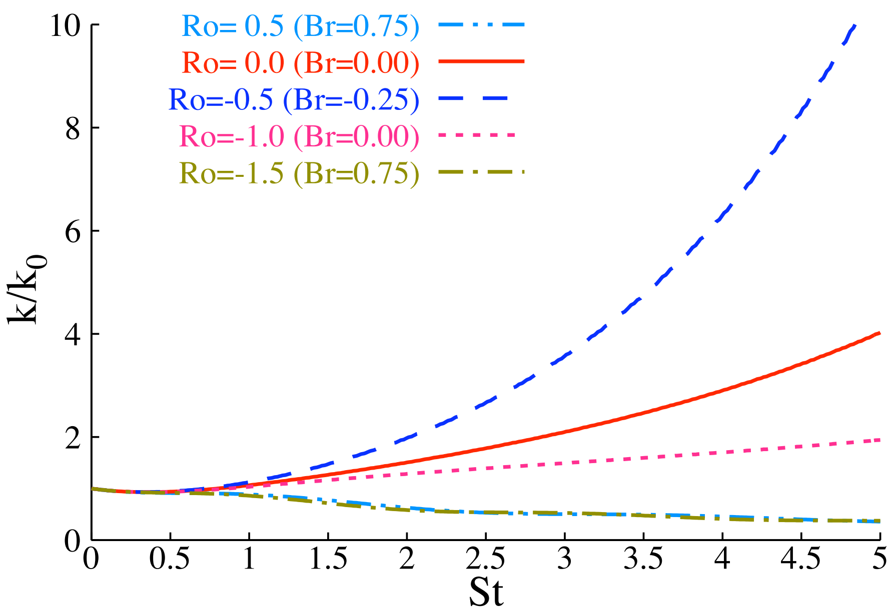
$$d_\tau\overline{uv} = {}^1\!/_5\,R(\overline{v^2} - \overline{u^2}) - {}^2\!/_5\,\overline{v^2}\ldots$$

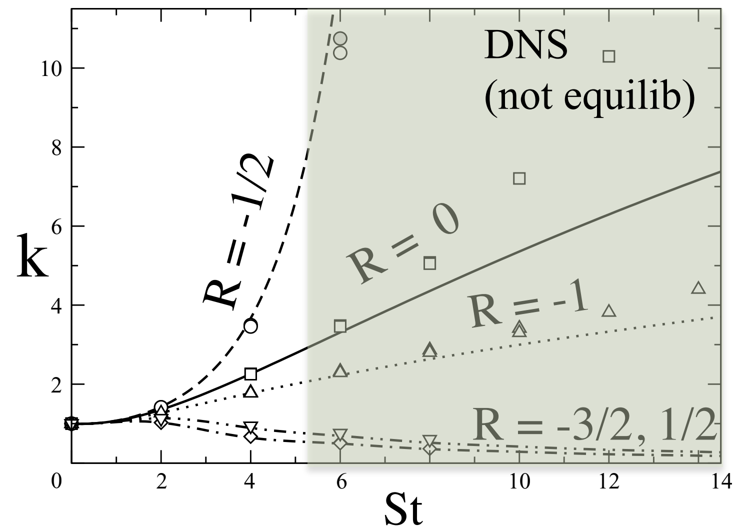
Note $\overline{uv} < 0$ in shear flow. If $R > 2/7$, $\overline{v^2}$ will be suppressed.

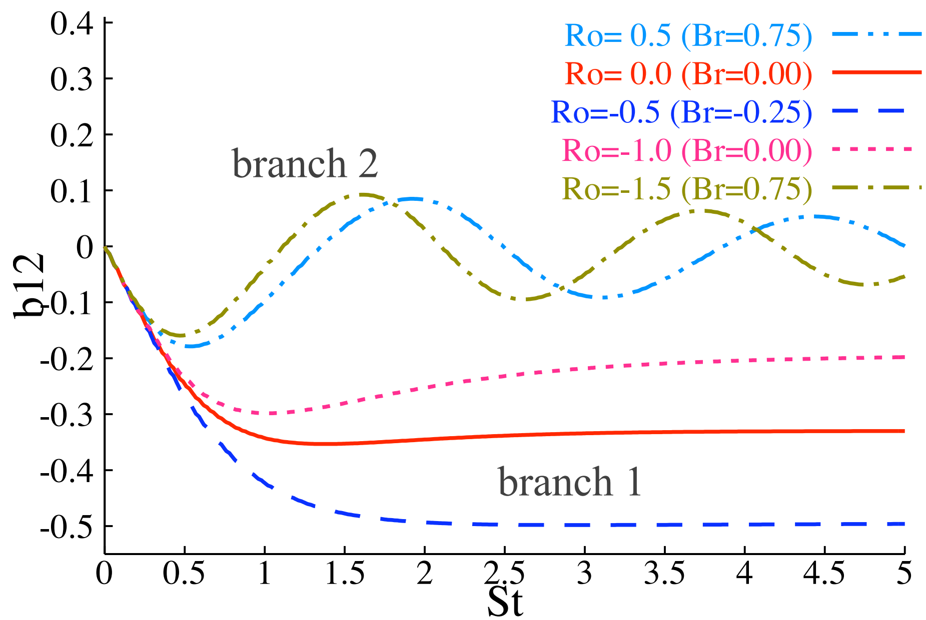
Reynolds stress equations capture the inviscid mechanism

Second moment closure

Rotating homogeneous shear







Bradshaw parameter

$Br = R(R+1) \geq -0.25$

Background: Equilibria of k-ε equations

Model in homogeneous shear

$$\begin{aligned} \frac{dk}{dt} &= \mathcal{P} - \varepsilon \\ \frac{d\varepsilon}{dt} &= \frac{C_{\varepsilon 1}\mathcal{P} - C_{\varepsilon 2}\varepsilon}{T} \end{aligned}$$

$$\mathcal{P} = -\overline{u_i u_j} S_{ij}$$

With eddy viscosity

$$\mathcal{P} = 2\nu_T |S|^2; \qquad \nu_T = C_\mu \frac{k}{\varepsilon}$$

N.B. unaffected by rotation

Moving equilibrium: k grows, but k/ε and $\mathcal{P}/\varepsilon$ reach constant levels

Approaches to 2-equation modeling

Pragmatic motivation: this is the type of model used in turbomachinery analysis and design

Basic concept: rotation can alter growth rate and can stabilize shear flow turbulence: how can this be incorporated?

At 2-equation level it corresponds to dependence of

production/dissipation : P/ε

on rotation

Do analysis to understand how models work:

Moving equilibrium

$$\frac{d}{dt}\left(\frac{\varepsilon}{k}\right) = \left(\frac{\varepsilon}{k}\right)^2 \left[(C_{\varepsilon 1} - 1)\frac{\mathcal{P}}{\varepsilon} - (C_{\varepsilon 2} - 1)\right] \to 0$$

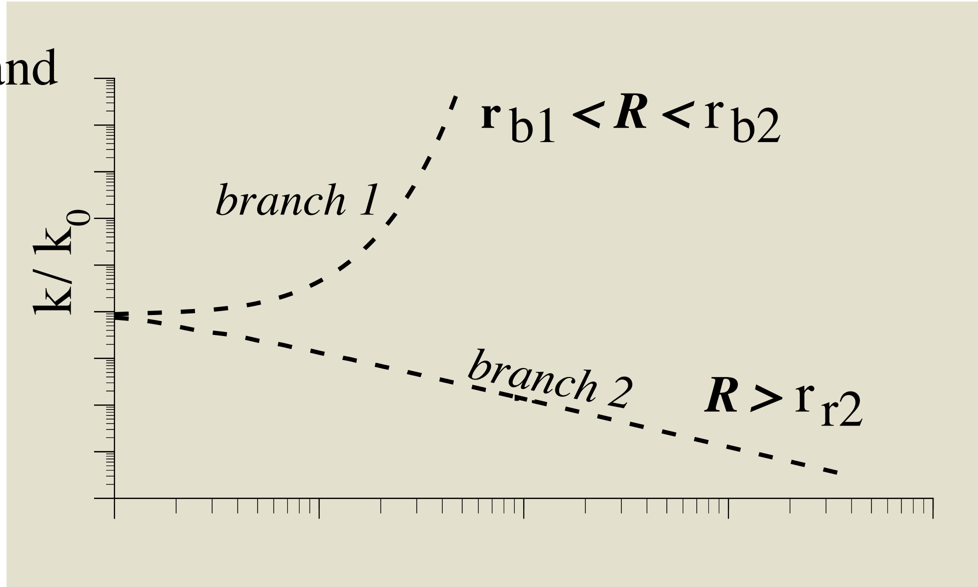
The 2 solutions are

$$\text{branch 1}: \quad \frac{\mathcal{P}}{\varepsilon} = \frac{C_{\varepsilon 2} - 1}{C_{\varepsilon 1} - 1} = \frac{2C_\mu |S|^2 k^2}{\varepsilon^2}$$

$$\text{branch 2}: \quad \frac{\varepsilon}{k} = 0$$

Roughly, these are growing (healthy) and decaying (unhealthy) states. Valid for Reynolds stress models if

$$\mathcal{P} = -\overline{u_i u_j} S_{ij}$$



Branch 1

$$k = k_\infty e^{\lambda t}, \quad \varepsilon = \varepsilon_\infty e^{\lambda t}$$

where

$$\lambda = \frac{C_{\varepsilon 2} - C_{\varepsilon 1}}{C_{\varepsilon 1} - 1} \left(\frac{\varepsilon}{k}\right)_\infty .$$

Finally

$$\left(\frac{\varepsilon}{k}\right)_\infty = \sqrt{2C_\mu |S|^2} \sqrt{\frac{C_{\varepsilon 1} - 1}{C_{\varepsilon 2} - 1}}$$

and

$$\lambda = \frac{C_{\varepsilon 2} - C_{\varepsilon 1}}{\sqrt{(C_{\varepsilon 1} - 1)(C_{\varepsilon 2} - 1)}} \sqrt{2C_\mu |S|^2}$$

Branch 2

$$k = A_\infty t^{-m}, \quad \varepsilon = B_\infty t^{-m-1}$$

N.B. $\varepsilon/k \propto 1/t$ as $t \to \infty$.

$$m = \frac{1 - \mathcal{P}/\varepsilon}{(C_{\varepsilon 2} - 1) - \mathcal{P}/\varepsilon(C_{\varepsilon 1} - 1)}.$$

If $\mathcal{P} < \varepsilon$ then $m > 0$ and turbulent energy decays

How can equilibrium analysis be used to develop models?

Modified coefficients

$C_{\varepsilon 1},\ C_{\varepsilon 2},\ C_\mu$

Recall the Bradshaw parameter from stability theory

$$Br = R(R+1)$$

Might parameterize rotation effects by functions of *Br*

$$C_{\varepsilon 1}(Br),\ C_{\varepsilon 2}(Br),\ C_\mu(Br)$$

$Br \geq -1/4$ and $Br < 0$ is exponentially unstable range; but algebraic growth occurs at $Br = 0$.

Analogue to instability: $\mathcal{P}/\varepsilon > 1$. Equilibrium solution

$$\frac{\mathcal{P}}{\varepsilon} = \frac{C_{\varepsilon 2} - 1}{C_{\varepsilon 1} - 1} \qquad \text{Branch 1}$$

provides connection to parameters. Introduce critical Bradshaw number, and parametric dependence:

$$1 = \frac{C_{\varepsilon 2}(Br_{crit}) - 1}{C_{\varepsilon 1}(Br_{crit}) - 1} \implies C_{\varepsilon 2}(Br_{crit}) = C_{\varepsilon 1}(Br_{crit})$$

Standard values are $C_{\varepsilon 1} = 1.44$,$C_{\varepsilon 2} = 1.92$. An early propsal: $C_{\varepsilon 2} = C^0_{\varepsilon 2}(1 - C_{sc}Br)$ with $C_{sc} \sim 2.5$. Then

$$Br_{crit} = \frac{C^0_{\varepsilon 2} - C_{\varepsilon 1}}{C^0_{\varepsilon 2} C_{sc}} = 0.1$$

Hellsten — translated from $k - \omega$ — is $(C_{\varepsilon 2} = C_{\omega 2} + 1)$

$$C_{\varepsilon 2} = \frac{C^0_{\varepsilon 2} + C_{sc} Br}{1 + C_{sc} Br}.$$

with $C_{sc} = 3.6$. So

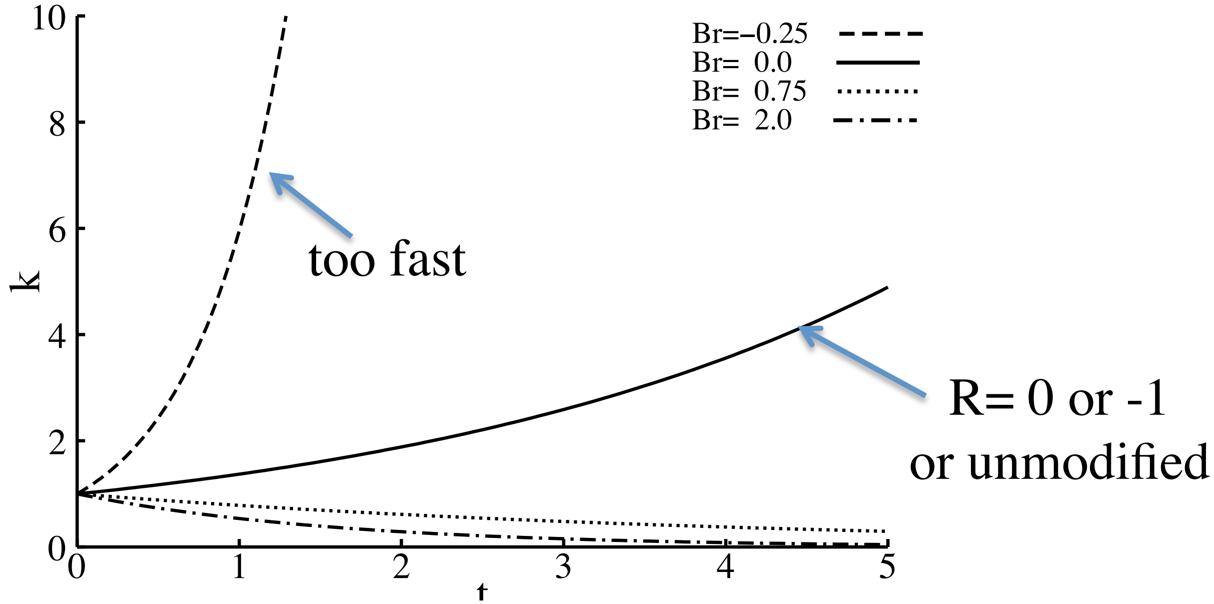
$$Br_{crit} = \frac{C^0_{\varepsilon 2} - C_{\varepsilon 1}}{(C_{\varepsilon 1} - 1) C_{sc}} = \frac{12}{11 C_{sc}} = 0.3$$

Corresponding range of rotation numbers (i.e. $R(1 + R) = 0.3$)

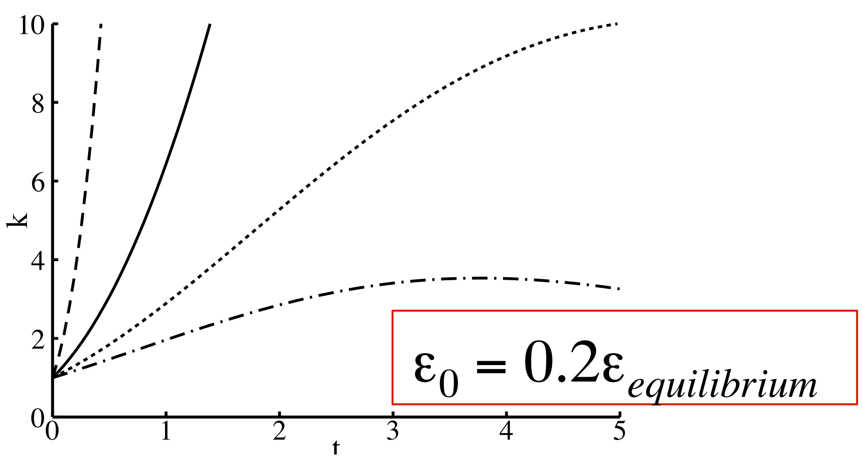
$$-1.24 < R < 0.24$$

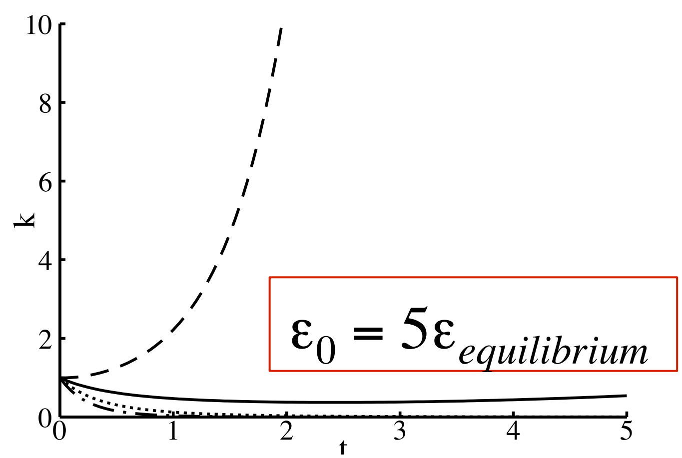
Rotating, homogeneous shear

I.c.: equilibrium ratio



Warning (c.f. LES, DNS data)





Comment on parameterization

To avoid singularity at $S = 0$, $C_{\varepsilon 2} = C^0_{\varepsilon 2}(1 - C_{sc} Br\,(|\boldsymbol{S}|k/\varepsilon)^2)$ with $C_{sc} = 0.4$ has been suggested (HBR model). Then

$$Br_{crit} = \frac{C^0_{\varepsilon 2} - C_{\varepsilon 1}}{A(C_{\varepsilon 1} - 1)} = 0.026.; \quad A = \frac{C_{sc} C^0_{\varepsilon 2}}{2C_\mu (C_{\varepsilon 1} - 1)}$$

However Sk/ε is imaginary for $C_{\varepsilon 2} < 1$ so this model is ill posed. In fact $C_{\varepsilon 2} < 0$ for $Br < 1/A = -0.103$ (Cazalbou)

Various definitions

$$Br = \frac{2\Omega^F(2\Omega^F - \partial_y U)}{\partial_y U^2}$$

$$\widetilde{Br} = \frac{2\Omega^F(2\Omega^F - \partial_y U)}{(\varepsilon/k)^2}$$

Consider rotor-stator: what is Ω^F, 0 or rotor velocity? Or, rotor computed in rotating (flow is steady) or inertial (flow is time-dependent) frame.

How to define ‘frame rotation’? Convective derivative of rate of strain (Spalart-Shur); if $\mathbf{e}^{(i)}$ are rate of strain eigenvectors

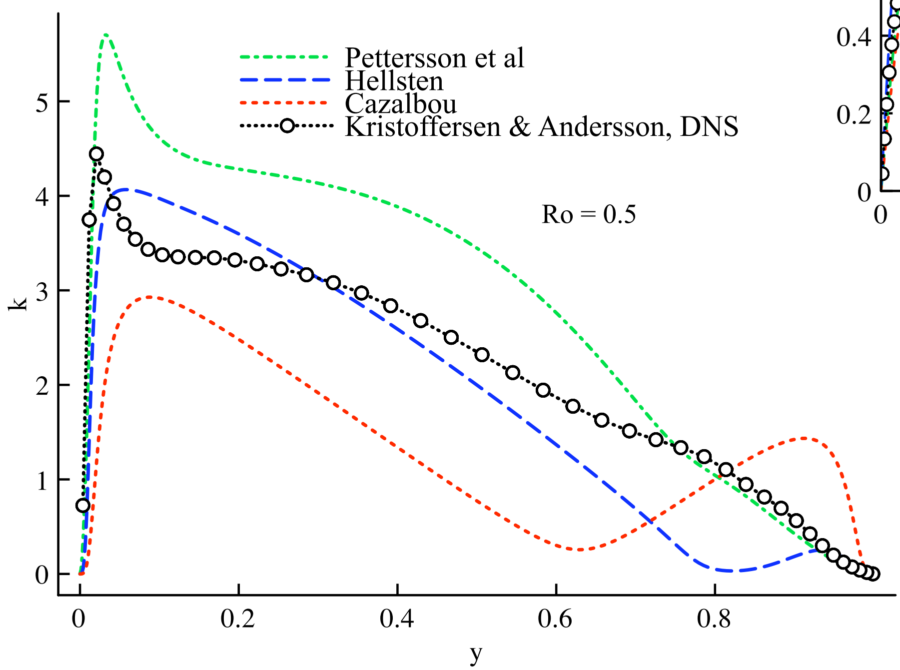
$$\Omega^F_{ij} = \boldsymbol{e}^{(i)} \cdot D_t \boldsymbol{e}^{(j)}$$

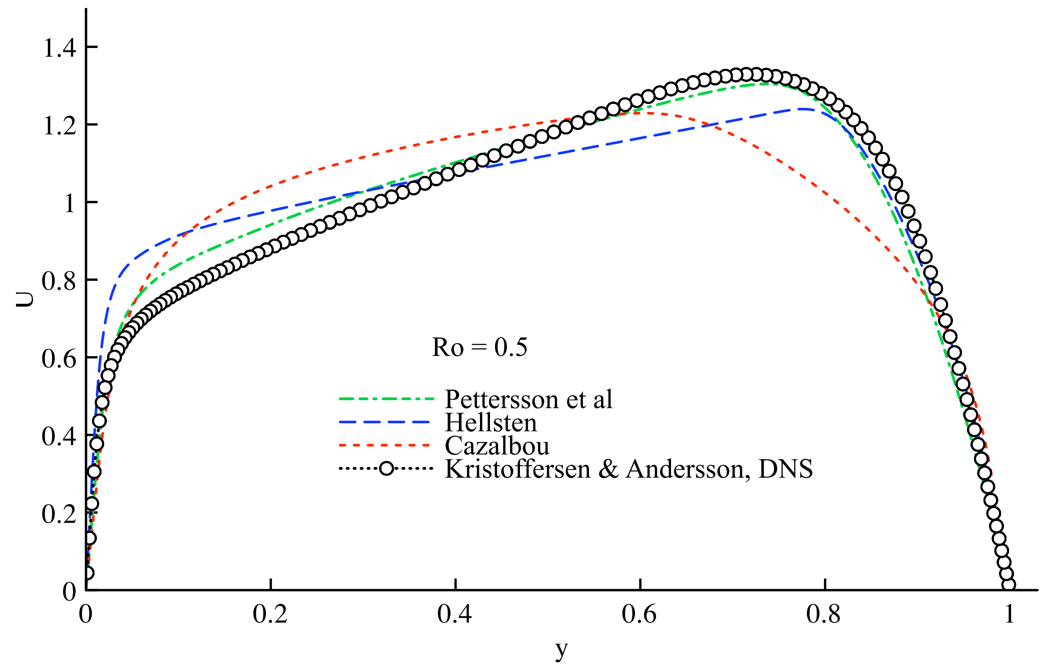
May be expensive, and is it right? More useable ansatz:

$$\Omega^F_{ij} \leftrightarrow (\boldsymbol{S} \cdot D_t \boldsymbol{S} - D_t \boldsymbol{S} \cdot \boldsymbol{S})/2|\boldsymbol{S}|^2$$

which is only frame rotation in 2-D

Rotating plane channel





Hellsten and Cazalbou are k-omega formulations

$$2\Omega H/U_b = 0.5$$

But: Physics are inviscid.
Modifying ε-equation coefficients is an artifice that increases dissipation (it probably should decrease with rotation).

P/ε should decrease because P is reduced by centrifugal stabilization

Bifurcation of SMC models

Recall equilibria of k–ε system:

$$branch\ 1: \quad \frac{\mathcal{P}}{\varepsilon} = \frac{C_{\varepsilon 2} - 1}{C_{\varepsilon 1} - 1}$$

and

$$branch\ 2: \quad \frac{\varepsilon}{k} = 0.$$

But, instead of eddy viscosity (2-equation closure)

$$\mathcal{P} = -\overline{u_i u_j}\partial_j U_i = -\overline{u_i u_j} S_{ij}$$

Solve SMC for Reynolds stress tensor

Equilibrium, algebraic stress

Moving equilibrium

$$d_t(\overline{u_iu_j}/k) = 0 \;\rightarrow\; d_t\overline{u_iu_j} = \frac{-\overline{u_iu_j}}{k} d_t k = \frac{-\overline{u_iu_j}}{k}(\mathcal{P} - \varepsilon)$$

[Aside: this gives a linear algebraic equation

$$0 = (1 - C_1 - cP/\varepsilon)\boldsymbol{b} - \frac{8}{15}\mathcal{S} - \boldsymbol{b}\cdot\mathcal{S} - \mathcal{S}\cdot\boldsymbol{b} + \frac{2}{3}\boldsymbol{\delta}\,\mathrm{trace}(\boldsymbol{b}\cdot\mathcal{S}) - \boldsymbol{b}\cdot\boldsymbol{W} + \boldsymbol{W}\cdot\boldsymbol{b}.$$

for $b_{ij} = \overline{u_iu_j}/k - {}^2\!/_3\,\delta_{ij}$]

Gives algebraic stress approximation (ASM); solution is called an explicit algebraic stress model (EASM). Rotation effects are captured through Reynolds stress equations.

$\exists$ closed form solution starting as

$$\overline{u_i u_j} = -F_\mu \boldsymbol{S} k^2/\varepsilon + {}^2\!/_3\, k\delta_{ij} \ldots$$

Aside: For General Linear closure model

$$F_\mu = \frac{{}^8\!/_{15}(C_1 - 1 + \mathcal{P}/\varepsilon)}{(C_1 - 1 + \mathcal{P}/\varepsilon)^2 - {}^2\!/_3(1 - C_2 - C_3)^2|\boldsymbol{S}k/\varepsilon|^2 + 2|\boldsymbol{W}k/\varepsilon|^2}$$

The remaining terms do not contribute to production:

$$\mathcal{P} = F_\mu(\boldsymbol{S}, \Omega^A, \Omega^F; k/\varepsilon)|\boldsymbol{S}|^2 k^2/\varepsilon$$

Hence not frame independent

This `constitutive' equation accompanies k and ε equations

On branch 1 $\mathcal{P}_R \equiv \mathcal{P}/\varepsilon = (C_{\varepsilon 2} - 1)/(C_{\varepsilon 1} - 1) \Rightarrow$

$$(\varepsilon/Sk)^2_\infty = \frac{2(1 - C_2 - C_3)^2}{3(C_1 - 1 + \mathcal{P}_R)^2} + \frac{8}{15(1 - C_2 - C_3)(C_1 - 1 + \mathcal{P}_R)\mathcal{P}_R} - \frac{2|\boldsymbol{W}|^2}{(C_1 - 1 + \mathcal{P}_R)^2 |\boldsymbol{S}|^2}$$

where

$$\frac{|\boldsymbol{W}|}{|\boldsymbol{S}|} = (1 - C_2 + C_3) + (2 - C_2 + C_3)R$$

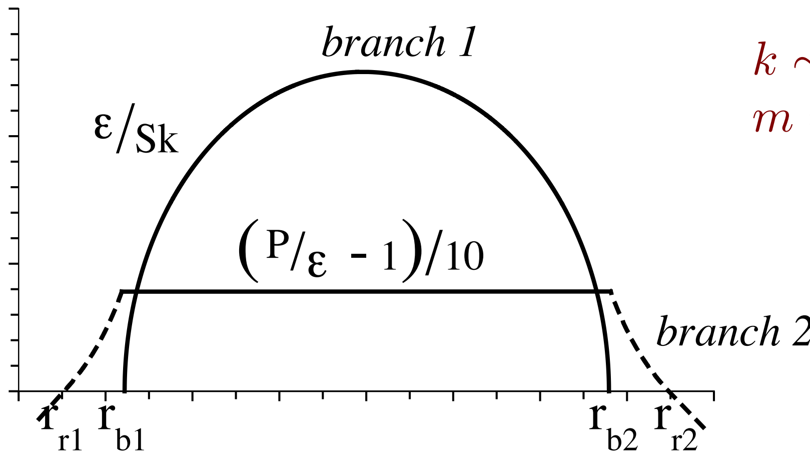
The 'bifurcation curve' is of the form

$$(\varepsilon/Sk)^2_\infty = A + B(R + C)^2$$

Where A, B and C are constants and *R* is the rotation number, as usual. Bifurcation points are $R_{\pm}$ satisfying

$$A + B(R + C)^2 = 0$$

Bifurcation diagram for homogeneous shear



$k \sim t^{-m}$

$m = \frac{1-\mathcal{P}/\varepsilon}{(C_{\varepsilon 2}-1)-\mathcal{P}/\varepsilon(C_{\varepsilon 1}-1)}$

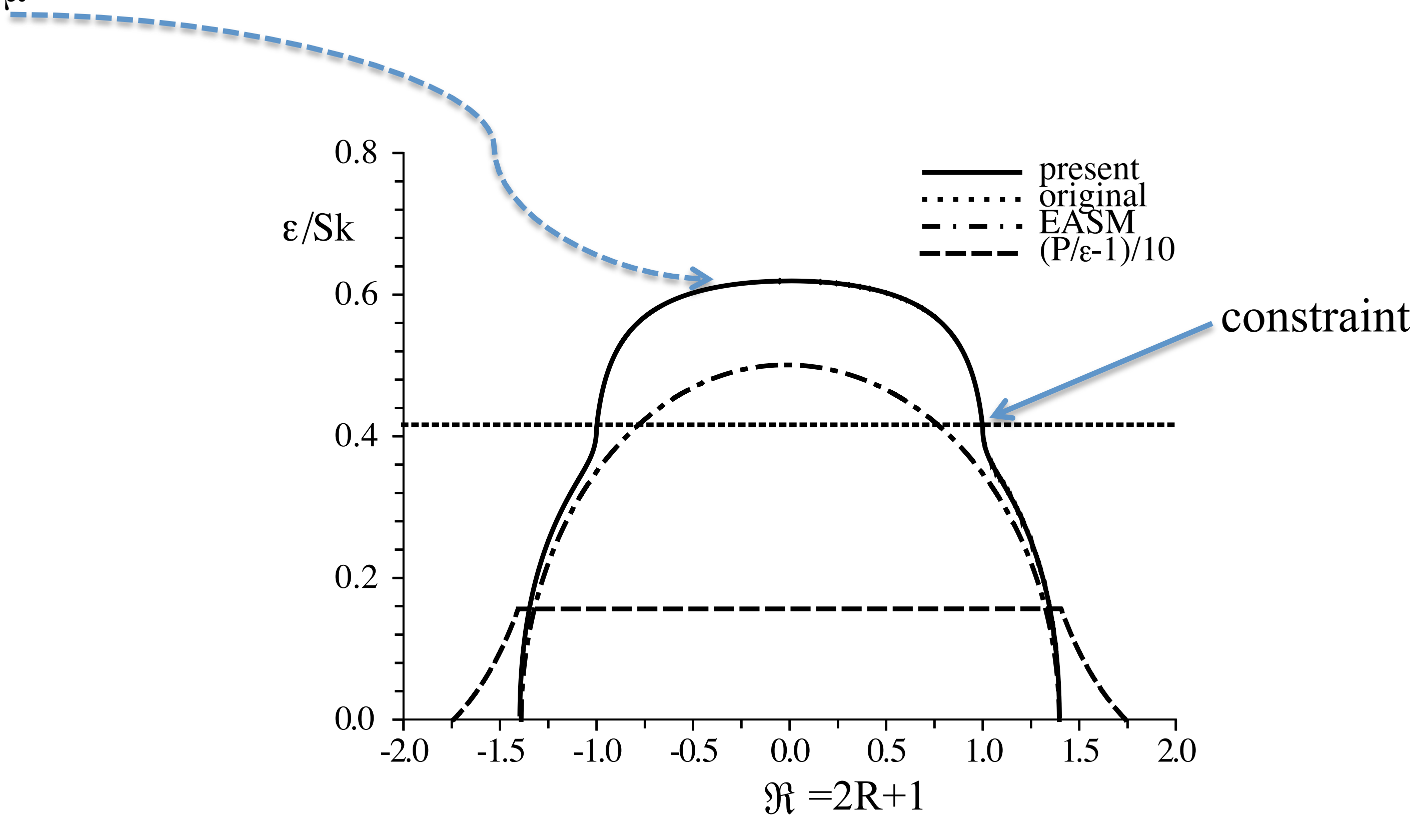
Model	$\mathcal{P}/\varepsilon$	R_-	R_+
— *Bifurcation*, r_b —			
SSG	2.09	−1.048	0.159
IP	2.09	−0.750	0.178
— *Stabilization*, r_r —			
SSG	1.00	−1.078	0.190
IP	1.00	−0.807	0.236

c.f. R_+=0 and R_-= -1

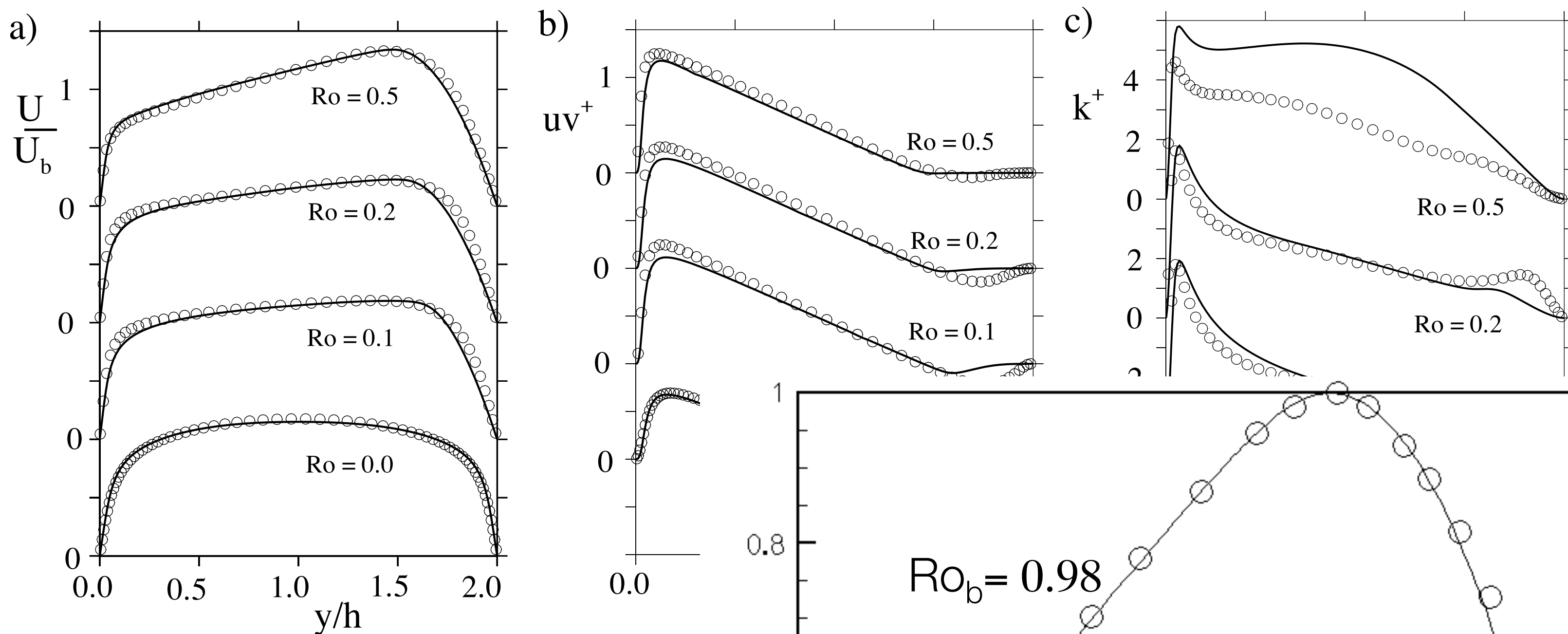
ε

One might use $\nu_T = F_\mu \mathbf{S}\, k^2/\varepsilon$ to capture bifurcation in eddy viscosity framework

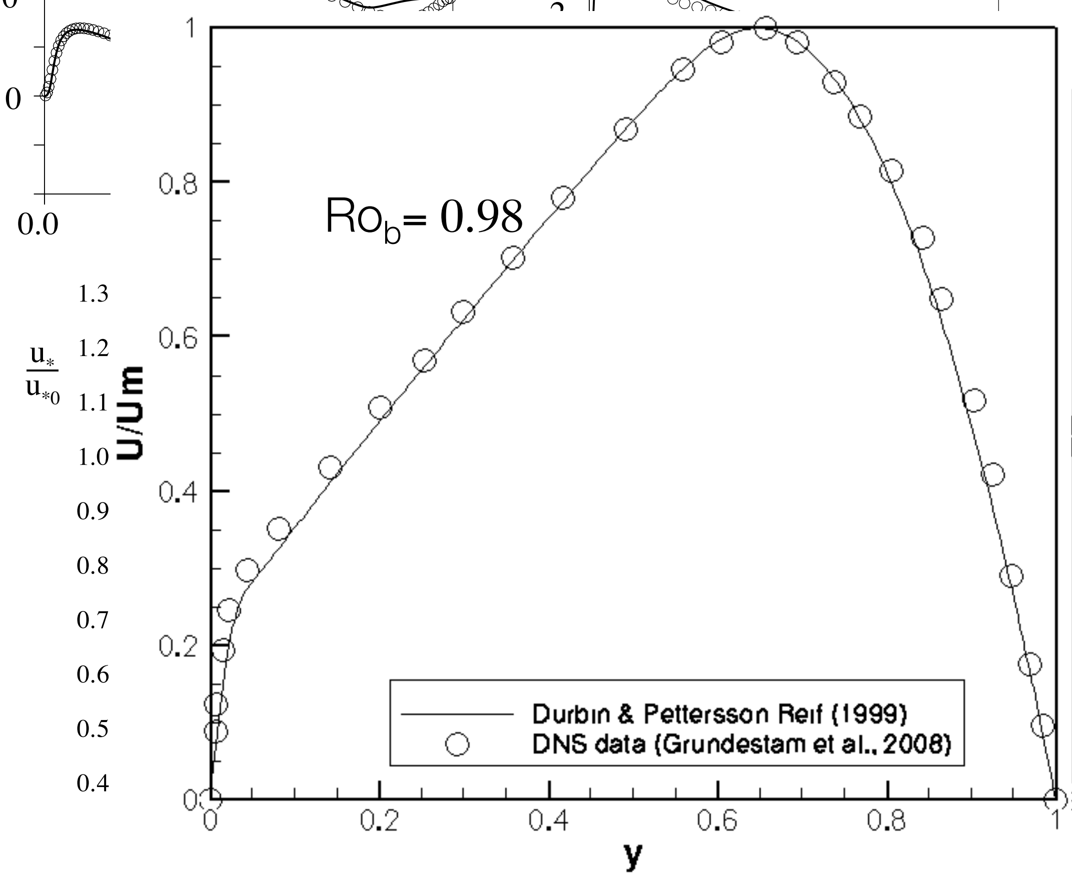
Caveat: EASM does not reproduce non-rotating k-ε solution. Modified F_μ is needed.



Rotating channel

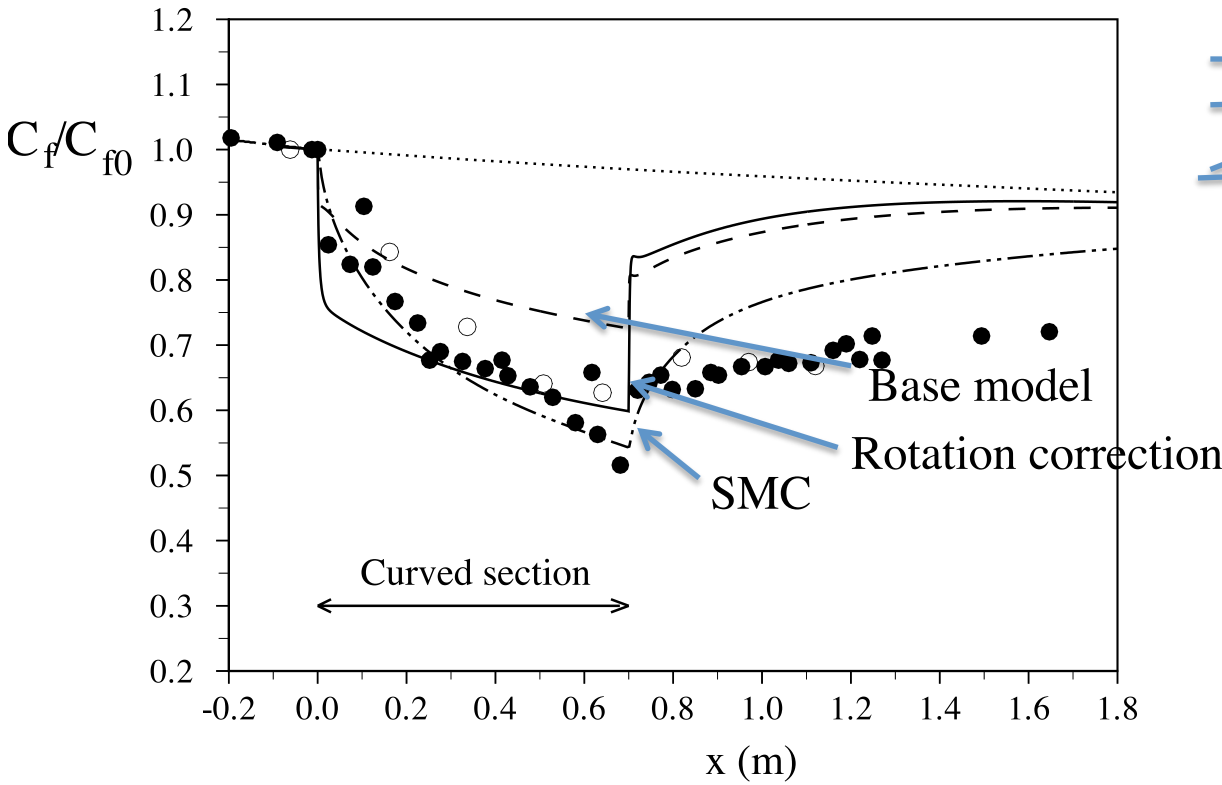


Kristoffersen & Andersson, DNS

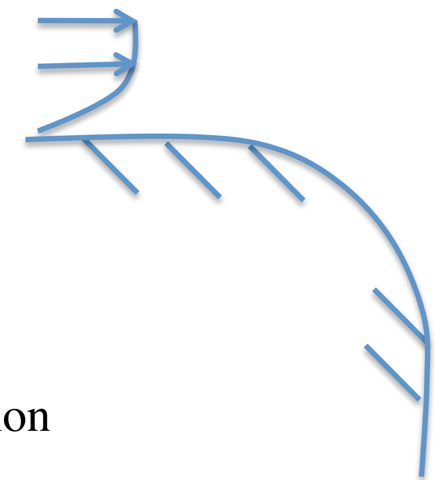


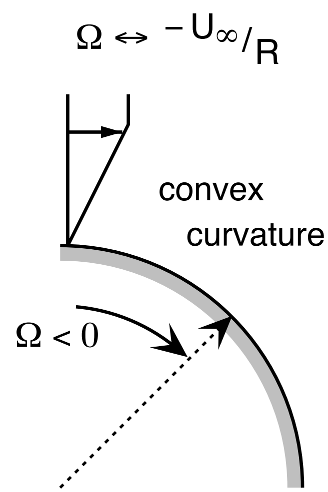
Pettersson et al
v^2-f model that bifurcates

Convex wall



1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
C_f/C_{f0}
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
x (m)
Curved section
Base model
Rotation correction
SMC





$\Omega \leftrightarrow -U_\infty/R$
convex
curvature
$\Omega < 0$

Summary, part II

- Equilibrium analysis relates P/ε to constants in the ε or ω equation
- Within the confines of eddy viscosity closure, P/ε can be reduced below unity by replacing these constants by functions of the Bradshaw parameter; but that is not consistent with physical mechanisms
- The equilibrium solution to a full Reynolds stress model bifurcates from healthy to decaying turbulence branches.
- Bifurcation is effected by adding a dependence of the eddy viscosity on rates of strain and rotation. This is another approach to incorporating rotation into eddy viscosity closure.