Turbulence As A Source of Sound

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Turbulence Makes Sound

Source

Sound
Turbulence Makes Sound

- Increasingly a concern...
The acoustic limit

Sources and sound

Turbulence: the acoustic analogy

Challenges in predicting sound from turbulence
- Complex turbulence statistic
- Phase velocity restriction
- Coupled process: different source components, refraction,…

Robustness as a criterion for formulation selection

Outlook
Sound Energies Are Small

- Acoustic energy radiated from a jet at take-off insufficient to boil an egg
- Double exit velocity: $\sim 250$ times more acoustic power
- Typically neglected in conservation of energy analysis of mechanical systems
Acoustic Limit
What is sound?

- A solution of the compressible flow equations

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0
\]

\[
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial p}{\partial x_i} = [\text{viscous terms}]
\]

- Approximately inviscid: interested in sound that propagates long distances, many wavelengths

\[
f = 4 \text{ kHz} \quad \Rightarrow \quad \lambda = a_o/f \approx 0.1 \text{ m}
\]
What is sound?

- Low energy $\rightarrow$ low amplitude $\rightarrow$ linearize

$$\rho(x, t) = \rho_o + \rho'(x, t) \quad u_i(x, t) = 0 + u_i'(x, t) \quad p(x, t) = p_0 + p'(x, t)$$

yielding

$$\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u_i'}{\partial x_i} = 0$$

$$\rho_o \frac{\partial u_i'}{\partial t} + \frac{\partial p'}{\partial x_i} = 0$$
What is sound?

- Eliminate velocity:

\[
\frac{\partial}{\partial t} [\text{mass}] \Rightarrow \frac{\partial^2 \rho'}{\partial t^2} + \rho_o \frac{\partial^2 u'_i}{\partial t \partial x_i} = 0
\]

\[
\frac{\partial}{\partial x_i} [\text{momentum}] \Rightarrow \rho_o \frac{\partial^2 u'_i}{\partial t \partial x_i} + \frac{\partial^2 p'}{\partial x_i \partial x_i} = 0
\]

and subtract

\[
\frac{\partial^2 \rho'}{\partial t^2} - \frac{\partial^2 p'}{\partial x_i \partial x_i} = 0
\]
What is sound?

- Speed of sound

\[
a_o = \left( \frac{\partial p}{\partial \rho} \right)_s \approx \frac{p'}{\rho'} \quad \Rightarrow \quad p' = a_o^2 \rho' + \text{h.o.t.}
\]

yielding the linear, scalar wave equation for \( \rho' \)

\[
\frac{\partial^2 \rho'}{\partial t^2} - a_o^2 \frac{\partial^2 \rho'}{\partial x_i \partial x_i} = 0
\]
What is sound?

- Speed of sound

\[ a_o = \left( \frac{\partial p}{\partial \rho} \right)_s \approx \frac{p'}{\rho'} \Rightarrow p' = a_o^2 \rho' + \text{h.o.t.} \]

yielding the linear, scalar wave equation for \( \rho' \)

\[ \frac{\partial^2 \rho'}{\partial t^2} - a_o^2 \frac{\partial^2 \rho'}{\partial x_i \partial x_i} = 0 \]

or for \( p' \)

\[ \frac{\partial^2 p'}{\partial t^2} - a_o^2 \frac{\partial^2 p'}{\partial x_i \partial x_i} = 0 \]
Solution Forms

- Plane waves: $\omega^2 = a_0^2 k^2$

  $$\rho' \sim \exp \left[ i (k \cdot x + \omega t) \right] = \exp \left[ ik (\hat{k} \cdot x \pm a_0 t) \right]$$
Solution Forms

- **Plane waves:** \( \omega^2 = a_o^2 k^2 \)

  \[ \rho' \sim \exp [i(k \cdot x + \omega t)] = \exp [ik(\hat{k} \cdot x \pm a_o t)] \]

- **Cylindrical waves (e.g. \( r^2 = x_1^2 + x_2^2 \))**

  \[ \rho' \sim H_0^{(1 \text{ or } 2)}(kr) \exp [ika_o t] \sim \left[ \frac{2}{\pi kr} \right]^{1/2} \exp \left[ \mp ik(r - a_o t) \pm \frac{i\pi}{4} \right] \sim \frac{1}{r^{1/2}} \]
Solution Forms

- **Plane waves:** \( \omega^2 = a_0^2 k^2 \)
  \[
  \rho' \sim \exp \left[ i(\mathbf{k} \cdot \mathbf{x} + \omega t) \right] = \exp \left[ i k (\hat{\mathbf{k}} \cdot \mathbf{x} \pm a_0 t) \right]
  \]

- **Cylindrical waves** (e.g. \( r^2 = x_1^2 + x_2^2 \))
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  \]

- **Spherical waves** (\( r = |\mathbf{x}| \))
  \[
  \rho' \sim \frac{1}{r} \exp[i k (r \pm a_0 t)] \sim \frac{1}{r}
  \]
Acoustic Perturbations Are Related

- Plane wave traveling in $+x_1$:

\[
\begin{align*}
\rho' &= \Re \left[ Ae^{ikx-i\omega t} \right] \\
u_1' &= \Re \left[ \frac{a_0}{\rho_o} Ae^{ikx-i\omega t} \right] \\
p' &= \Re \left[ Aa_0^2 e^{ikx-i\omega t} \right]
\end{align*}
\]

\[
\begin{align*}
u_2' &= u_3' = 0
\end{align*}
\]

So

\[
\begin{align*}
p' &= \rho_o a_0 u_1' \\
\rho' &= \frac{\rho_o}{a_0} u_1'
\end{align*}
\]
Acoustic Intensity

- Acoustic intensity, mean power flux

\[ I = \langle p' u' \rangle = \frac{a_o^3}{\rho_o} \langle (\rho')^2 \rangle \]

- Large \( r \):
  
  cylindrical: \( I \sim \frac{1}{r} \)  
  spherical: \( I \sim \frac{1}{r^2} \)

- Intensity usually metric of practical interest
Sources of Sound
Sources of Sound

● A mass source...

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = Q(x, t)
\]

\[
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial p}{\partial x_i} = \text{viscous terms}
\]

● Linearize, differentiate, form wave equation,....

\[
\frac{\partial^2 \rho'}{\partial t^2} - a_o^2 \frac{\partial^2 \rho'}{\partial x_i \partial x_i} = \frac{\partial Q}{\partial t} \equiv q(x, t)
\]
Green’s Function Solution

- Greens function:

\[
\frac{\partial^2 G}{\partial t^2} - a_o^2 \frac{\partial^2 G}{\partial x_i \partial x_i} = \delta(x) \delta(t)
\]

has solution

\[
G(x, t) = \frac{\delta(x - a_o t)}{4\pi a_o |x|} \sim \frac{1}{r}
\]

- Solution of

\[
\frac{\partial^2 \rho'}{\partial t^2} - a_o^2 \frac{\partial^2 \rho'}{\partial x_i \partial x_i} = q(x, t)
\]

is

\[
\rho'(x, t) = \frac{1}{4\pi a_o^2} \int q\left(y, t - \frac{|x-y|}{a_o}\right) \frac{dy}{|x-y|}
\]
\[ \rho'(x, t) = \frac{1}{4\pi a_o^2} \int \frac{q(y, t - \frac{|x-y|}{a_o})}{|x-y|} \, dy \]

Source scales: \( \ell, u, \rho_o \)

- \( q \sim \rho_o u^2 / \ell^2 \)
Compact Source Approximation

\[ \rho'(x, t) = \frac{1}{4\pi a_o^2} \int \frac{q(y, t - \frac{|x-y|}{a_o})}{|x-y|} \, dy \]

- **Source scales**: \( \ell, u, \rho_o \)
- **\( q \sim \rho_o u^2 / \ell^2 \)**
- **Difference in emission times across source** \( \tau_{\text{emission}} = \ell/a_o \)
- **Source changes on time scale** \( \tau_{\text{source}} = \ell/u \)
- **Consider** \( \tau_{\text{emission}} \ll \tau_{\text{source}} \):
  - \( \tau_{\text{emission}}/\tau_{\text{source}} = u/a_o \equiv m \ll 1 \) — low Mach number
  - **integrand**: \( q(y, t - \frac{|x-y|}{a_o}) \approx q(y, t - \frac{|x|}{a_o}) \)
Far-field Intensity

- Consider far field $|x| \gg \ell$, so

\[
\frac{1}{|x - y|} \approx \frac{1}{|x|}
\]

- Compact-source and far-field approximations

\[
\rho'(x, t) = \frac{1}{4\pi a_o^2 |x|} \int q \left( y, t - \frac{|x|}{a_o} \right) dy \sim 1/r a_o^2 \sim \rho_o u^2/\ell^2 \sim \ell^3
\]

thus:

\[
\rho' \sim \rho_o \frac{\ell}{r} m^2
\]

- Intensity

\[
I \sim (\rho')^2 \sim m^4
\]
Sources of Sound: Force

● A momentum source...

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0
\]

\[
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial p}{\partial x_i} = F_i(x, t) + \text{[vis. terms]}
\]

● Linearize, differentiate, inviscid, form acoustic equations,....

\[
\frac{\partial^2 \rho'}{\partial t^2} - a_o^2 \frac{\partial^2 \rho'}{\partial x_i \partial x_i} = \frac{\partial F_i}{\partial y_i}
\]
Far-field, Compact

- Same Green’s function solution

\[ \rho'(x, t) = \frac{1}{4\pi a_o^2} \int \frac{\partial F_i \left( y, t - \frac{|x-y|}{a_o} \right)}{\partial y_i} \frac{1}{|x-y|} \, dy \]

- Far-field, compact \((m \ll 1)\):

\[ \rho'(x, t) = \frac{1}{4\pi a_o^2 |x|} \int \frac{\partial F_i \left( y, t - \frac{|x|}{a_o} \right)}{\partial y_i} \, dy \]

- Same source scaling \((F \sim \rho_o u^2/\ell)\) also yields

\[ \rho' \sim \rho_o \frac{\ell}{r} m^2 \quad \text{and} \quad I \sim \rho^2 \sim \rho_o^2 \left( \frac{\ell}{r} \right)^2 m^4 \]
Far-field, Compact

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but this is very wrong....
A Missed Cancellation

\[ \rho'(\mathbf{x}, t) = \frac{1}{4\pi a_0^2|x|} \int \frac{\partial F_i(y, t - \frac{|\mathbf{x}|}{a_0})}{\partial y_i} \, dy \]

- Divergence theorem for compact region of finite \( F_i \)

\[ \int \frac{\partial F_i(y, t - \frac{|\mathbf{x}|}{a_0})}{\partial y_i} \, dy = 0 \]
Build Near-Cancellation into Formulation

\[ \rho'(x, t) = \frac{1}{4\pi a_o^2} \int \frac{\partial F \left( y, t - \frac{|x-y|}{a_o} \right)}{\partial y_i} \frac{1}{|x-y|} \ dy \]

\[ = \frac{1}{4\pi a_o^2} \frac{\partial}{\partial x_i} \int \frac{F \left( y, t - \frac{|x-y|}{a_o} \right)}{|x-y|} \ dy \]

via

\[ \frac{\partial}{\partial x_i} \int f(y)g(x-y) \ dy = \int f(y) \frac{\partial}{\partial x_i} g(x-y) \ dy \]

\[ = - \int f(y) \frac{\partial}{\partial y_i} g(x-y) \ dy \]

\[ = + \int \frac{\partial f(y)}{\partial y_i} g(x-y) \ dy - \int \frac{\partial}{\partial y_i} [f(y)g(x-y)] \ dy \]
\[ \rho'(x, t) = \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \int \frac{F(y, t - \frac{|x-y|}{a_0})}{|x-y|} dy \]

\[ \approx \frac{1}{4\pi a_0^2} \frac{\partial}{\partial x_i} \int \frac{F(y, t - \frac{|x|}{a_0})}{|x|} dy \]

\[ \approx \frac{1}{4\pi a_0^2} \int \left[ \frac{\partial F(y, t - \frac{|x|}{a_0})}{\partial x_i} \frac{1}{|x|} + F(y, t - \frac{|x|}{a_0}) \frac{\partial}{\partial x_i} \left( \frac{1}{|x|} \right) \right] dy \]

\[ = -\frac{1}{4\pi a_0^2 |x|} \int \frac{\partial F}{\partial t} \left( y, t - \frac{|x|}{a_0} \right) \frac{1}{a_0} \frac{\partial |x|}{\partial x_i} dy + O \left( \frac{1}{|x|^2} \right) \]
Far-field, Compact (again)

\[ \rho'(\mathbf{x}, t) = \frac{1}{4\pi a_o^2} \frac{\partial}{\partial x_i} \int \frac{F\left(\mathbf{y}, t - \frac{\lvert \mathbf{x} - \mathbf{y} \rvert}{a_o}\right)}{\lvert \mathbf{x} - \mathbf{y} \rvert} \, d\mathbf{y} \]

\[ \approx \frac{1}{4\pi a_o^2} \frac{\partial}{\partial x_i} \int \frac{F\left(\mathbf{y}, t - \frac{\lvert \mathbf{x} \rvert}{a_o}\right)}{\lvert \mathbf{x} \rvert} \, d\mathbf{y} \]

\[ \approx \frac{1}{4\pi a_o^2} \int \left[ \frac{\partial F\left(\mathbf{y}, t - \frac{\lvert \mathbf{x} \rvert}{a_o}\right)}{\partial x_i} \frac{1}{\lvert \mathbf{x} \rvert} + F\left(\mathbf{y}, t - \frac{\lvert \mathbf{x} \rvert}{a_o}\right) \frac{\partial}{\partial x_i} \left( \frac{1}{\lvert \mathbf{x} \rvert} \right) \right] \, d\mathbf{y} \]

\[ = -\frac{1}{4\pi a_o^2 \lvert \mathbf{x} \rvert} \int \frac{\partial F}{\partial t} \left(\mathbf{y}, t - \frac{\lvert \mathbf{x} \rvert}{a_o}\right) \frac{1}{a_o} \frac{\partial \lvert \mathbf{x} \rvert}{\partial x_i} \, d\mathbf{y} + O\left(\frac{1}{\lvert \mathbf{x} \rvert^2}\right) \]

\[ \frac{\partial \lvert \mathbf{x} \rvert}{\partial x_i} = ? \ldots \]
Far-field, Compact (again)

\[
\rho'(x, t) = \frac{1}{4\pi a_o^2 |x|} \int \frac{\partial F}{\partial t} \left( y, t - \frac{|x|}{a_o} \right) \frac{1}{a_o} \frac{\partial |x|}{\partial x_i} \, dy
\]

\[
\text{Noting that } \frac{\partial |x|}{\partial x_i} = \frac{\partial}{\partial x_i} \sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{x_i}{|x|}
\]

yields

\[
\rho'(x, t) = \frac{1}{4\pi a_o^3 |x| |x|} \int \frac{\partial F}{\partial t} \left( y, t - \frac{|x|}{a_o} \right) \left[ \text{ } \right] \, dy_{\ell^3}
\]

F \sim \rho_o u^2/\ell; t \sim \ell/u

so

\[
\rho' \sim \rho_o \left( \frac{u}{a_o} \right)^3 \frac{l}{r} \sim m^3 \quad \text{and} \quad I \sim m^6
\]
Dipole Character

- Dipole — equivalent to nearly canceling equal and opposite $q$’s

\[ I \propto \frac{x_i}{|x|} \propto \cos^2(\theta) \]

- Initial wrong approach missed cancellation (or got zero)

- Space derivative $\partial y_i$ of source was key factor

- This also affects how turbulence makes sound...
Turbulence As A Source of Sound
Source and sound are intuitively obvious
A Turbulent Jet

- No obvious length/time-scale separation to clarify distinction
- $\frac{u'u'}{U^2} = O(1)$ — beyond weakly nonlinear
- Simplifications of standard acoustics do not apply
So what do we know for sure...? A short list:
\mathcal{N}(\vec{q}) = 0
Our Only Truth

\[ \mathcal{N}(\vec{q}) = 0 \]

- The flow equations \( \mathcal{N} \) govern the flow variables \( \vec{q} \)
Our Only Truth

\[ \mathcal{N}(\vec{q}) = 0 \]

- So what can we do...?
Give $\vec{q}$ a dual role

- Seems that we must use $\vec{q}$ in two ways *simultaneously*

- Rearrange $\mathcal{N}(\vec{q}) = 0$ into $\mathcal{L}\vec{q} = S(\vec{q})$
  - $\mathcal{N}$ — compressible flow equations
  - $\mathcal{L}$ — wave propagation operator (usually linear)
  - $S$ — nominal noise source (usually nonlinear)
\[ \mathcal{L} = \partial_{tt} - a_o^2 \nabla^2 \]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0
\]

\[
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial p}{\partial x_i} = \text{viscous terms}
\]

- Turbulence fluctuations are not small... can’t just linearize

- \( \partial_t [\text{mass}] - \partial_{x_i} [\text{momentum}] \) as before:

\[
\frac{\partial^2 \rho}{\partial t^2} - a_o^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}
\]

the Lighthill equation, where the Lighthill stress is

\[
T_{ij} = \rho u_i u_j + (p - a_o^2 \rho) + \tau_{\text{viscous}}
\]

- Note: this is an \textit{exact} re-arrangement of the flow equations
**Acoustic Analogy**

\[
\frac{\partial^2 \rho}{\partial t^2} - \alpha_o^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}
\]

- Treat \( T_{ij} \) as analogous to externally applied stress
- Same solution procedure....

\[
\rho(x, t) = \frac{1}{4\pi \alpha_o^2} \int \frac{\partial^2 T_{ij}(y, t - \frac{|x-y|}{\alpha_o})}{\partial y_i \partial y_j} \frac{1}{|x-y|} \, dy
\]

\[
= \frac{1}{4\pi \alpha_o^2} \frac{\partial^2}{\partial x_i \partial x_j} \int T_{ij}(y, t - \frac{|x-y|}{\alpha_o}) \frac{1}{|x-y|} \, dy
\]
Acoustic Analogy

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\[
\rho(\mathbf{x}, t) = \frac{1}{4\pi a_o^2} \int \frac{\partial^2 T_{ij}(\mathbf{y}, t - \frac{|\mathbf{x} - \mathbf{y}|}{a_o})}{\partial y_i \partial y_j} \frac{1}{|\mathbf{x} - \mathbf{y}|} \ d\mathbf{y}
= \frac{1}{4\pi a_o^2} \frac{\partial^2}{\partial x_i \partial x_j} \int T_{ij}(\mathbf{y}, t - \frac{|\mathbf{x} - \mathbf{y}|}{a_o}) \frac{1}{|\mathbf{x} - \mathbf{y}|} \ d\mathbf{y}
\]

- Compact source, far field....
Acoustic Analogy

\[
\frac{\partial^2 \rho}{\partial t^2} - a_o^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}
\]

- Compact-source, far-field assumptions...

\[
\rho(x, t) = \frac{1}{4\pi a_o^2} \frac{\partial^2}{\partial x_i \partial x_j} \int T_{ij} \left( y, t - \frac{|x|}{a_o} \right) \frac{1}{|x|} \, dy
\]

\[
= \frac{1}{4\pi a_o^4 |x|} \int \frac{x_i x_j}{|x|^2} \frac{\partial^2 T_{ij} \left( y, t - \frac{|x|}{a_o} \right)}{\partial t^2} \, dy + O \left( \frac{1}{|x|^2} \right)
\]

- Scaling: \( T_{ij} \approx \rho u_i u_j \sim \rho_o u^2 \ldots \)

\[
\rho \sim \rho_o \frac{\ell}{r} m^4 \quad \text{and} \quad I \sim m^8
\]
Quadrupole Character

- Quadrupole — equivalent to nearly canceling equal and opposite \( q \)'s

\[
I \propto \frac{x_i x_j}{|x|^2} \quad \propto \cos^4(\theta)
\]

\[
I \propto \frac{x_i x_j}{|x|^2} \quad \propto \cos(\theta)^2 \sin^2(\theta)
\]

- Far-field exact in the Mach number \( M \to 0 \) limit...
Consequence: $U^8$

- Predicts that jet-noise power should scale as $U^8$
Consequence: $U^8$

- Predicts that jet-noise power should scale as $U^8$

UIUC Jet Noise Facility
Consequence: $U^8$

- Predicts that jet-noise power should scale as $U^8$

Anechoic
Consequence: $U^8$

- Predicts that jet-noise power should scale as $U^8$

Sound Power versus Exit Velocity
Predictions

● Gross features:
  ✦ $U^8$ even for $M$ approaching unity
  ✦ $U^6$ with surfaces
  ✦ $U^4$ with mass-source-like features
Predictions

- Gross features:
  - \( U^8 \) even for \( M \) approaching unity
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  - \( U^4 \) with mass-source-like features

- Detailed quantitative predictions
  - Can calculate sound given prediction of \( \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \) ... 
  - Challenging....
    - depends upon turbulence
Predictions

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  - $U^8$ even for $M$ approaching unity
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    - $T_{ij,ij}$ includes non-source effects (refraction)
Predictions

- Gross features:
  - \( U^8 \) even for \( M \) approaching unity
  - \( U^6 \) with surfaces
  - \( U^4 \) with mass-source-like features

- Detailed quantitative predictions
  - Can calculate sound given prediction of \( \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \) ...
  - Challenging....
    - depends upon turbulence ... in more complex manner than needed in most turbulence modeling
    - \( T_{ij,ij} \) includes non-source effects (refraction)
    - most of \( T_{ij} \) is non-radiating
Complex Turbulence Statistic

- Can predict sound via prediction of \( \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \)...

- Mean intensity: \( (I = \langle \rho^2 \rangle) \)

- Compact source, far field

\[
I(x) = \frac{x_i x_j x_k x_l}{16\pi^2 a_\infty^5 |x|^5} \int_\infty^\infty \int_\infty^\infty \frac{\partial^4}{\partial \tau^4} T_{ij}(y, t)T_{kl}(y + \xi, t + \tau) \, d\xi \, dy ,
\]

fourth-order space retarded-time covariance tensor....
Complex Turbulence Statistic

- Can predict sound via prediction of $\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \ldots$

- Mean intensity: $(I = \langle \rho^2 \rangle)$

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I(x) = \frac{x_i x_j x_k x_l}{16\pi^2 a_\infty^5 |x|^5} \int_{\infty}^{\infty} \int_{\infty}^{\infty} \frac{\partial^4 T_{ij}(y, t)}{\partial \tau^4} T_{kl}(y + \xi, t + \tau) \, d\xi \, dy,
\]

fourth-order space retarded-time covariance tensor....


- Components have been measured

- No universal character
Refraction

- Appears as a source in:

\[
\frac{\partial^2 \rho}{\partial t^2} - a_o^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}
\]

**Refraction**

- Appears as a source in:

\[
\frac{\partial^2 \rho}{\partial t^2} - a_o^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}
\]

- Freund & Fleischman, *Int. J. Aeroac. (2002)*: direct assessment of refraction in the high frequency limit
Refraction

Appears as a source in:

\[
\frac{\partial^2 \rho}{\partial t^2} - a_o^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}
\]

Refraction

- Appears as a source in:

$$\frac{\partial^2 \rho}{\partial t^2} - a_o^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

Lighthill Source

- $T_{ij,ij}$ for $M = 0.9$, $Re = 3600$ DNS (Freund, 2001)
Only ‘modes’ with supersonic phase velocity radiate:
Mostly Non-radiating

- Only ‘modes’ with supersonic phase velocity radiate:
- Consider two-dimensional example:

\[
\frac{\partial^2 \rho}{\partial t^2} - a_o^2 \left( \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} \right) = S(x, y, t)
\]
Mostly Non-radiating

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Source
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● Fourier transform

\[
f(x, y, t) = \int \int \hat{f}(k, y, \omega) e^{ikx} e^{i\omega t} \, dk \, d\omega
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Mostly Non-radiating

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\]

- Fourier transform

\[
f(x, y, t) = \int \int \hat{f}(k, y, \omega) e^{ikx} e^{i\omega t} \, dk \, d\omega
\]

\[
\frac{d^2 \hat{\rho}}{dy^2} + (\omega^2 - a_o^2 k^2) \hat{\rho} = -\hat{S}
\]
Mostly Non-radiating

- Solutions
  \[ \hat{\rho}(k, y, \omega) = [\cdots] e^{\pm y \sqrt{a_o^2 k^2 - \omega^2}} \]
  decay (not waves) in \( \pm y \) unless \( \omega^2 > a_o^2 k^2 \)

- \( \omega^2 > a_o^2 k^2 \) corresponds to supersonic phase velocity
  \[ \left| \frac{\omega}{k} \right| > a_o \]

- Most of turbulence in a \( M \approx 1 \) jet is moving with convection velocity (phase velocity) \( U_c \lesssim a_o \):
  - subsonic phase velocity
\[ M = 0.9 \]
Don’t ‘see’ what makes the sound
$M = 2.5$: Supersonic Convection

- Eddies emit shock waves
$M_c > 1$ Basic Mechanism
Basic Mechanism

- Like sonic boom
$M_c > 1$ Basic Mechanism

- Like sonic boom
- ... but the aircraft–eddy appears and disappears
\( M = 0.9 \) Jet: Lighthill Source

- \( T_{ij,ij} \) for \( M = 0.9, \ Re = 3600 \) DNS (Freund, 2001)
$M = 0.9$ Jet: Lighthill Source

- Source in wavenumber-frequency space

$S(x, t) \Rightarrow \hat{S}(k, \omega)$

- Radiating
- Non-radiating

$M_c = 0.05$
$M_c = 0.3$
$M_c = 0.5$
$M_c = 1.0$
$M_c = -1.0$
\( M = 0.9 \) **Jet: Lighthill Source**

- Source in wavenumber-frequency space

\[ S(x, t) \Rightarrow \hat{S}(k, \omega) \]

- Filtering down to radiating-only modes breaks locality

- Flows can look the same and yet have very different sound
Large Turbulent Structures

- Very similar looking ‘turbulence’ can have entirely different sound
- Two-dimensional mixing layer (Wei & Freund, JFM, 2006)
- Controlled flow is ≥ 6 dB quieter

No Control  \( v \)-control
Large-scale Structures
● Need to predict subtle aspects of turbulence....

● $\omega = 1.5a_\infty/r_0$:

● Need to faithfully represent components with small energy
Common mean + perturbation turbulence decomposition

\[ T_{ij} = \bar{T}_{ij} + \rho (\bar{u}_i u'_j + u'_i \bar{u}_j) + \rho u'_i u'_j \]

\[ + (p' - a^2_{\infty} \rho') \delta_{ij} - \tau'_{ij} \]

Neglect viscous source (universally accepted)

Implicit result of Colonius & Freund (2000) even for \( Re = 2000 \)
Directivity ($M = 0.9$ Jet)

\[ \frac{p'p'}{(\rho^2 U^4)} \times 10^{10} \]

- Total
- All $\rho u_i u_j$
- Self
- Shear
- ‘Entropy’

$\alpha$ (degrees)

6 dB
Net Power

<table>
<thead>
<tr>
<th>Component</th>
<th>Power/$\rho_j U_j^3 A_j$</th>
<th>Power/Power $T_{ij}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total:</td>
<td>$T_{ij}$</td>
<td>$8.3 \times 10^{-5}$</td>
</tr>
<tr>
<td>Shear:</td>
<td>$T_{ij}^l$</td>
<td>$8.7 \times 10^{-5}$</td>
</tr>
<tr>
<td>Self:</td>
<td>$T_{ij}^n$</td>
<td>$6.9 \times 10^{-5}$</td>
</tr>
<tr>
<td>Entropy:</td>
<td>$T_{ij}^s$</td>
<td>$2.0 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

- Net powers of different components do not “add up”
Correlation Coefficients

\[ C_{\beta-\gamma} = \frac{\rho^\beta \rho^\gamma}{\rho_{rms}^\beta \rho_{rms}^\gamma} \]

- Need to model terms and correlations
Non-local

Should be local if quadrupole: \( p = T_{ij} \ast G_{ij} \)

\[ \frac{p'p}{(\rho_j^2 U_j^4)} \times 10^{10} \]

\[ \frac{\rho_0 u'_i u'_j}{(\rho_j^2 U_j^4)} \times 10^{10} \]
Non-local

Should be local if quadrupole: \( p = T_{ij} \ast G_{ij} \)

\[
\begin{align*}
\text{All } T_{ij} & \quad \text{and} \\
\rho_{0}u'_{i}u'_{j} & \quad \text{but not for all details}
\end{align*}
\]
Other options for $\mathcal{L}\vec{q} = S(\vec{q})$?

\[ \mathcal{N}(\vec{q}) = 0 \quad \Rightarrow \quad \mathcal{L}\vec{q} = S(\vec{q}) \]
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\[
\mathcal{N}(\vec{q}) = 0 \quad \Rightarrow \quad \mathcal{L}\vec{q} = S(\vec{q})
\]

- Make prediction easier with more propagation physics in $\mathcal{L}$?
- Common choices:
  - Lighthill: $\mathcal{L}$ – homogeneous-medium wave operator
  - Lilley (linearized): $\mathcal{L}$ – refraction due to parallel shear flow
  - Goldstein: $\mathcal{L}$ – refraction due to mean flow (e.g.)
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- One alternative: Ad hoc source/propagation combination
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- One alternative: *Ad hoc* source/propagation combination
  - Tam & Auriault: $\mathcal{L}$ has locally parallel flow with made up $S$

- Why when exact relations can be a starting point? – unjustified
Which $\mathcal{L}\vec{q} = S(\vec{q})$ best?

- All $\mathcal{L}\vec{q} = S(\vec{q})$ are exact
  - Given $S(\vec{q})$, $\mathcal{L}^{-1}S(\vec{q})$ gives sound
  - So how to choose?
Which $\mathcal{L}\vec{q} = S(\vec{q})$ best?

- All $\mathcal{L}\vec{q} = S(\vec{q})$ are exact
  - Given $S(\vec{q})$, $\mathcal{L}^{-1}S(\vec{q})$ gives sound
  - So how to choose?

- Simplest? $\longrightarrow$ Lighthill (or related)
  - $\mathcal{L}$ easily inverted
  - $S(\vec{q})$ seems no more complex than others
  - solutions of $\mathcal{L}\vec{q} = 0$ well behaved
  - disturbing that so much non-source stuff is in $S$
Anything Simpler?

- Better differentiation of source and propagation?
  - complicates $\mathcal{L}^{-1}S(\vec{q})$
  - may simplify $S(\vec{q})$ – more like true source (unexplored)

- Inconvenient truth: the turbulent flow that constitutes $S(\vec{q})$ remains mysterious
Anything Simpler?

● Better differentiation of source and propagation?
  ✦ complicates $L^{-1}S(\vec{q})$
  ✦ may simplify $S(\vec{q})$ – more like true source (unexplored)

● Inconvenient truth: the turbulent flow that constitutes $S(\vec{q})$ remains mysterious

● Most robust?
Robustness

- $S$ is never known exactly
Robustness

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- Acoustic inefficiency allows far-field $\vec{q}$ errors $\gg S(\vec{q})$ errors
  - e.g. errors potentially disrupt cancellations
Robustness

- $S$ is never known exactly

- Acoustic inefficiency allows far-field $\vec{q}$ errors $\gg S(\vec{q})$ errors
  
  ✦ e.g. errors potentially disrupt cancellations

- Use formulation most robust to unavoidable errors in $S$
  
Robustness

- Many potential ways to evaluate robustness...

- For now: empirical robustness evaluation using DNS data

- Work with time dependent formulations
  - SGS noise models
  - large-scale dynamics models (POD Galerkin projection, PSE)
Goldstein (2003) general acoustic analogy $L\vec{q} = S(\vec{q})$:

$$\bar{\rho} \frac{\bar{D}}{Dt} \frac{\rho'}{\bar{\rho}} + \frac{\partial}{\partial x_j} \bar{\rho} u_j' = 0$$

$$\bar{\rho} \left( \frac{\bar{D}}{Dt} u_i' + u_j' \frac{\partial \tilde{v}_i}{\partial x_j} \right) + \frac{\partial p_e'}{\partial x_i} - \frac{\rho'}{\bar{\rho}} \frac{\partial \tilde{\tau}_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} (e'_{ij} - \tilde{e}_{ij})$$

$$\frac{1}{\gamma - 1} \left( \frac{\bar{D}p_e'}{Dt} + \gamma p_e' \frac{\partial \tilde{v}_j}{\partial x_j} + \gamma \frac{\partial}{\partial x_j} \bar{p} u_j' \right) - u_i' \frac{\partial \tilde{\tau}_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} (\eta_j' - \tilde{\eta}_j) + (e'_{ij} - \tilde{e}_{ij}) \frac{\partial \tilde{v}_i}{\partial x_j}$$

Exact consequence of flow equations
Formulation

● Base flow \((\bar{\rho}, \bar{p}, \bar{v}_i)\)
  
  ✦ "user" specified
  ✦ for explicit mean-flow refraction \((\text{e.g.})\)
  ✦ satisfies exact equations with sources \(\tilde{T}_{ij}, \tilde{H}_{ij}\) and \(\tilde{H}_0\)

● Introduced new dependent variables

\[
p'_e \equiv p' + \frac{\gamma - 1}{2} \rho v_i v_i + (\gamma - 1) \tilde{H}_0 \quad \text{and} \quad u'_i \equiv \rho \frac{v'_i}{\bar{\rho}},
\]
Formulation

- Noise source $S(\bar{q})$:

\[
e'_{ij} \equiv -\rho v'_i v'_j + \frac{\gamma - 1}{2} \delta_{ij} \rho v'_k v'_k + \sigma'_{ij}
\]
\[
\tilde{e}_{ij} \equiv \tilde{T}_{ij} - \delta_{ij}(\gamma - 1) \tilde{H}_0
\]
\[
\eta'_i \equiv -\rho v'_i h'_0 - q'_i + \sigma_{ij} v'_j
\]
\[
\tilde{\eta}_i \equiv \tilde{H}_i - \tilde{T}_{ij} \tilde{v}_j
\]

- zero mean for time averaged base flow
Formulation Summary

● Step I: pick base flow

✦ uniform (Lighthill-like)
✦ globally parallel flow (Lilley)
✦ spreading mean flow
Formulation Summary

- **Step I: pick base flow**
  - uniform (Lighthill-like)
  - globally parallel flow (Lilley)
  - spreading mean flow

- **Step II: base flow defines source** \( S(\tilde{q}) \)

\[
e'_{ij} - \tilde{e}_{ij} \quad \text{and} \quad \eta'_{i} - \tilde{\eta}_{i}
\]
Formulation Summary

- **Step I:** pick base flow
  - uniform (Lighthill-like)
  - globally parallel flow (Lilley)
  - spreading mean flow

- **Step II:** base flow defines source $S(\vec{q})$
  \[ e'_{ij} - \tilde{e}_{ij} \quad \text{and} \quad \eta'_i - \tilde{\eta}_i \]

- **Step III:** solve $L\vec{q} = S(\vec{q})$
  - same high-order schemes at DNS
  - same mesh
  - same wave-equation extrapolation to far field
  - no special treatment of $L\vec{q} = 0$ solutions (!?!?)
  - neglect diffusive transport
Locally Parallel Base Flow

- Mean-flow base flow but neglect streamwise derivatives

\[ \frac{\partial \bar{q}}{\partial x_1} = 0 \]

- Rational approximation of mean-flow analogy

- Used by Tam & Auriault with *ad hoc* \( S(\bar{q}) \)

- Analyze in same way as true acoustic analogies using actual \( S \) subject to same approximation
- Two-dimensional mixing layer

- Randomly excited
- 3907 fields stored every $4\Delta t$
Source Errors

- Decompose DNS flow into empirical eigenfunctions (POD modes)
  \[ \bar{q}(x, t) = \sum_{i=1}^{N} a_i(t) \bar{\psi}_i(x) \quad N = 587 \]
  where \( \bar{\psi} \) modes are constructed using snapshots and KE norm
  \[ E = \int_{V} \rho u_i u_i \, dx \]

- Expect:
  ✦ low modes: large scale, low frequency, high energy
  ✦ high modes: small scale, high frequency, low energy
Mode Spectrum

\[ \frac{\lambda}{\sum_j \lambda_j} \]

Mode #

\[ 10^0 \quad 10^1 \quad 10^2 \]

\[ 10^0 \quad 10^1 \quad 10^2 \]

\[ \lambda / \sum_j \lambda_j \]

Mode #
Mode Shapes and $a(t)$
Mode Shapes and $a(t)$

MODE 1

MODE 128
Lower modes: larger scale, lower frequency, higher energy

Higher modes: smaller scale, higher frequency, lower energy
Errors to Assess Robustness

- High-frequency / small-scale errors: truncate series

\[ \vec{q}_e(x, t) = \sum_{i=1}^{N_t} a_i(t) \vec{\psi}_i(x) \]

- e.g. missing scales in LES

- Low-frequency / large-scale errors: mess with mode 1 and/or 2

\[ \vec{q}_e(x, t) = \vec{q}(x, t) - \frac{a_1(t)}{2} \vec{\psi}_1(x) \]

\[ \vec{q}_e(x, t) = \vec{q}(x, t) - \frac{a_1(t)}{2} \vec{\psi}_1(x) - \frac{a_2(t)}{2} \vec{\psi}_2(x) \]

- e.g. POD dynamical model, PSE
<table>
<thead>
<tr>
<th>Case</th>
<th>Energy Retained</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>100.0%</td>
<td>Full source</td>
</tr>
<tr>
<td>B</td>
<td>99.3%</td>
<td>128 modes</td>
</tr>
<tr>
<td>C</td>
<td>91.3%</td>
<td>32 modes</td>
</tr>
<tr>
<td>D</td>
<td>92.5%</td>
<td>$a_1' = a_1/2$</td>
</tr>
<tr>
<td>E</td>
<td>85.5%</td>
<td>$a_{1,2}' = a_{1,2}/2$</td>
</tr>
</tbody>
</table>
Sound Pressure Spectra: $\phi = 50^\circ$

$\hat{p}^* / \rho_\infty^2 a_\infty^2 \delta^2$

DNS
 DNS-mean base flow
 Uniform base flow

$\omega \delta / a_\infty$
Sound Pressure Spectra: $\phi = 50^\circ$

DNS
DNS-mean base flow
Uniform base flow

Case A: Full source
Sound Pressure Spectra: $\phi = 50^\circ$

DNS
DNS-mean base flow
Uniform base flow

Case B: 128 modes
Sound Pressure Spectra: $\phi = 50^\circ$

DNS
DNS-mean base flow
Uniform base flow

Case C: 32 modes
Sound Pressure Spectra: $\phi = 50^\circ$

DNS
DNS-mean base flow
Uniform base flow

Case D: $a'_1 = a_1/2$
Sound Pressure Spectra: $\phi = 50^\circ$

\[
\hat{p}^* / \rho_\infty a_\infty^2 \delta^2
\]

DNS
DNS-mean base flow
Uniform base flow

Case E: $a'_{1,2} = a_{1,2}/2$
Sound Pressure Spectra: $\phi = 50^\circ$

DNS
Parallel base flow
Locally parallel flow
Sound Spectra: $\phi = 50^\circ$

None clearly more robust at $50^\circ$
Sound Pressure Spectra: $\phi = 130^\circ$

\[ \frac{\hat{p}^*}{\rho^2 a^2 \delta^2} \]

\[ \omega \delta / a_\infty \]

**DNS**
- DNS-mean base flow
- Uniform base flow
Sound Pressure Spectra: $\phi = 130^\circ$

- DNS
- DNS-mean base flow
- Uniform base flow

Case A: Full source
Sound Pressure Spectra: $\phi = 130^\circ$

DNS
DNS-mean base flow
Uniform base flow

Case B: 128 modes
Sound Pressure Spectra: $\phi = 130^\circ$

DNS
DNS-mean base flow
Uniform base flow

Case C: 32 modes
Sound Pressure Spectra: $\phi = 130^\circ$

- DNS
- DNS-mean base flow
- Uniform base flow

Case D: $a'_1 = a_1/2$
Sound Pressure Spectra: \( \phi = 130^\circ \)

DNS
DNs-mean base flow
Uniform base flow

Case E: \( a'_{1,2} = a_{1,2}/2 \)
Sound Spectra: $\phi = 130^\circ$

DNS
Parallel base flow
Locally parallel flow
Sound Spectra: $\phi = 130^\circ$

Lighthill-like analogy pathologically sensitive to $S$ errors
Sound Field Visualization

- **DNS**
- DNS-mean base flow, $a_1/2$
- DNS-mean base flow, 128 modes
- Uniform base flow, $a_1/2$
Filter flow variables:
\[ \beta \hat{f}_{i-2} + \alpha \hat{f}_{i-1} + \hat{f}_i + \alpha \hat{f}_{i+1} + \beta \hat{f}_{i+2} = \sum_{j=0}^{N} a_j (f_{i-j} + f_{i+j}) \]

Transfer function: 
\[ T(k \Delta x) = \frac{\sum_{n=0}^{N} a_n \cos(nk \Delta x)}{1 + 2\alpha \cos(k \Delta x) + 2\beta \cos(2k \Delta x)} \]

\[ T(k_1 \Delta x) = s_1 \]
\[ T(k_2 \Delta x) = s_2 \]

\( \beta = 0.16645, \quad \alpha = -0.66645 \)

\( a_0 = \frac{1}{4} (2 + 3\alpha), \quad a_1 = \frac{1}{16} (9 + 16\alpha + 10\beta), \)

\( a_2 = \frac{1}{4} (\alpha + 4\beta), \quad a_3 = \frac{1}{16} (6\beta - 1). \)
\( T(\lambda) = 0.5 \quad \text{for} \quad \lambda = 15.7\delta_\omega \)

- Filter applied directly to DNS data
- Filtered data used to compute means, correlations and sources
Sound Pressure Spectra

\[
\frac{\hat{p}^*}{\rho_\infty a_\infty^2 \delta^2} = \frac{\omega \delta}{a_\infty} \alpha
\]

- \( \alpha = 50^\circ \) (DNS-mean base flow)
- \( \alpha = 130^\circ \) (Uniform base flow)
Why is the Lighthill-like (uniform flow) so sensitive?

- DNS-mean base flow analogy
- Uniform base flow analogy

Bubbly Structure

Wavy Structure
A Crude Model

- \( a_1(t) = -\sin \omega t \) and \( a_2(t) = \cos \omega t \) ⇒ form suggested by actual POD analysis

- Convected harmonic wave modulated by a Gaussian envelope:
  \[
y_p = e^{-\eta x^2} [a_1(t) \cos kx + a_2(t) \sin kx]
  \]

- Velocity field: \( u_1 = \frac{1}{2}(M_1 - M_2)[\tanh(\sigma(y - y_p)) + 1] + M_2 \)

- Construct \( T_{11} \)

- Solve Lighthill's equation:
  \[
  \left( \nabla^2 - \frac{1}{a_\infty^2} \frac{\partial^2}{\partial t^2} \right) \rho(x, t) = -\frac{1}{a_\infty^2} \frac{\partial^2 \rho u_1 u_1}{\partial x \partial x_1}
  \]

- Solution: \( \rho(x, \omega) = -\frac{1}{4i} \int S(y, \omega) H_0^{(1)}(k_\omega | x - y |) dy \)

- Two cases: (1) \( a_1, a_2 \); (2) \( a_1/2, a_2 \)
$M_1 = 0.9$

$M_2 = 0.2$
Model Source

**Full source**

![Graph of Full source](image1)

**Source with error**

![Graph of Source with error](image2)
Model Source

Numerical simulation

Model

\( \alpha \) in degrees

Intensity

\( \alpha \) in degrees

Intensity
Summary

- Sound constitutes a tiny amount of a flow’s energy
- Defining sound involves splitting the flow solution into source and propagation
  - resulting formulas for prediction
    - predict $U^8$ scaling observed
    - challenging: turbulence complexity, phase velocity,....
  - source simplifications have not significantly improved predictions
  - $\mathcal{N}(\vec{q}) = 0 \Rightarrow \mathcal{L}\vec{q} = \mathcal{S}(\vec{q})$ not unique
Summary

- Assessed choice of $\mathcal{L}\tilde{q} = S(\tilde{q})$ based upon robustness criterion
- Small-scale errors
  - all analogies behaved similarly
  - potential implications for SGS-noise modeling
- Large-scale errors
  - large errors for uniform-flow base flow (Lighthill)
  - analogies with principal shear in $\mathcal{L}$ were similarly robust
  - potential implication for POD-dynamic, PSE models
- The high sensitivity of uniform base flow due to non-compact wavy character
- Homogeneous solutions ($\mathcal{L}\tilde{q} = 0$) did not hinder predictions
Wither Prediction?

- Large-eddy simulation..... we are at the dawn of affordability
Wither Prediction?

- Large-eddy simulation..... we are at the dawn of affordability

- Without engineering insights, there still wont be guidance regarding what to do with predictions.......
Wither Prediction?

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- Without engineering insights, there still wont be guidance regarding what to do with predictions....... next talk