EPTT 2012
SÃO PAULO

A variational framework for flow optimization: an introduction to adjoints

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Generalizations

recap: time-periodic flow

$$\frac{d}{dt}q = L(t)q \qquad\qquad L(t+T) = L(t)$$

period T

with the formal solution $q(t) = A(t)q_0$ initial condition

propagator

from periodicity $A(t+T) = A(t)\ A(T) = A(t)\,C$

monodromy matrix
(mapping over one period)

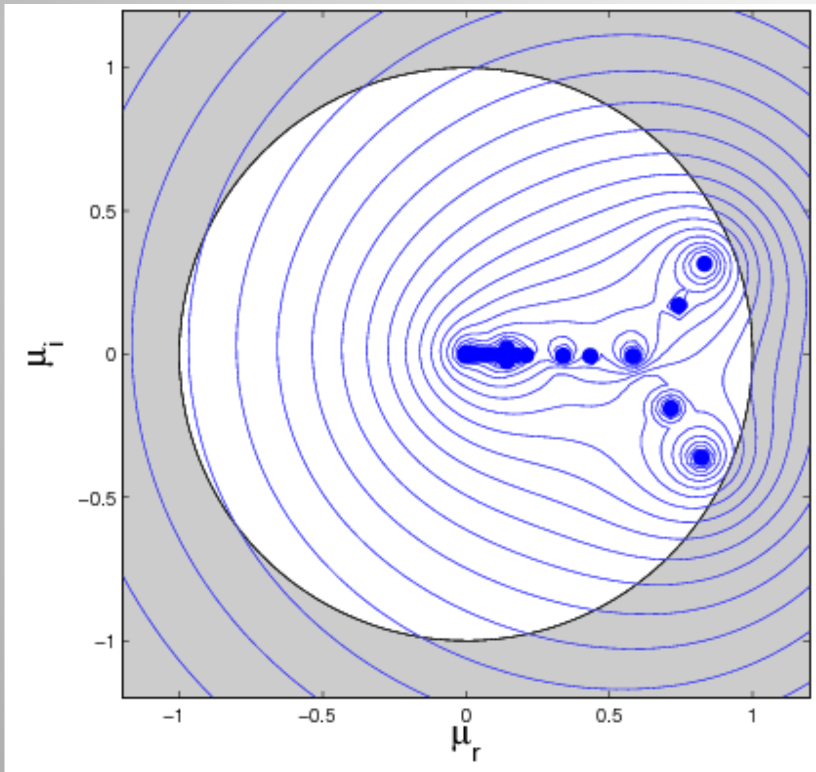
$$q_n = C\ q_{n-1} = C^n\ q_0$$

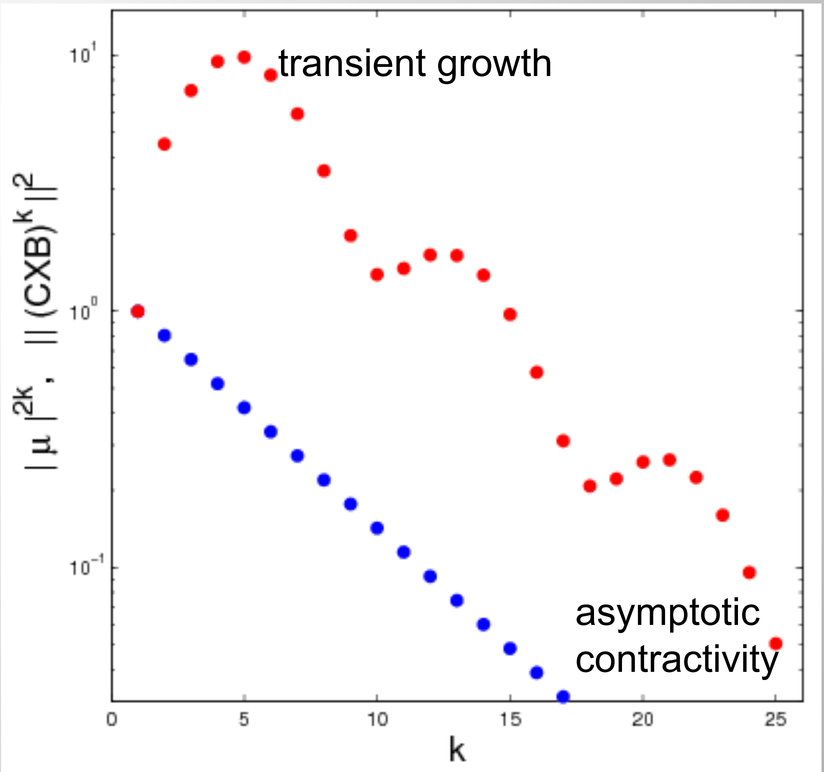
initial state

Generalizations

recap: time-periodic flow

Example: pulsatile channel flow

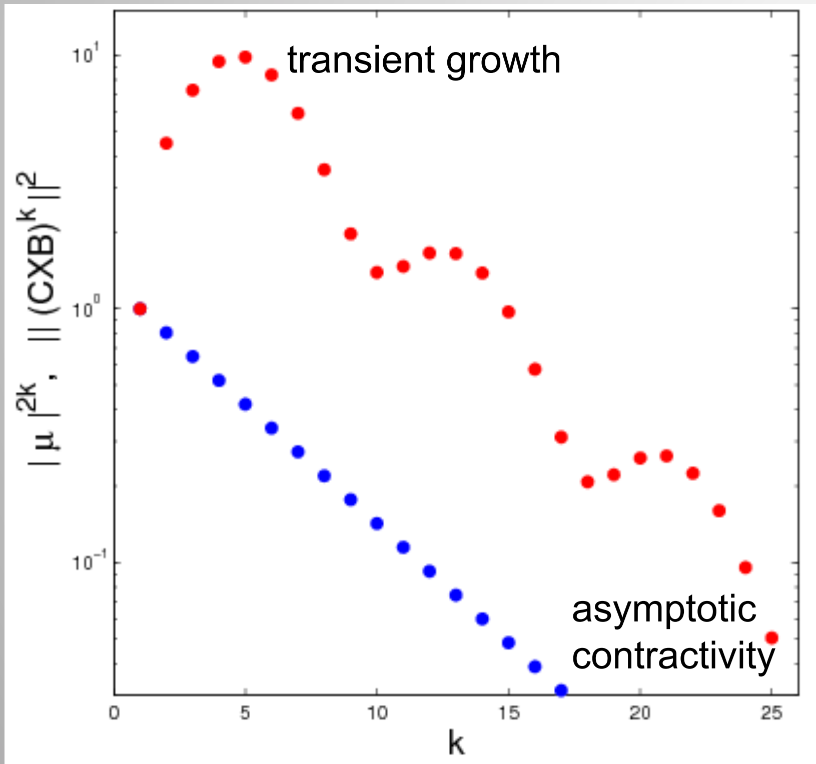


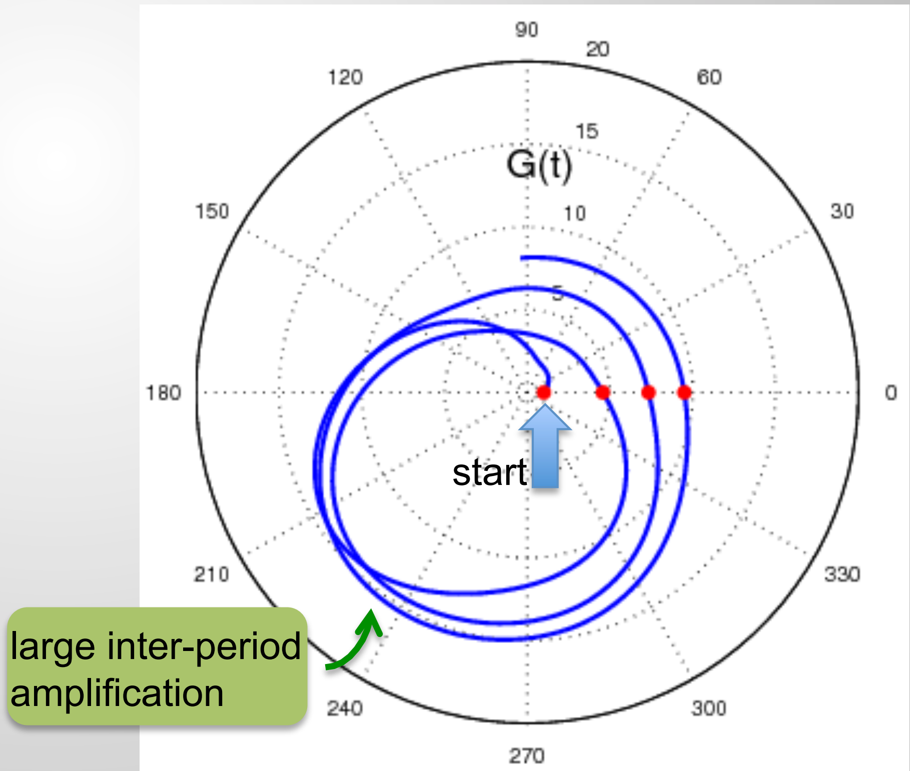


Generalizations

recap: time-periodic flow

Example: pulsatile channel flow





Generalizations

time-periodic and generally time-dependent flow $\Rightarrow$ pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

Can we analyze the amplification of energy between one period, i.e., for a non-periodic system matrix ?

We have

$$\frac{d}{dt} q = L(t) q$$

with the formal solution

$$\underbrace{q(t)}_{\text{final solution}} = \underbrace{A(t)}_{\text{propagator}} \ \underbrace{q_0}_{\text{initial condition}}$$

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

We can formulate the optimal amplification of energy as

$$G(t)^2 = \max_{q_0} \frac{\langle q, q \rangle}{\langle q_0, q_0 \rangle}$$

$$= \max_{q_0} \frac{\langle A(t) q_0, A(t) q_0 \rangle}{\langle q_0, q_0 \rangle}$$

$$= \max_{q_0} \frac{\langle A^H(t) A(t) q_0, q_0 \rangle}{\langle q_0, q_0 \rangle}$$

Generalizations

time-periodic and generally time-dependent flow ➡ pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

$$G(t)^2 = \max_{q_0} \frac{\langle \boxed{A^H(t)A(t)} q_0, q_0 \rangle}{\langle q_0, q_0 \rangle}$$

$A^H A$ is a ***normal*** matrix

- the maximum is achieved for the principal eigenvector of $A^H A$
- the principal eigenvector (and eigenvalue) can be found by power iteration

$$q_0^{(n+1)} = \rho^{(n)} A^H A \; q_0^{(n)}$$

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

$$q_0^{(n+1)} = \rho^{(n)} A^H \boxed{A \; q_0^{(n)}}$$

break the power iteration into two pieces

first step

$$w(t) = A \; q_0^{(n)}$$

propagation of initial condition forward in time

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

$$q_0^{(n+1)} = \rho^{(n)} \boxed{A^H} A \; q_0^{(n)}$$

break the power iteration into two pieces

second step

$$q_0^{(n+1)} = \rho^{(n)} A^H(t) w(t)$$

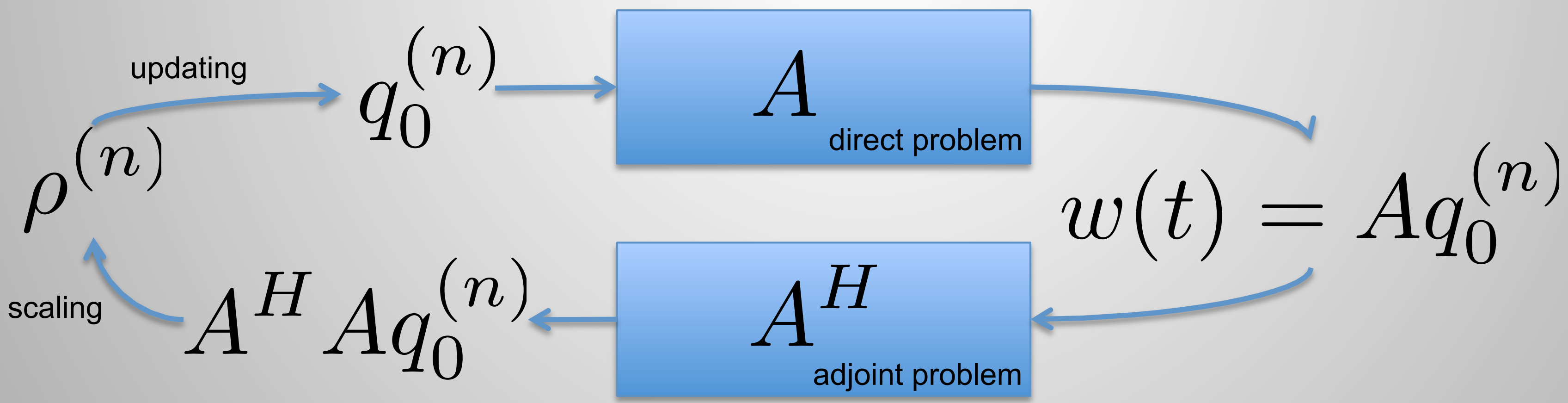
propagation of final condition backward in time

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis, adjoint analysis

Example: pulsatile channel flow

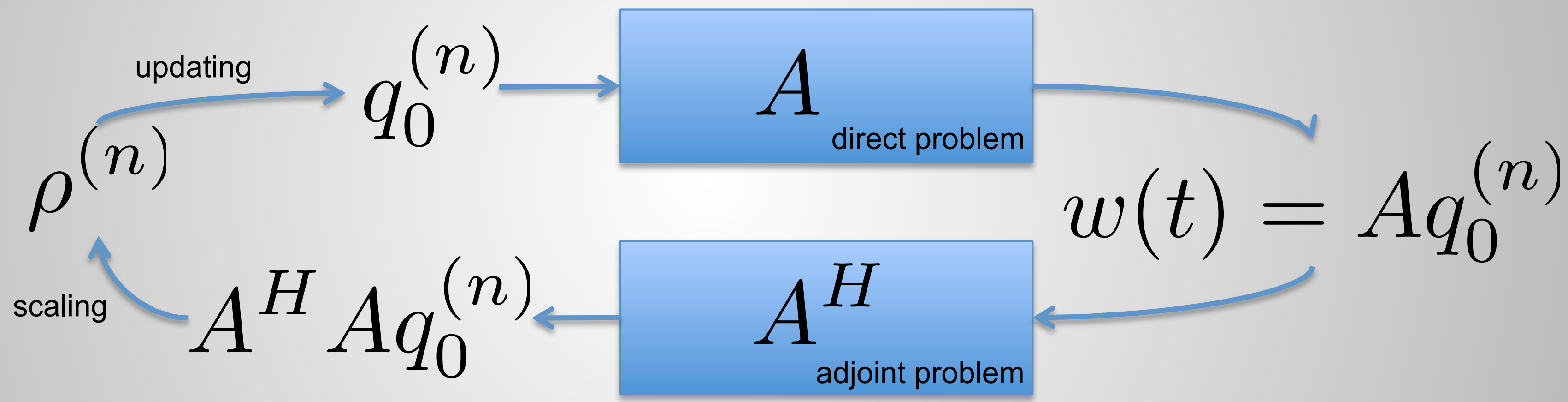
$$q_0^{(n+1)} = \rho^{(n)} A^H A \, q_0^{(n)}$$



Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow



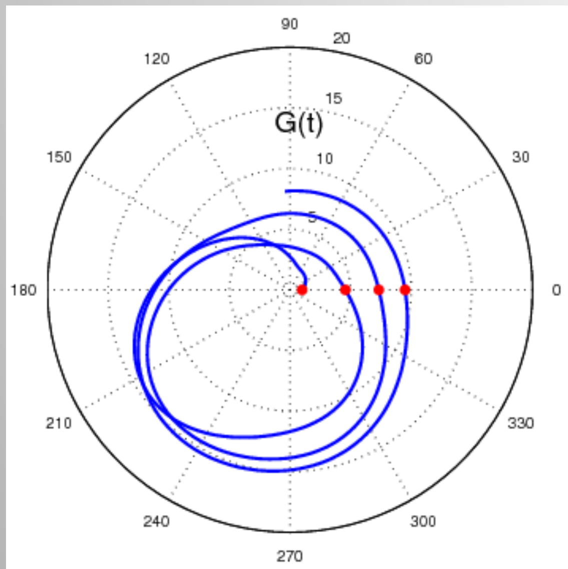
A can be any discretized solution operator. The above technique (adjoint looping) can be applied to general time-dependent stability problems.

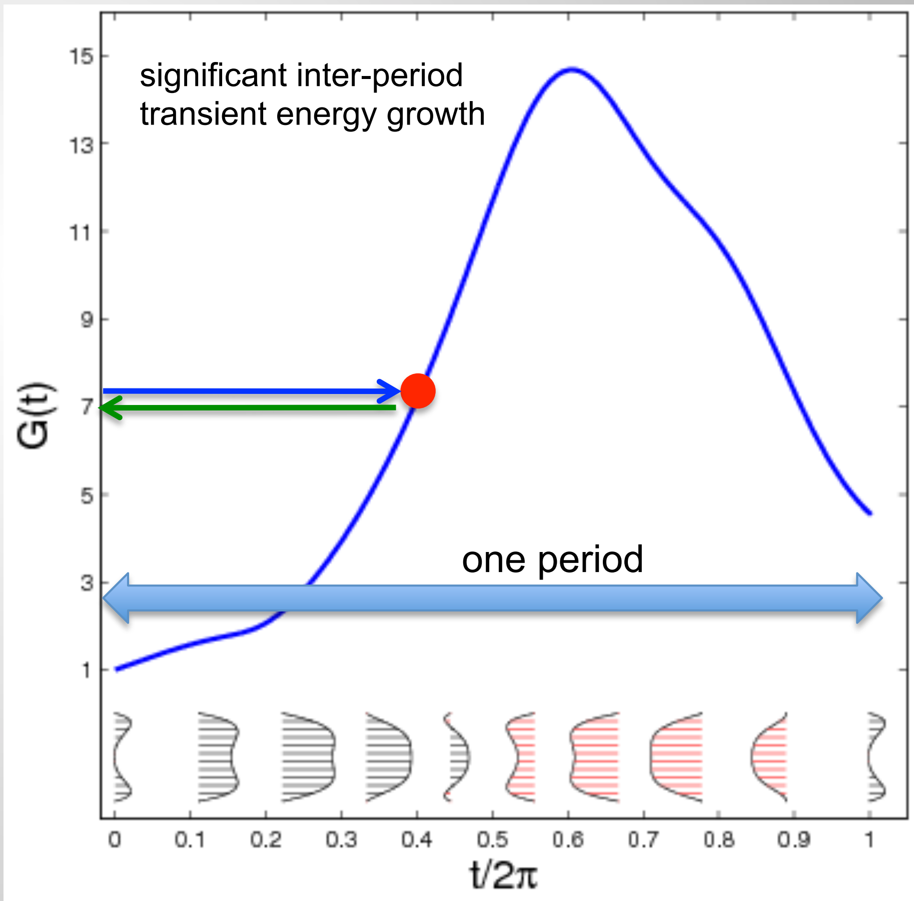
Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: pulsatile channel flow

applying adjoint looping to the pulsatile (inter-period) stability problem

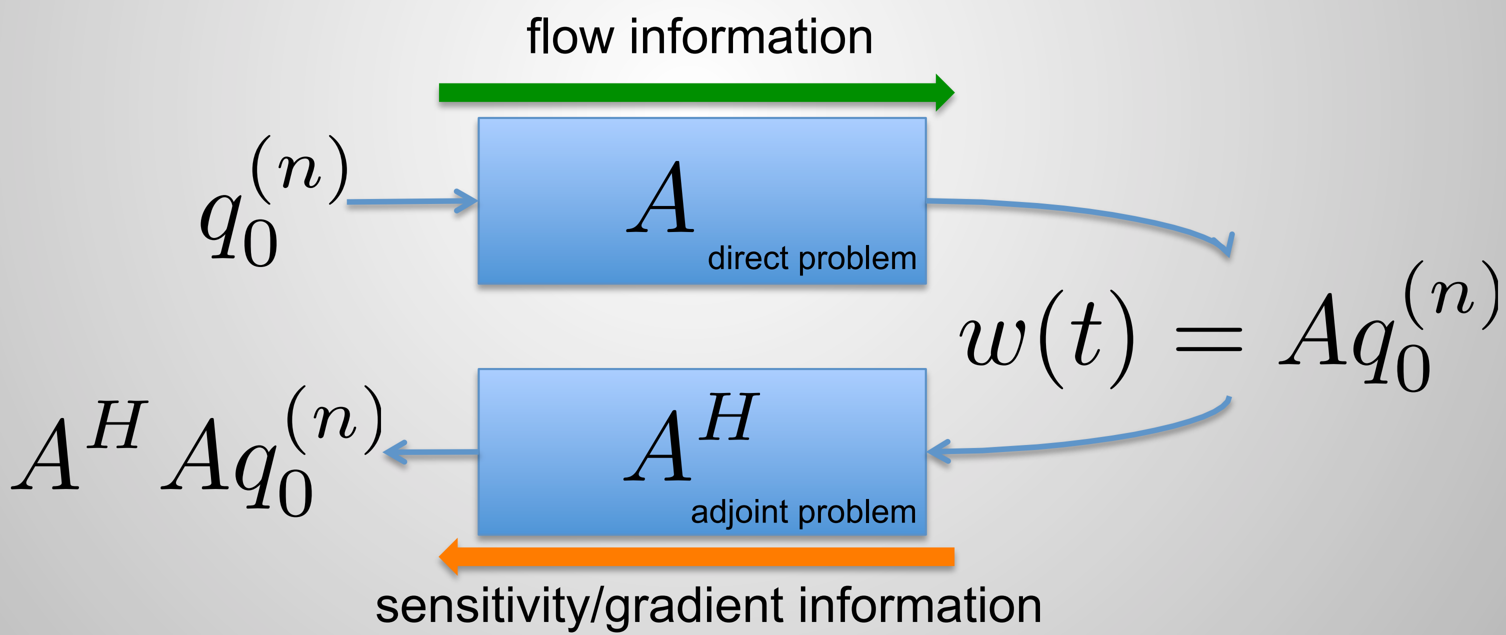




Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis, adjoint analysis

Another look at the direct-adjoint system



$I - A^H A$ correction

Generalizations

time-periodic and generally time-dependent flow $\Rightarrow$ pseudo-Floquet analysis
adjoint analysis

reformulate the optimal growth problem variationally

we wish to optimize

$$J = \frac{\|q\|^2}{\|q_0\|^2} \rightarrow \max$$

subject to the constraint

$$\frac{d}{dt}q - Lq = 0$$

Generalizations

time-periodic and generally time-dependent flow ➡

pseudo-Floquet analysis
adjoint analysis

rather than substituting the constraint directly into the cost functional …

$$J = \frac{\|q\|^2}{\|q_0\|^2} = \frac{\|\exp(tL)q_0\|^2}{\|q_0\|^2} \rightarrow \max$$

$$\frac{d}{dt}q - Lq = 0$$

only valid for LTI systems

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

… we enforce the equation via a Lagrange multiplier $\tilde{q}$

$$J = \frac{\|q\|^2}{\|q_0\|^2} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - L q \right) \right\rangle \rightarrow \max$$

This has the advantage that the solution to the governing equation does not have to be known *explicitly*.

Other constraints (such as initial and boundary conditions) can be added.

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

for an optimum we have to require all first variations of J to be zero

$$J = \frac{\|q\|^2}{\|q_0\|^2} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \to \max$$

$$\frac{\delta J}{\delta \tilde{q}} = 0 \quad \Rightarrow \quad \left\langle \delta\tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle = 0$$

$$\frac{\delta J}{\delta q} = 0 \quad \Rightarrow \quad \left\langle \tilde{q}, \left(\frac{d}{dt} \delta q - L\, \delta q \right) \right\rangle = 0$$

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

for an optimum we have to require all first variations of J to be zero

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Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

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$$\frac{\delta J}{\delta \tilde{q}} = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt} q - Lq = 0}$$

$$\frac{\delta J}{\delta q} = 0 \quad \Rightarrow \quad \left\langle \left(-\frac{d}{dt} \tilde{q} - L^H \tilde{q} \right), \delta q \right\rangle = 0$$

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

for an optimum we have to require all first variations of J to be zero

$$J = \frac{\|q\|^2}{\|q_0\|^2} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \rightarrow \max$$

$$\frac{\delta J}{\delta \tilde{q}} = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt} q - Lq = 0}$$ direct problem

$$\frac{\delta J}{\delta q} = 0 \quad \Rightarrow \quad \boxed{-\frac{d}{dt} \tilde{q} - L^H \tilde{q} = 0}$$ adjoint problem

KKT-condition

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

adjoint variables can be interpreted as sensitivities

$$J = \mathrm{obj} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \rightarrow \max$$

let us add an external body force to the governing equations

$$\frac{d}{dt} q - Lq = f$$

external force

$$\delta J = -\langle \tilde{q}, \delta f \rangle$$

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

adjoint variables can be interpreted as sensitivities

$$J = \mathrm{obj} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - Lq \right) \right\rangle \rightarrow \max$$

let us add an external body force to the governing equations

$$\frac{d}{dt} q - Lq = f$$

external force

$$\boxed{\nabla_f J = -\tilde{q}}$$

sensitivity to external body force

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: which adjoint variable measures the sensitivity to a mass source/sink?

$$J = \mathrm{obj} - \underbrace{\langle \tilde{\mathbf{u}}, NS(\mathbf{u}) \rangle}_{\text{enforcing momentum conservation}} - \underbrace{\langle \xi, \nabla \cdot \mathbf{u} \rangle}_{\text{enforcing mass conservation}}$$

$$-\langle \xi, \nabla \cdot \mathbf{u} \rangle \xrightarrow[\text{by parts}]{\text{integration}} \langle \nabla \xi, \delta \mathbf{u} \rangle$$

ξ is the adjoint pressure

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

Example: which adjoint variable measures the sensitivity to a mass source/sink?

$$J = \mathrm{obj} - \langle \tilde{\mathbf{u}}, NS(\mathbf{u}) \rangle - \langle \xi, \nabla \cdot \mathbf{u} \rangle$$

enforcing momentum conservation

enforcing mass conservation

assuming a mass source/sink

$$\nabla \cdot \mathbf{u} = Q$$

$$\delta J = \langle \xi, \delta Q \rangle$$

adjoint pressure = sensitivity to a mass source/sink

Generalizations

time-periodic and generally time-dependent flow → pseudo-Floquet analysis
adjoint analysis

for the incompressible Navier-Stokes equations

⇦ forcing | sensitivity ⇨

$$\frac{\partial \mathbf{u}}{\partial t} + \mathrm{advdiff}(\mathbf{U}, \mathbf{u}) + \nabla p = \mathbf{F} \qquad \nabla_{\mathbf{F}} J = \tilde{\mathbf{u}}$$

$$\nabla \cdot \mathbf{u} = Q \qquad \nabla_Q J = \tilde{p}$$

$$\mathbf{u} = \mathbf{u}_w \quad \text{on} \quad y = 0 \qquad \nabla_{\mathbf{u}_w} J = \tilde{\sigma}\big|_w$$

Generalizations

Sensitivity to internal changes
(changes of specific eigenvalues with respect to parameter variations)

general formulation

$$A(p)q = \lambda B q$$

p Reynolds number Re
wave number α, β
base-flow $U(y)$

perturbation expansion

$$(A + \delta A)(q + \delta q) = (\lambda + \delta\lambda) B (q + \delta q)$$

Generalizations

Sensitivity to internal changes
(changes of specific eigenvalues with respect to parameter variations)

general formulation

$$(A+\delta A)(q+\delta q) = (\lambda+\delta\lambda)B(q+\delta q)$$

$$\underbrace{(A-\lambda B)q}_{0} + (A-\lambda B)\delta q + (\delta A - \delta\lambda B)q + \underbrace{(\delta A - \delta\lambda B)\delta q}_{\approx 0 \text{ (higher order)}} = 0$$

Generalizations

Sensitivity to internal changes
(changes of specific eigenvalues with respect to parameter variations)

general formulation

$$(A + \delta A)(q + \delta q) = (\lambda + \delta\lambda)B(q + \delta q)$$

$$(A - \lambda B)\delta q + (\delta A - \delta\lambda B)q \approx 0$$

Generalizations

Sensitivity to internal changes
(changes of specific eigenvalues with respect to parameter variations)

general formulation

$$(A - \lambda B)\delta q + (\delta A - \delta\lambda B)q \approx 0$$

use adjoint $\quad q^+(A - \lambda B) = 0 \quad \Leftrightarrow \quad (A^+ - \lambda^* B^+)q^+ = 0$

$$q^+(A \cancelto{0}{- \lambda B})\delta q + q^+(\delta A - \delta\lambda B)q \approx 0$$

Generalizations

Sensitivity to internal changes
(changes of specific eigenvalues with respect to parameter variations)

general formulation

$$A(p)q = \lambda Bq$$

perturbation expansion

$$\boxed{\delta\lambda = \frac{q^+ \delta A q}{q^+ B q}}$$

gradient

$$\boxed{\nabla_p \lambda = \frac{q^+ \nabla_p A q}{q^+ B q}}$$

Generalizations

Example: sensitivity to a scalar parameter

complex Ginzburg-Landau $$u_t = \underbrace{\left(-\nu\partial_x + \gamma\partial_{xx} + \mu(x)\right)}_{A} u$$

$$\nu = U + 2ic_u$$

$$\nabla_U A = -\partial_x$$

eigenvalue sensitivity $$\nabla_U \lambda = \tilde{u}^+ \nabla_U A \tilde{u}$$

$$= -\tilde{u}^+ \partial_x \tilde{u}$$

$$A\tilde{u} = \lambda\tilde{u}$$

$$A^+\tilde{u}^+ = \lambda^*\tilde{u}^+$$

$$\lambda = \sigma + i\omega$$

Generalizations

Example: sensitivity to a scalar parameter

complex Ginzburg-Landau

$$u_t = \underbrace{\left(-\nu\partial_x + \gamma\partial_{xx} + \mu(x)\right)}_{A} u$$

$$\nu = U + 2ic_u$$

$$\nabla_U A = -\partial_x$$

eigenvalue sensitivity

$$\nabla_U \lambda = \tilde{u}^+ \nabla_U A \tilde{u}$$

$$A\tilde{u} = \lambda\tilde{u}$$

$$A^+\tilde{u}^+ = \lambda^*\tilde{u}^+$$

$$\lambda = \sigma + i\omega$$

$$\nabla_U \sigma = \mathrm{Real}(\nabla_U \lambda)$$

sensitivity of growth rate

$$\nabla_U \omega = \mathrm{Imag}(\nabla_U \lambda)$$

sensitivity of frequency

Generalizations

Sensitivity to internal changes
(changes of specific eigenvalues with respect to parameter variations)

Example: choose base flow profile as control variable

p Reynolds number Re
wave number α, β
$\boxed{\text{base-flow } U(y)}$

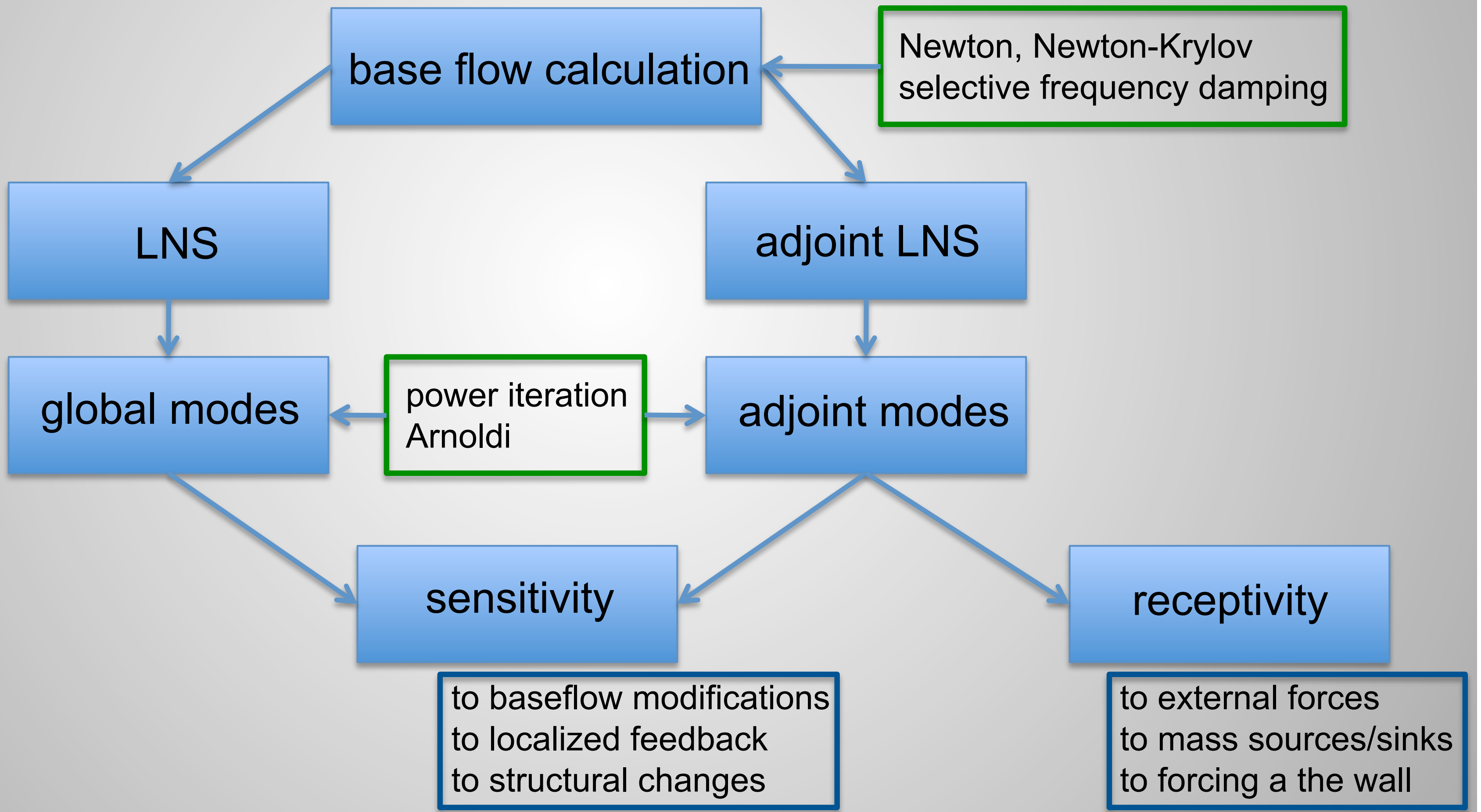
$$\nabla_{\mathbf{U}} \lambda = -(\nabla \mathbf{u})^H \tilde{\mathbf{u}} + \nabla \tilde{\mathbf{u}} \cdot \mathbf{u}^*$$

relate mean flow modification to small control forces

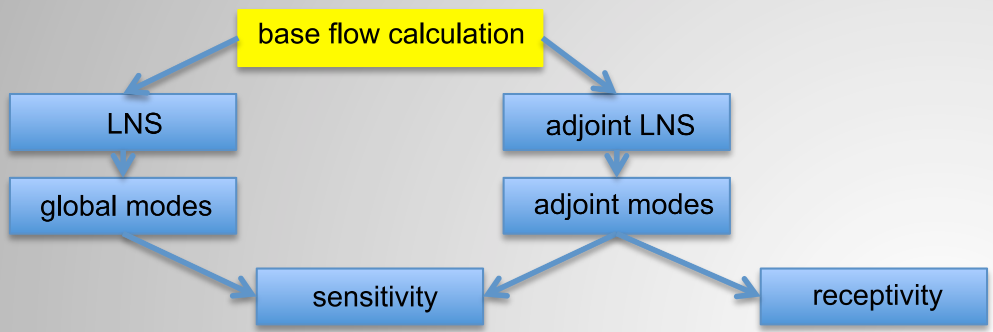
delay onset of instabilities to higher Reynolds numbers; increase stability margins

Generalizations

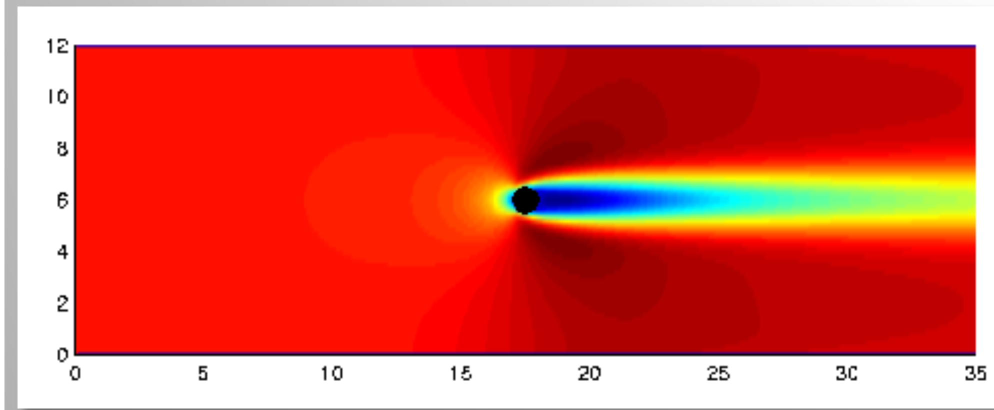
Flow chart for sensitivity/receptivity analysis



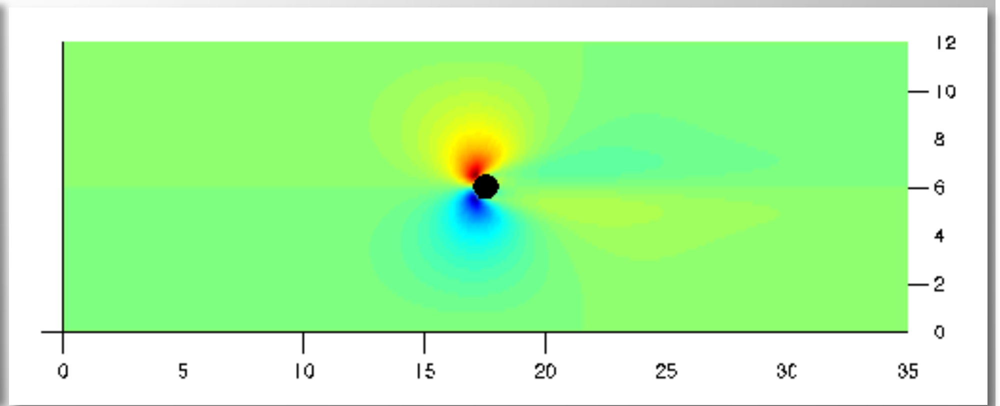
Generalizations



Example: flow around a cylinder

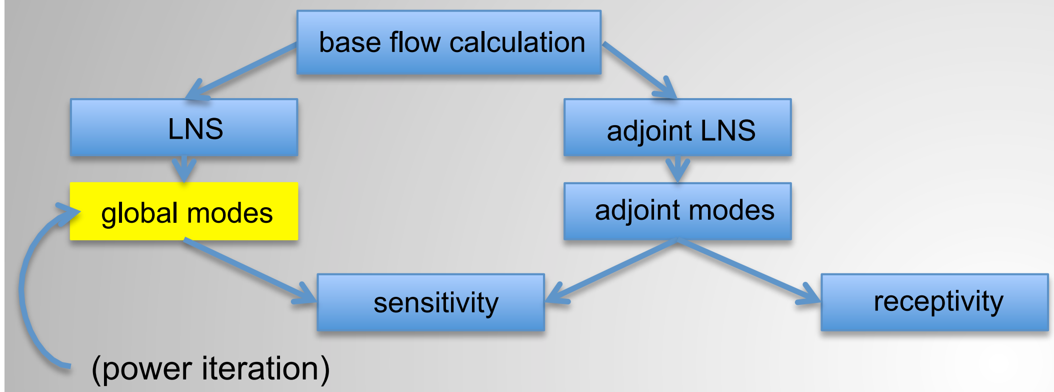


U_{base}



V_{base}

Generalizations



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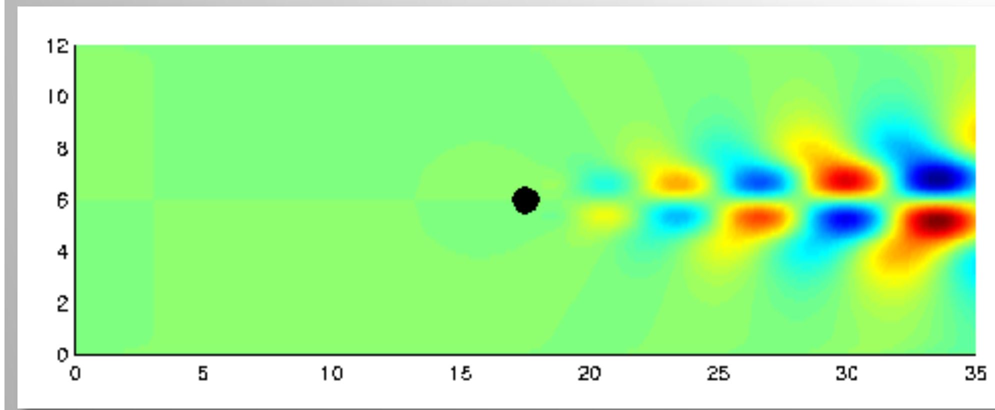
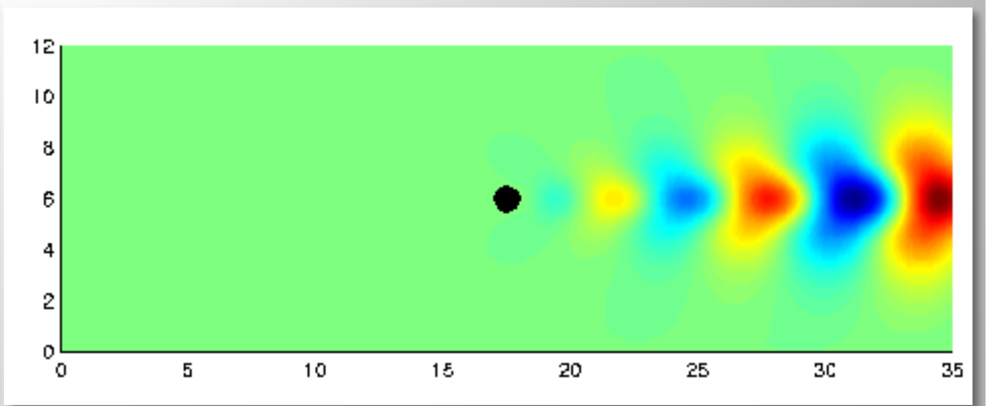


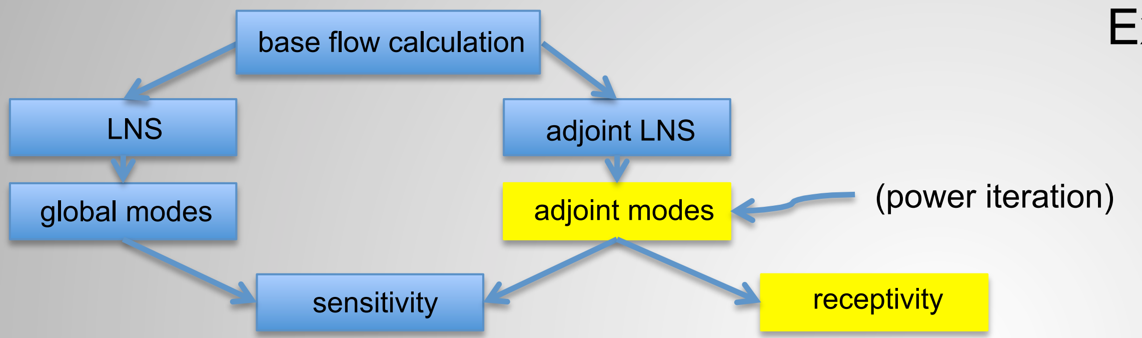
$$\tilde{u}_{\mathrm{direct}}$$

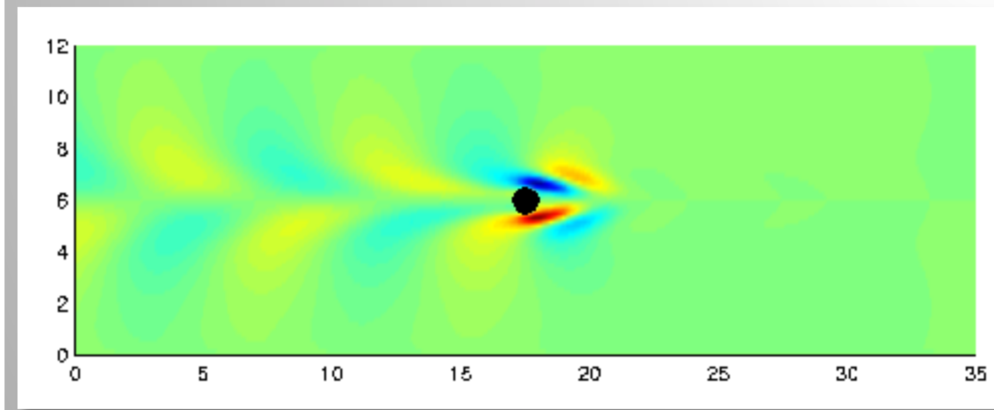


$$\tilde{v}_{\mathrm{direct}}$$

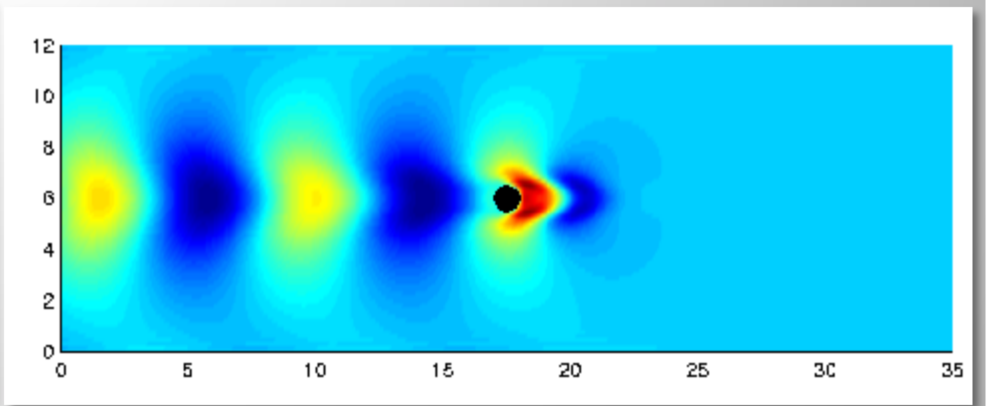
Generalizations

Example: flow around a cylinder



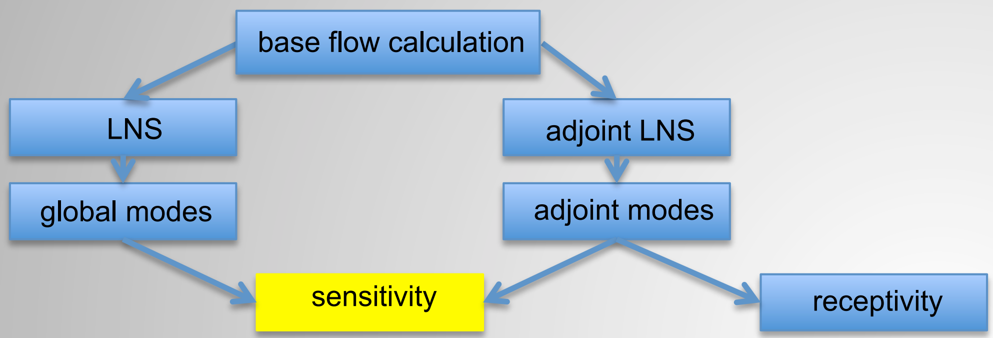


$\tilde{u}_{\text{adjoint}}$



$\tilde{v}_{\text{adjoint}}$

Generalizations



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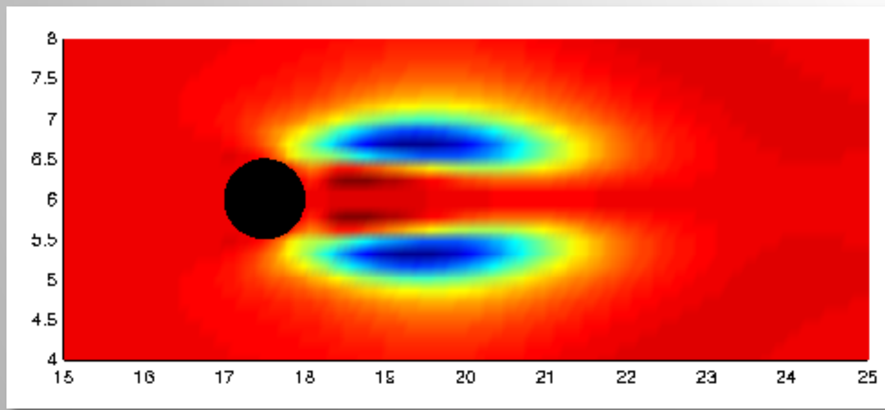
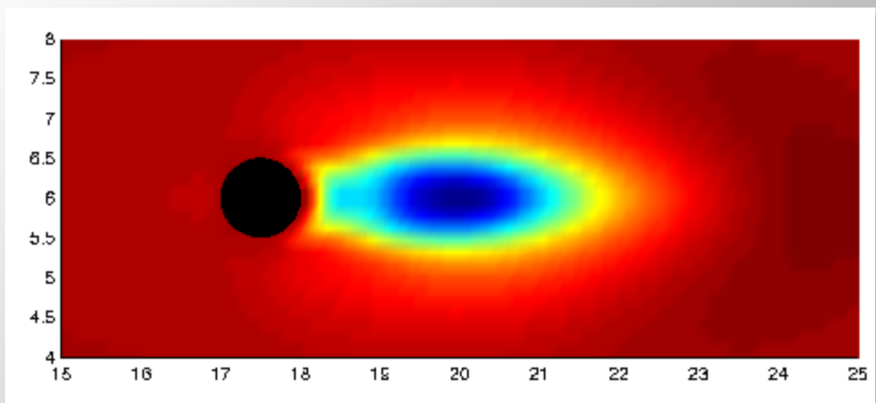
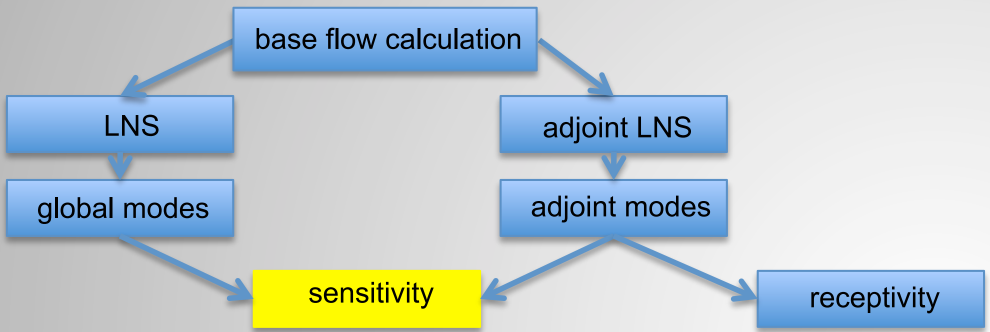


$u_{\text{wavemaker}}$



$v_{\text{wavemaker}}$

Generalizations



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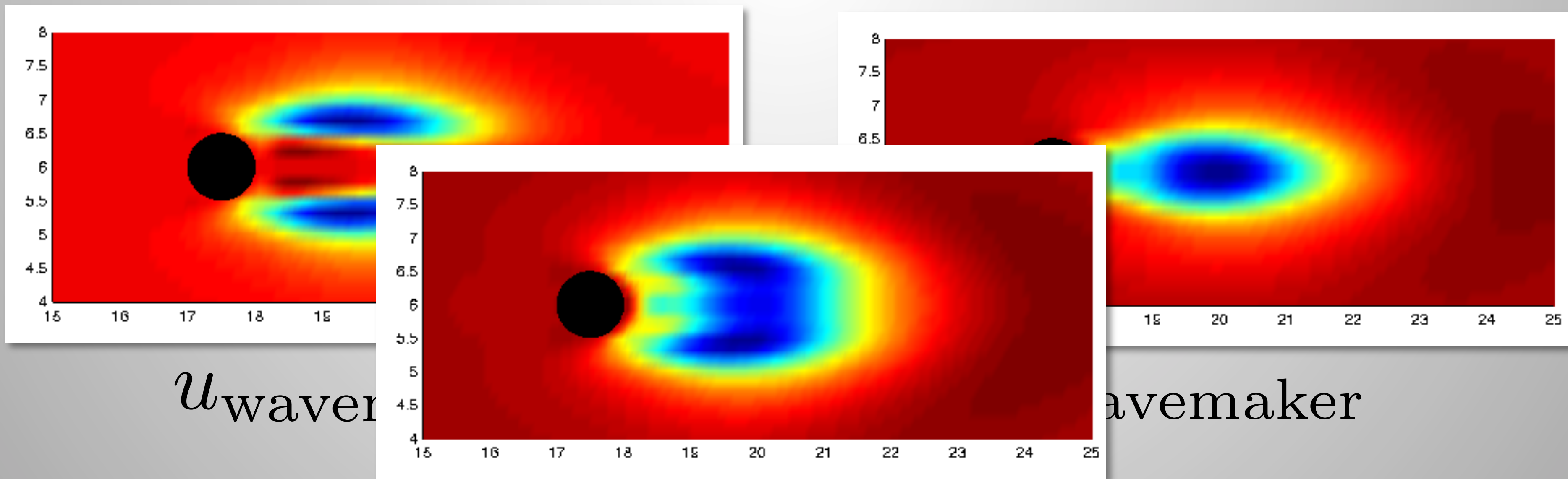
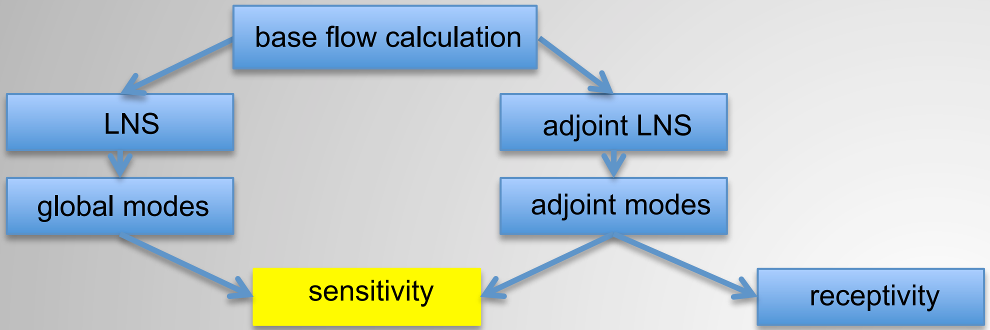


u_{waver} ... avemaker

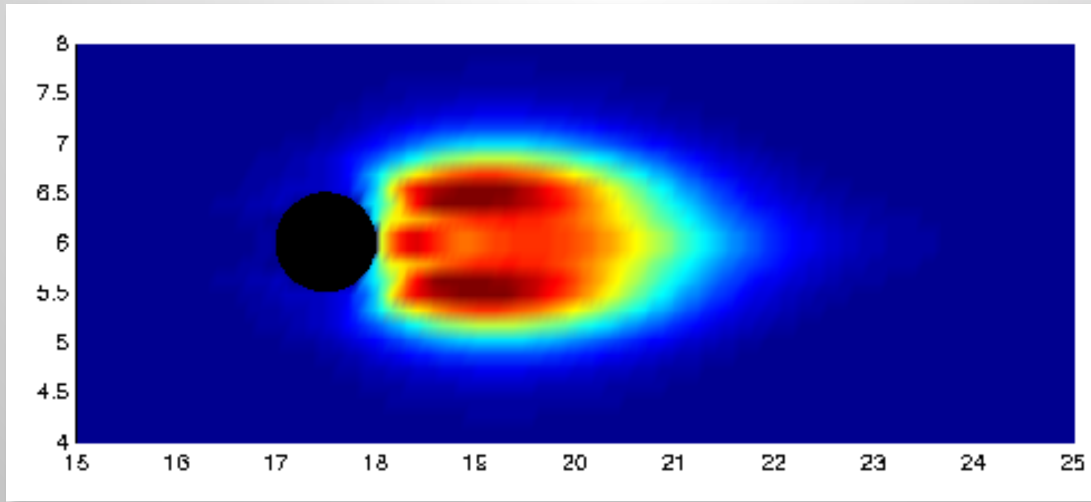
sensitivity to localized spatial feedback

Generalizations



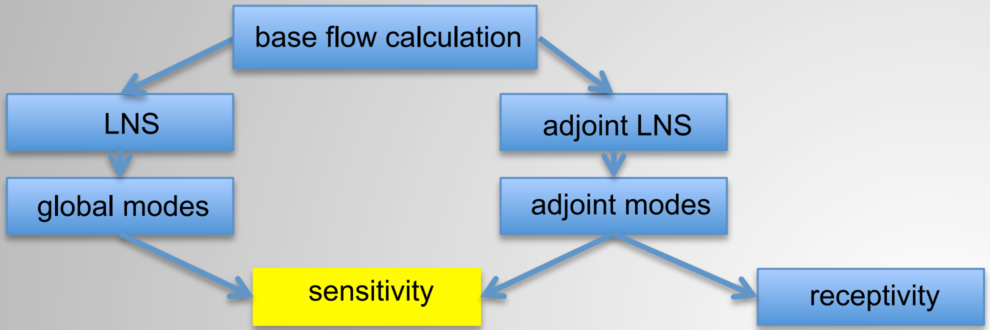
Example: flow around a cylinder

$$\nabla_{\mathbf{U}}\lambda = -(\nabla \mathbf{u})^{H}\tilde{\mathbf{u}} + \nabla\tilde{\mathbf{u}} \cdot \mathbf{u}^{*}$$



structural sensitivity

Generalizations



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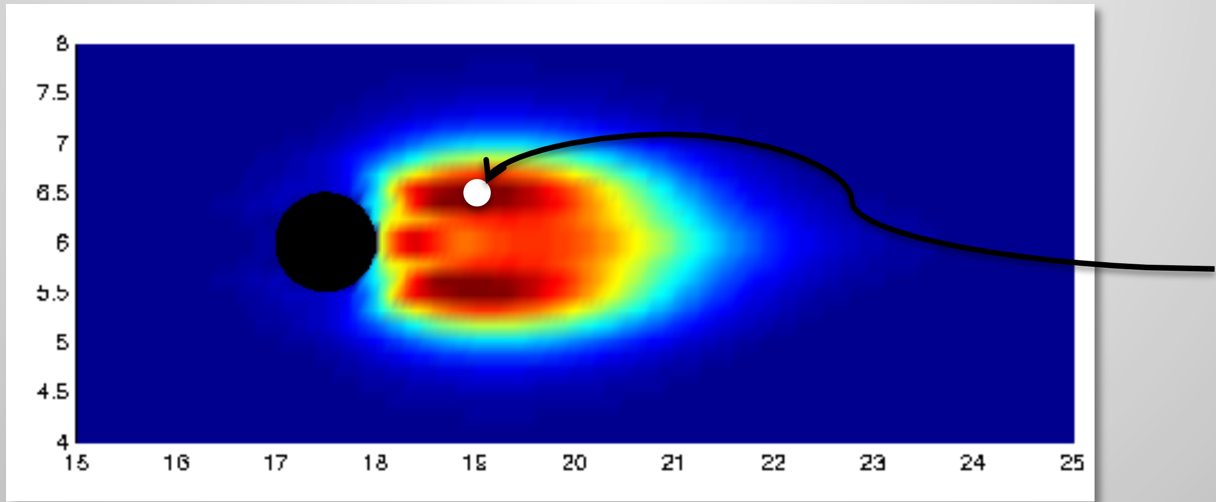


place a small cylinder here to increase Re_c to 70

structural sensitivity

Generalizations

What if the operator perturbation is stochastic ?

$$\frac{d}{dt}q - Lq = 0$$

$$L(t) = L_S + \epsilon\mu(t)S$$

statistically steady part

uncertain part

stochastic process $$d\mu = -\nu\mu dt + dW$$

ν ~ auto-correlation time

Generalizations

What if the operator perturbation is stochastic ?

We have to describe the solution statistically: propagation of covariance (second-order moments)

$$K = \mathcal{E}(qq^H)$$

evolution equation for the covariance matrix (expansion of propagator A(t))

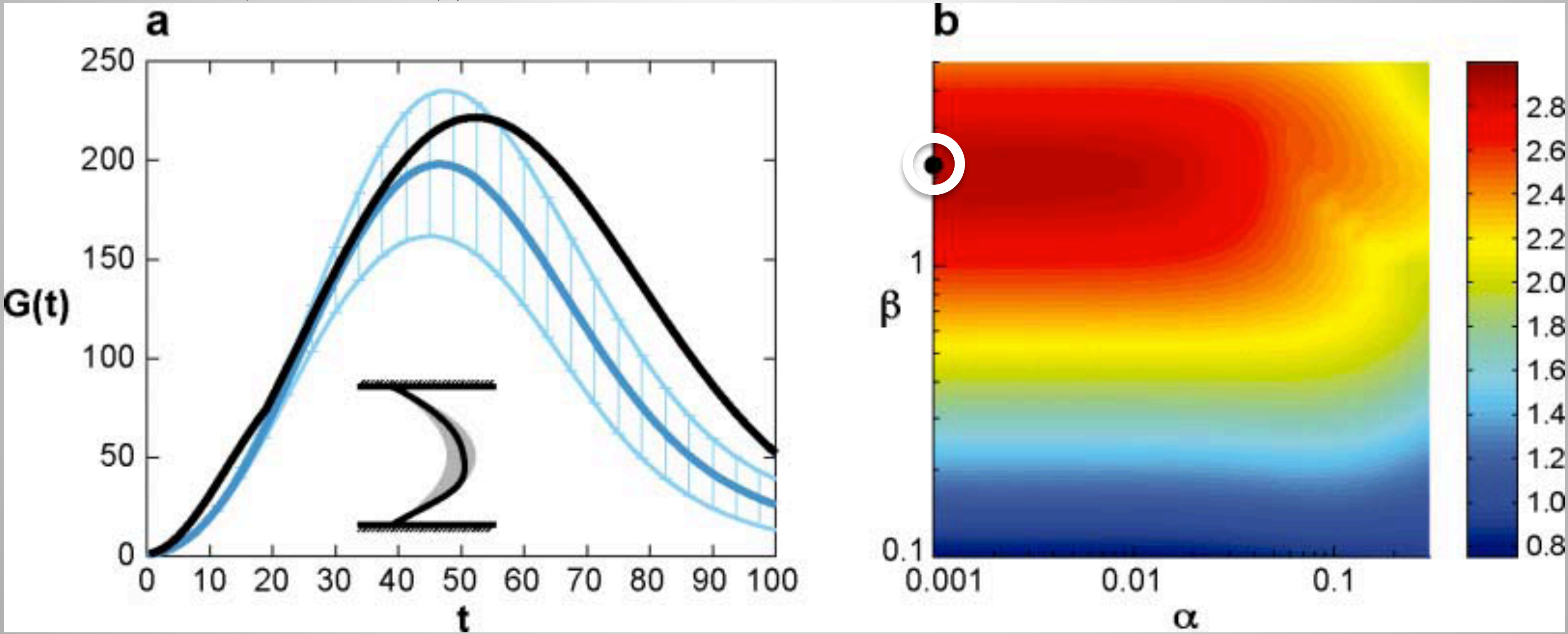
$$\boxed{\frac{d}{dt}K = (L_S + \epsilon^2 SD)K + K(L_S + \epsilon^2 SD)^H + \epsilon^2(SKD + DKS^H)}$$

with $\quad D = \int_0^t \exp(\tau L_s) S \exp(-\tau L_S) \exp(-\nu\tau)\ d\tau$

Generalizations

stochastic channel flow (perturbed base flow profile) $\nu = 1/5 \quad \epsilon = 0.2$

$Re = 2000, \alpha = 0.2, \beta = 2$



Generalizations

nonlinear perturbation dynamics → adjoint analysis with check-pointing

the variational formulation also allows us to add ***nonlinear*** constraints to the cost functional

$$J = \mathrm{obj} - \left\langle \tilde{q}, \left(\frac{d}{dt} q - N(q) \right) \right\rangle \to \max$$

nonlinear Navier-Stokes equations

How does this affect the adjoint looping ?

Generalizations

nonlinear perturbation dynamics → adjoint analysis with check-pointing

Example: nonlinear advective terms

$$\langle \tilde{\mathbf{u}}, \mathbf{u}\nabla\mathbf{u} \rangle \xrightarrow{\text{first variation}} \langle -\mathbf{u}\nabla\tilde{\mathbf{u}}, \delta\mathbf{u} \rangle$$

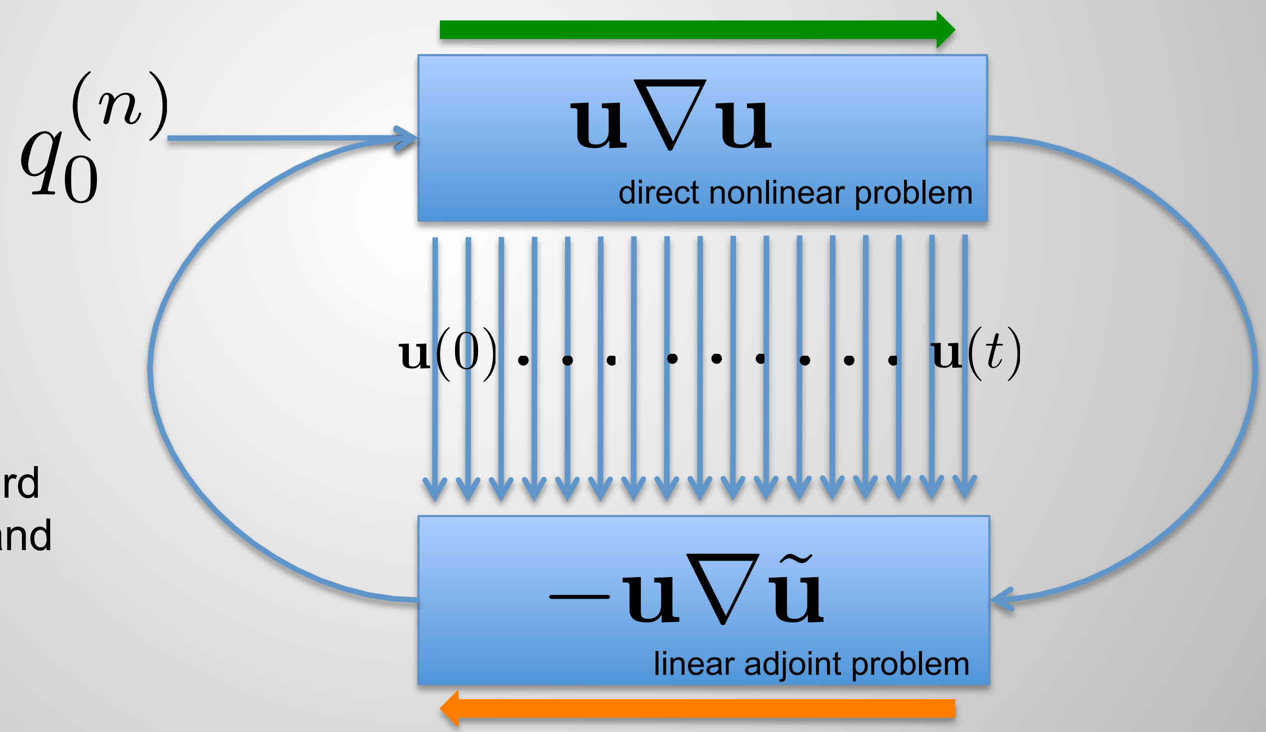
We have **direct** terms appearing in the **adjoint** equation.

Adjoint equation is a variable-coefficient linear equation.

Generalizations

nonlinear perturbation dynamics

adjoint analysis
with check-pointing

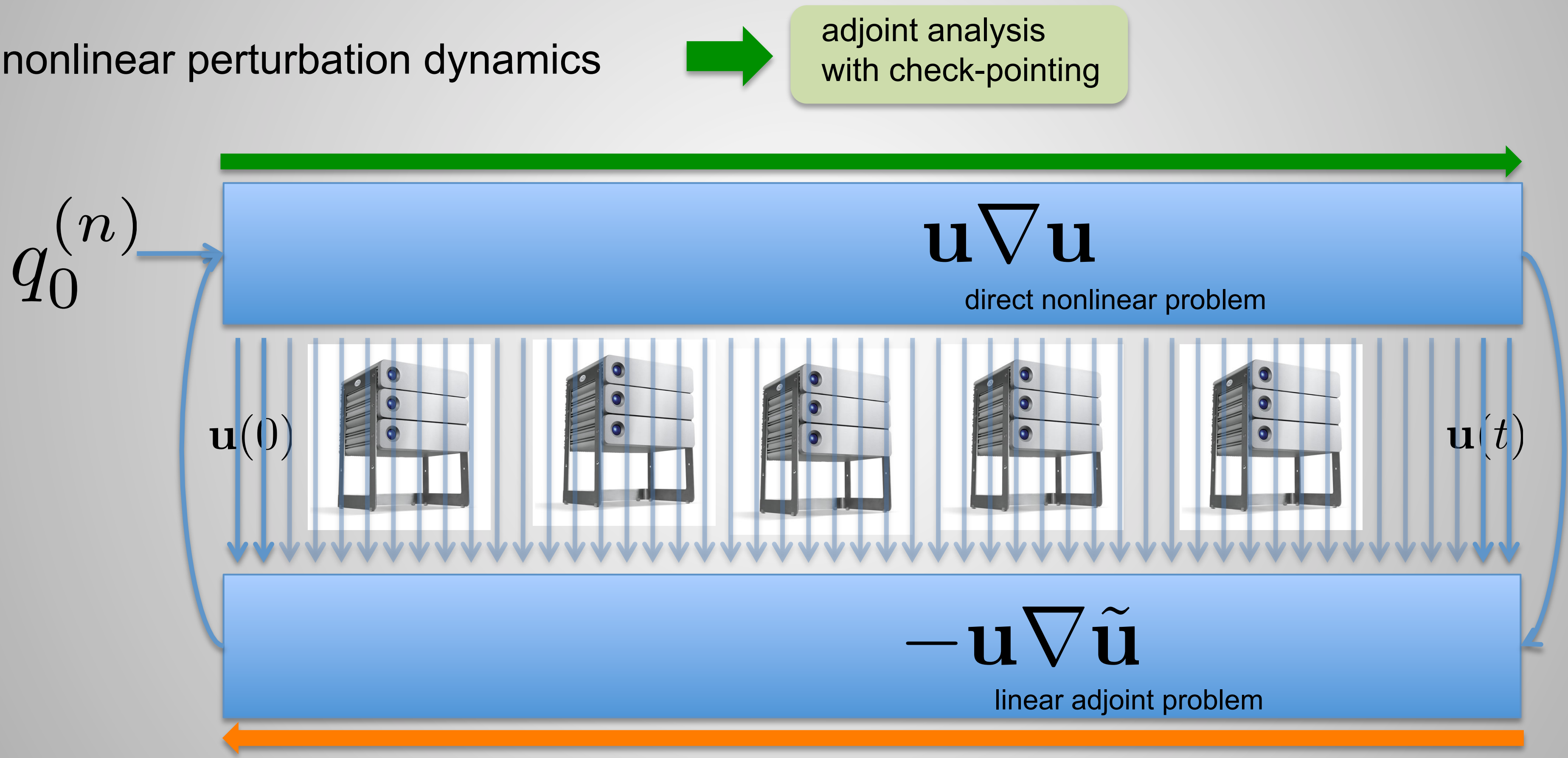


checkpointing

the flow fields at the forward sweep have to be saved and injected into the backward sweep

Generalizations

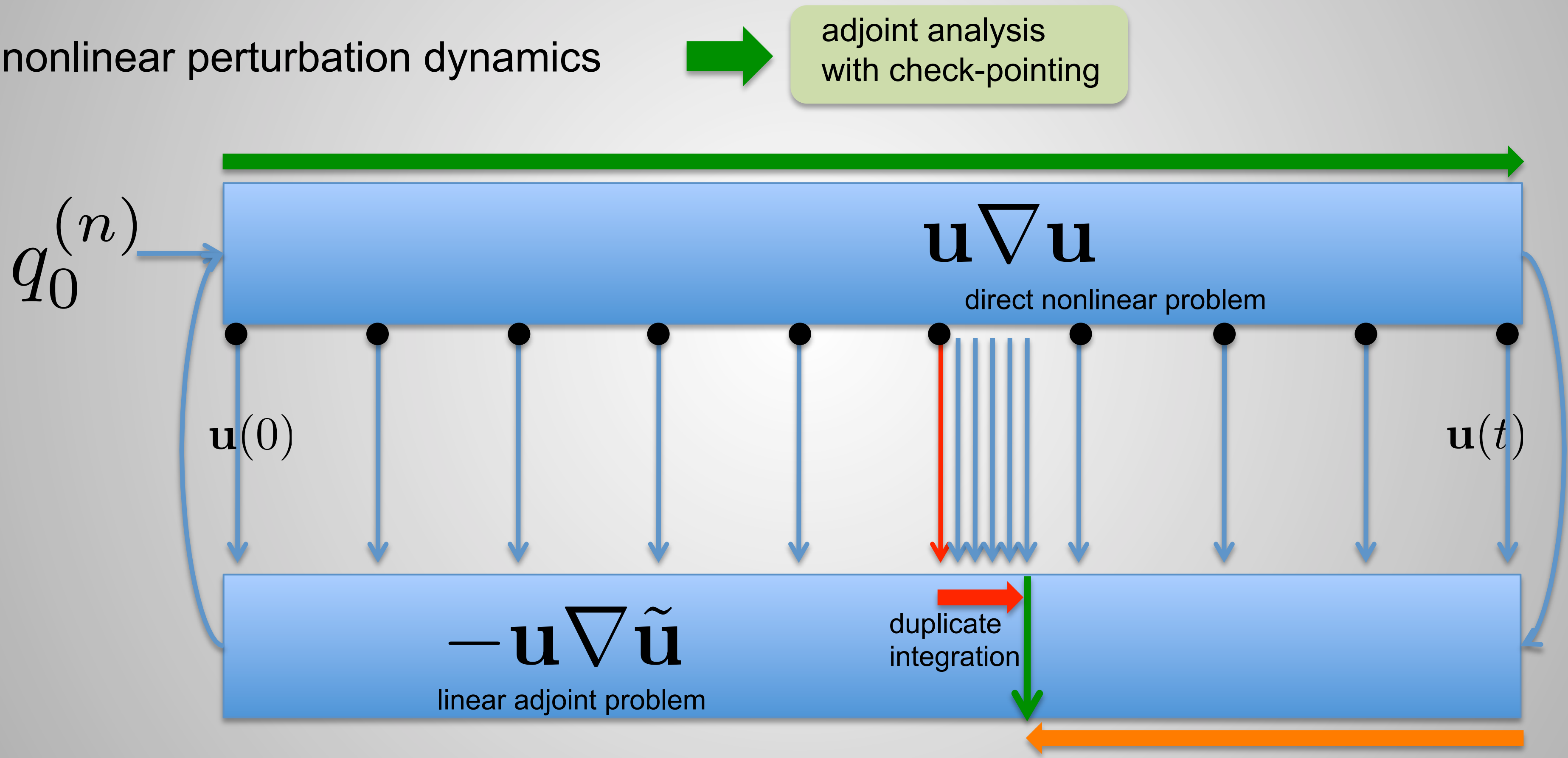
nonlinear perturbation dynamics



For long-time integrations and high-dimensional problems we quickly reach the limits of storage devices.

Generalizations

nonlinear perturbation dynamics

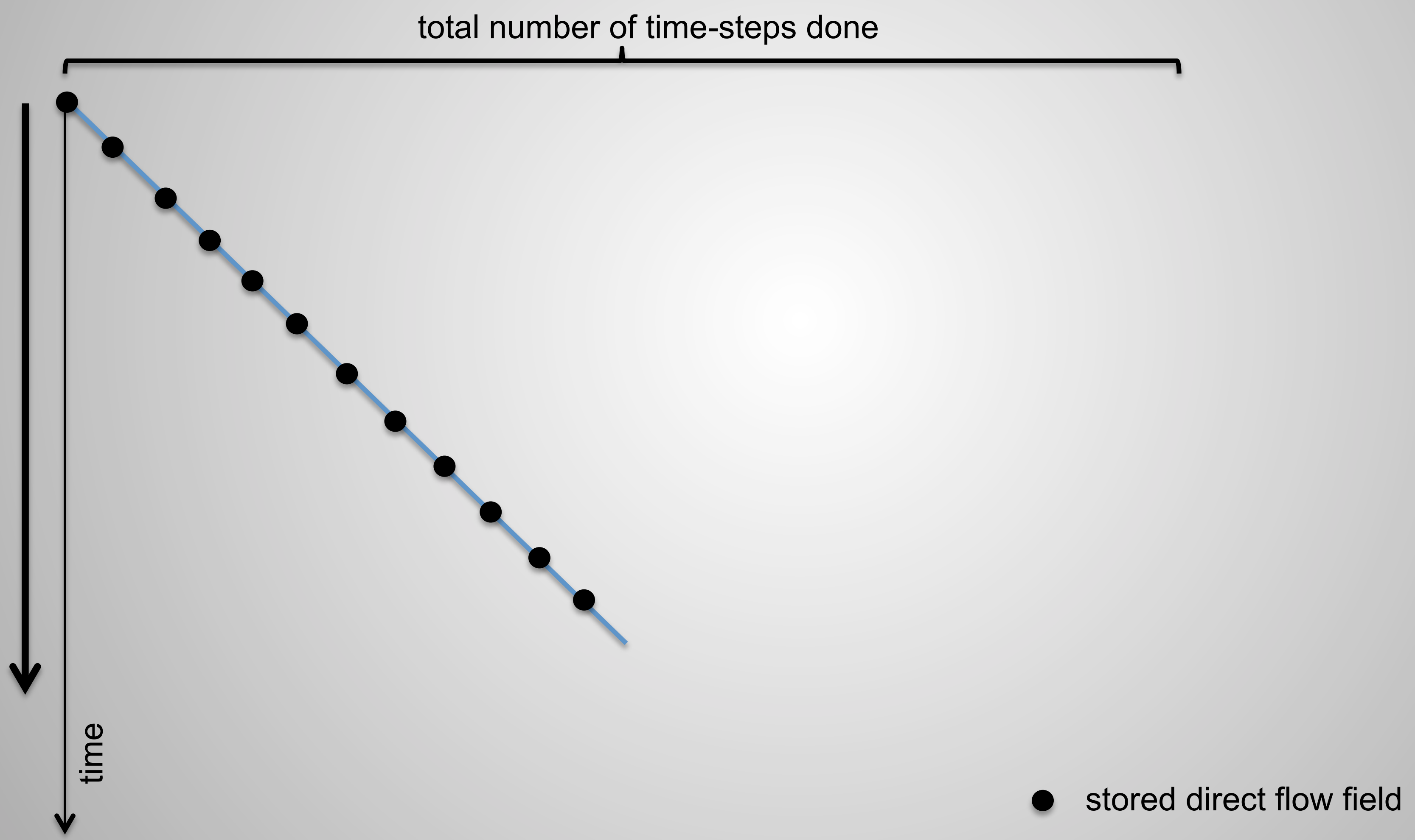


optimized checkpointing

store flow fields at coarse intervals ●●● and use as initial conditions for repeated forward integrations

Generalizations

no checkpointing (store everything)



Generalizations

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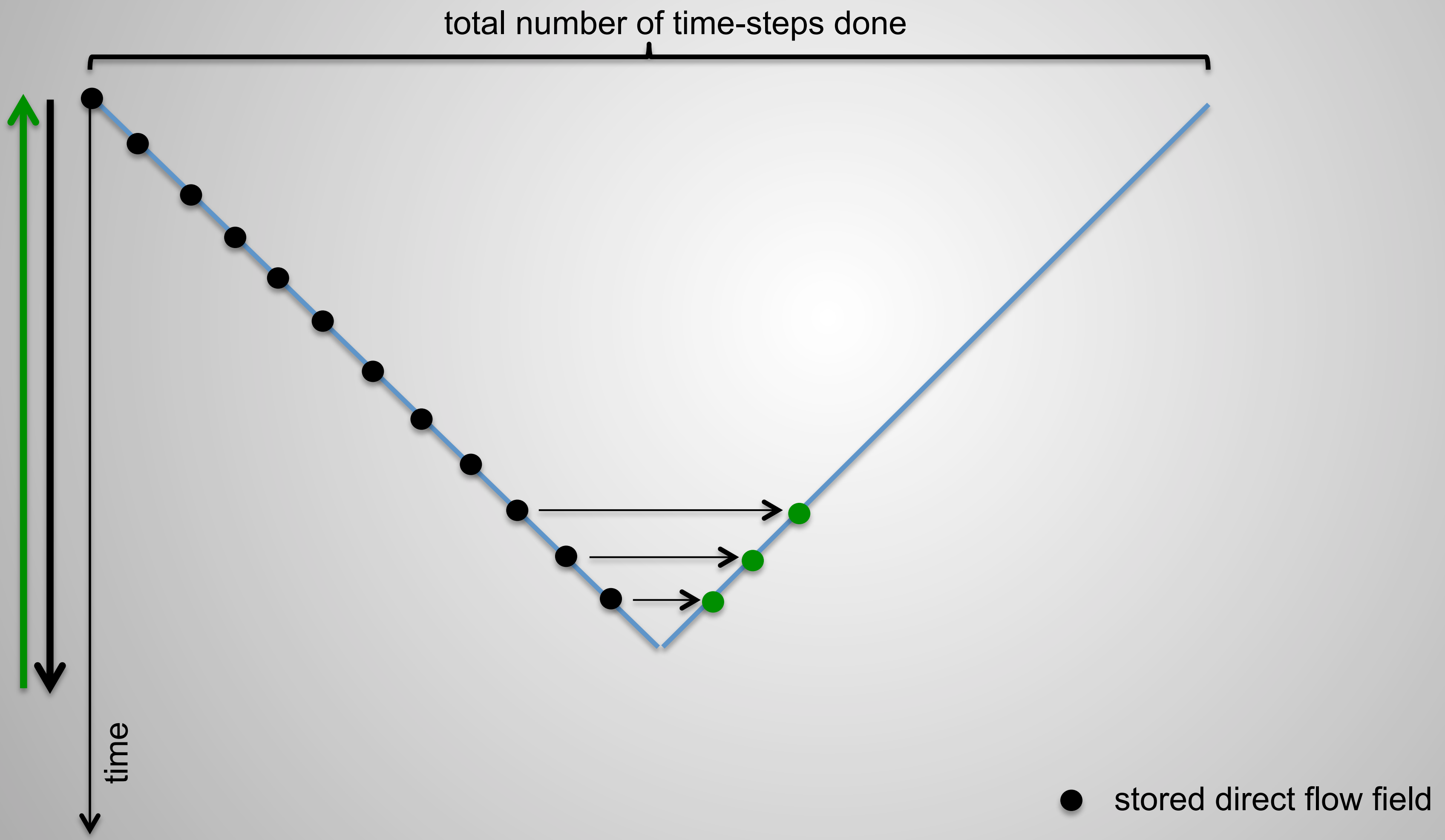

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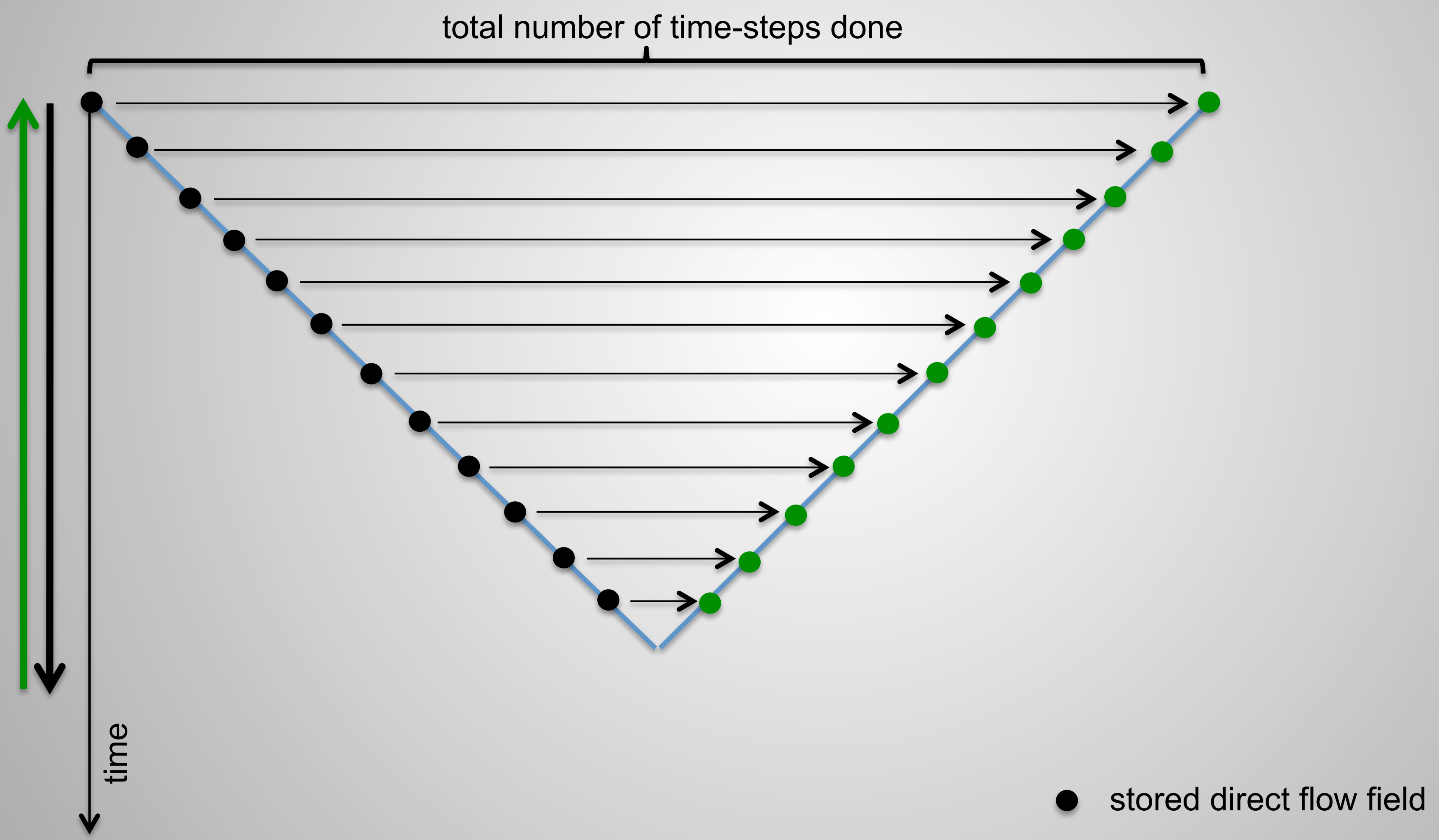
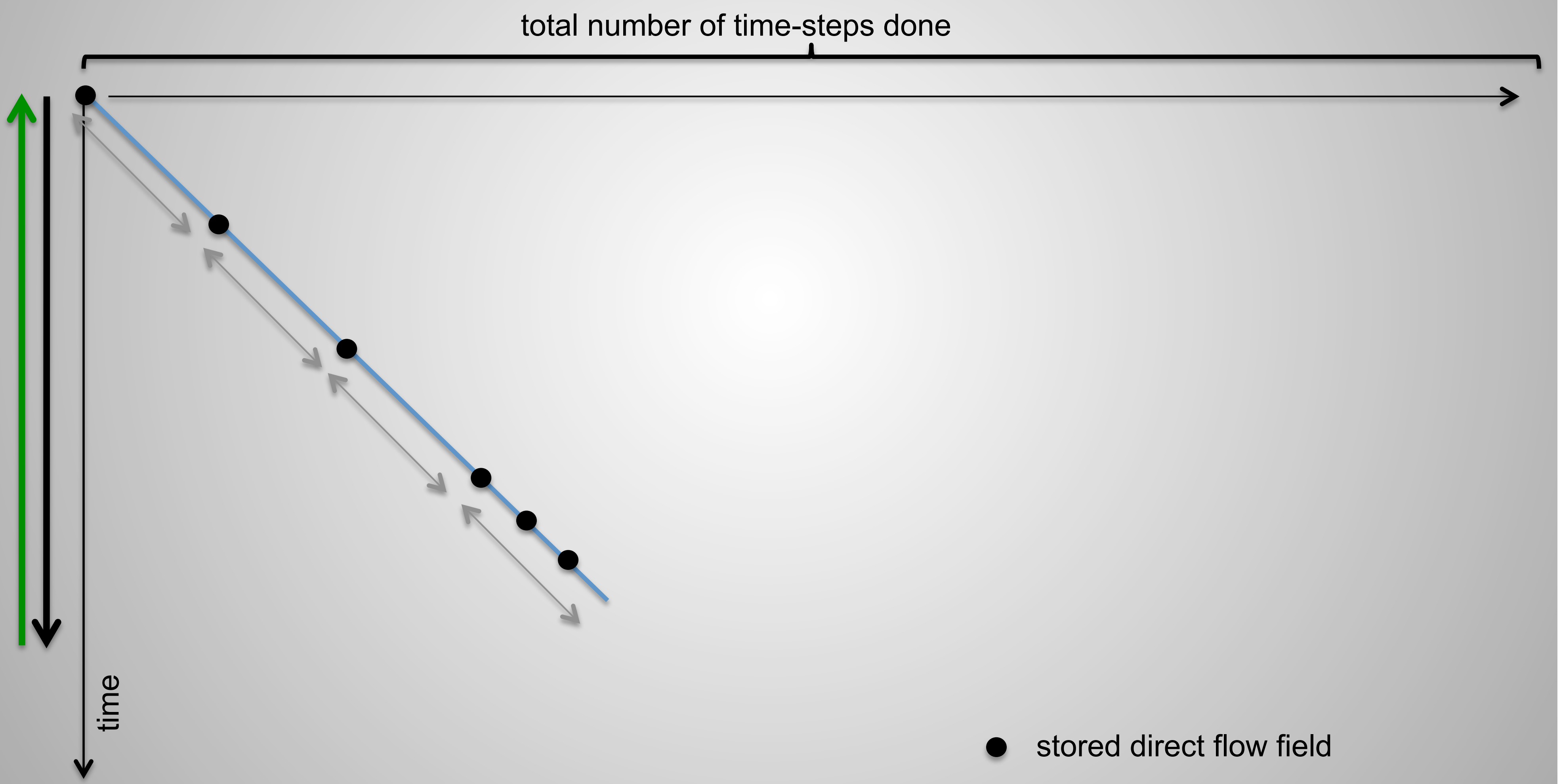


Generalizations

uniform checkpointing (store at a few equispaced locations)



Generalizations

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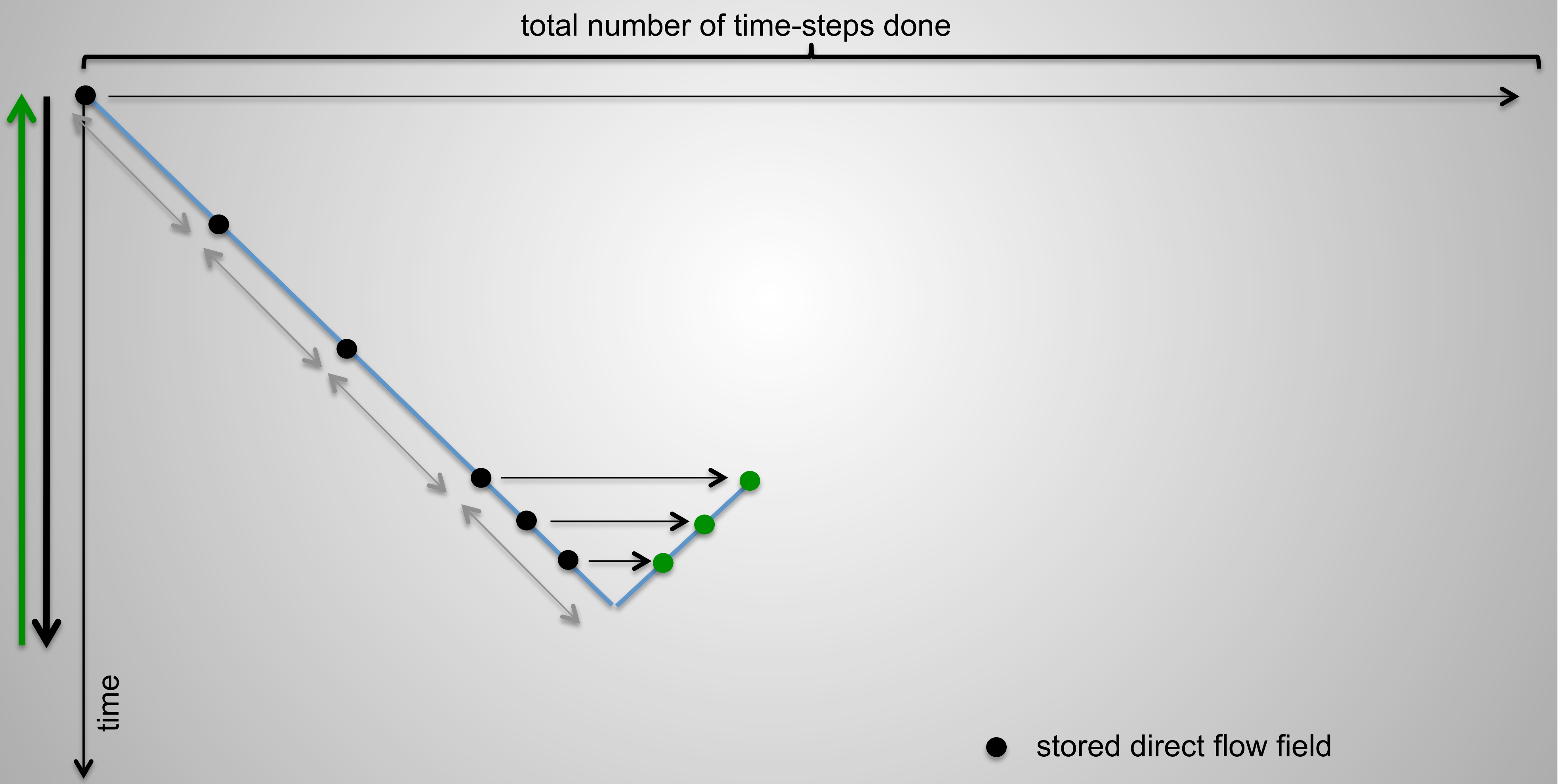

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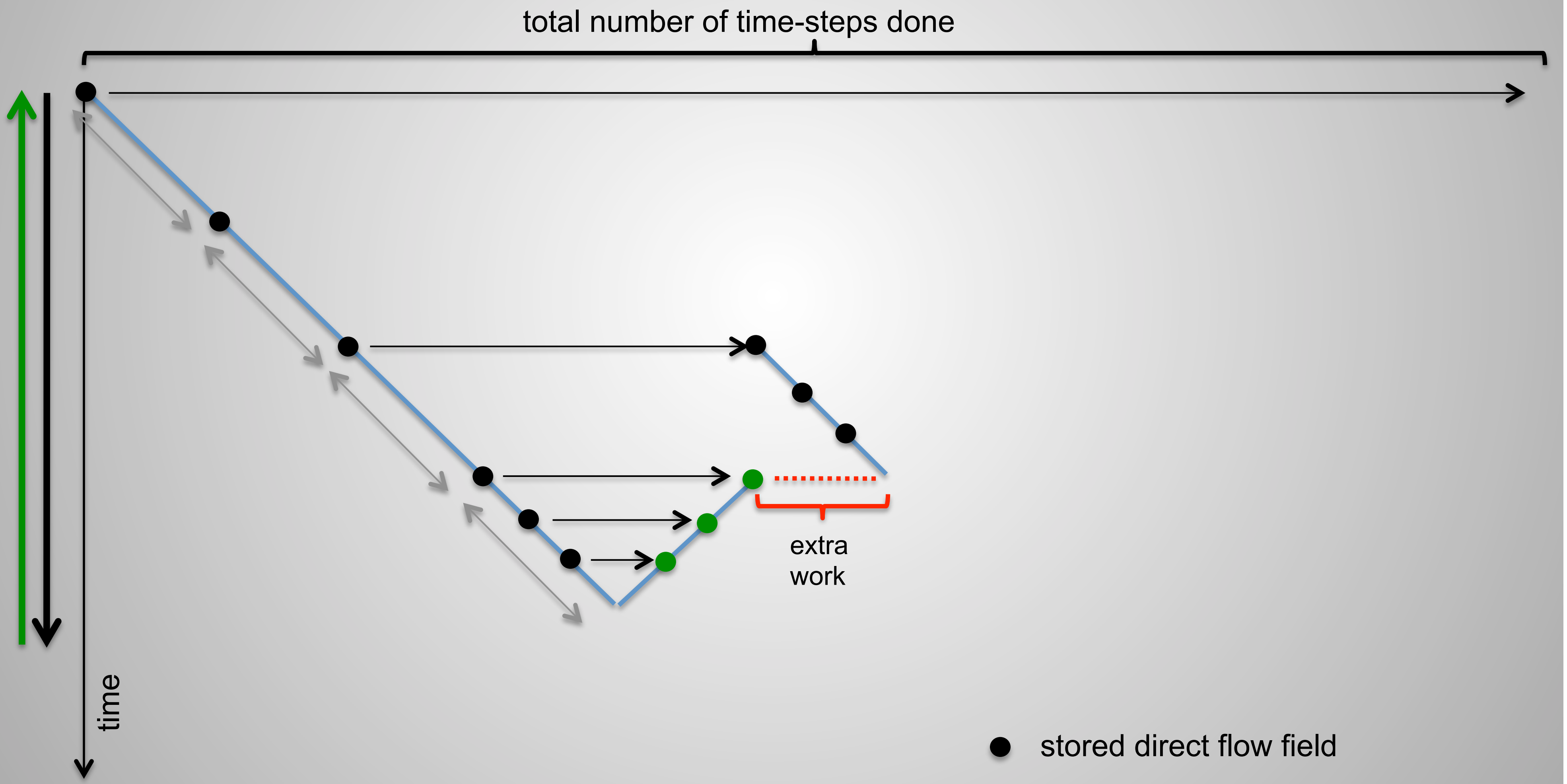

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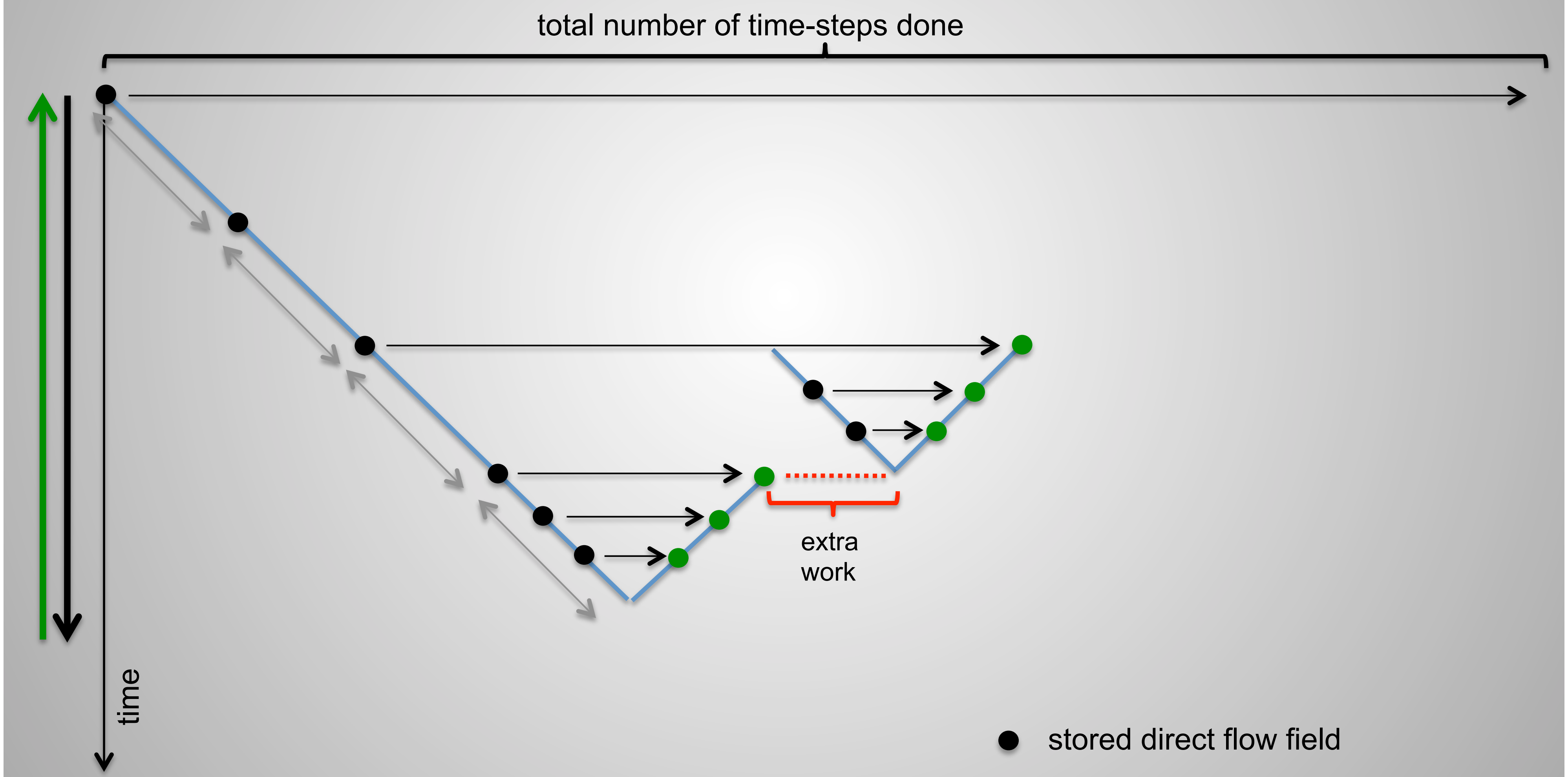

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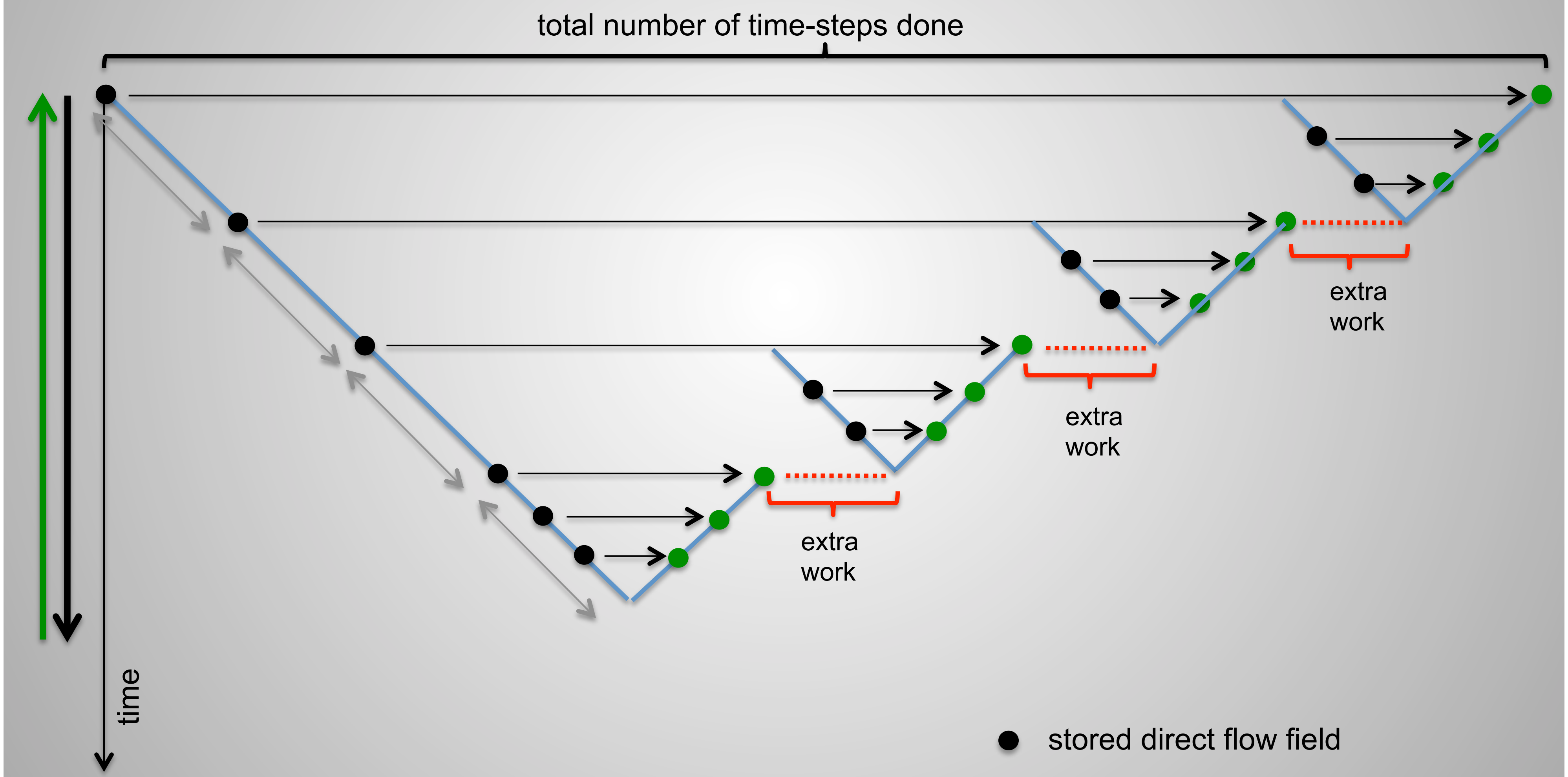

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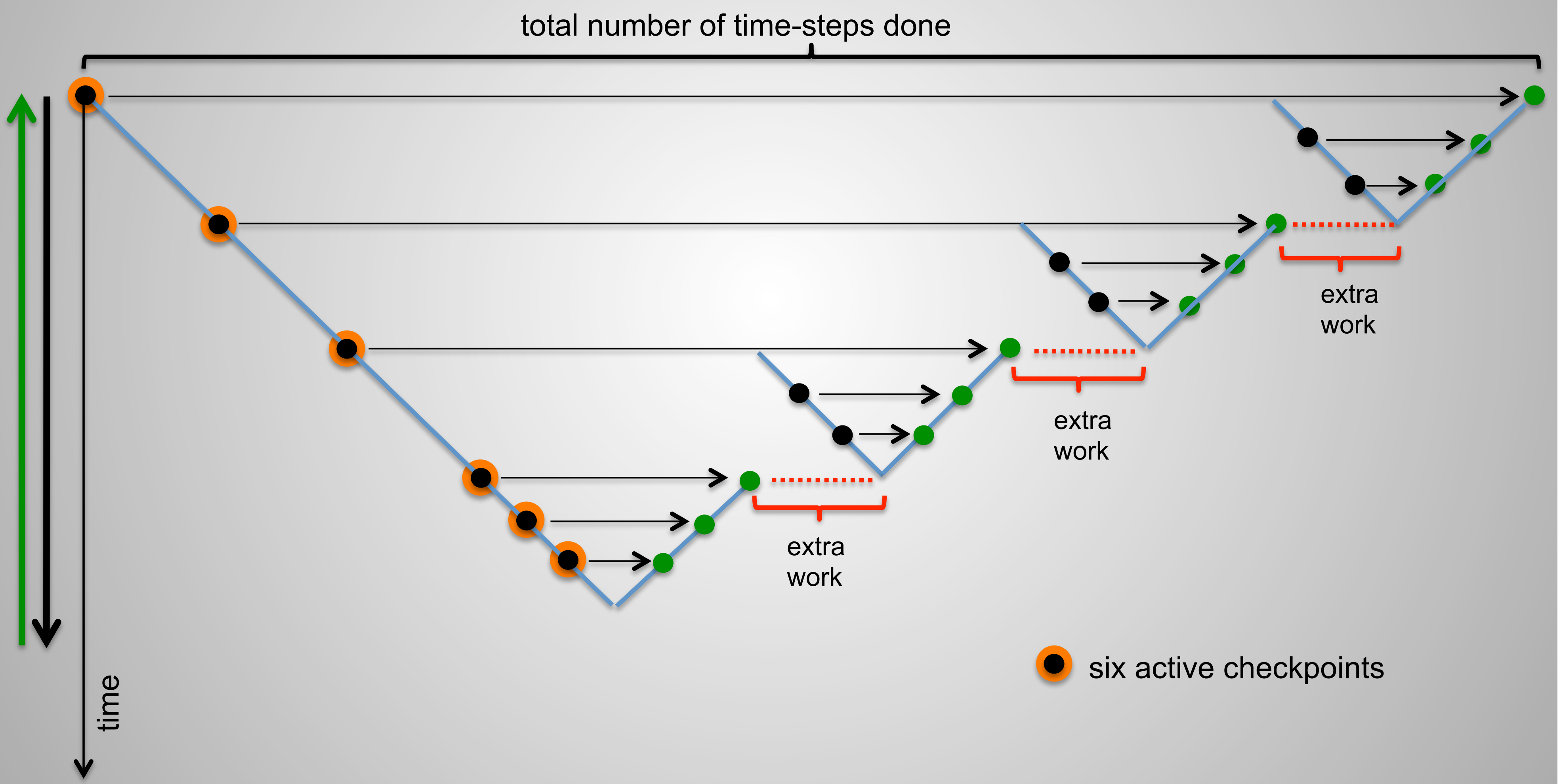

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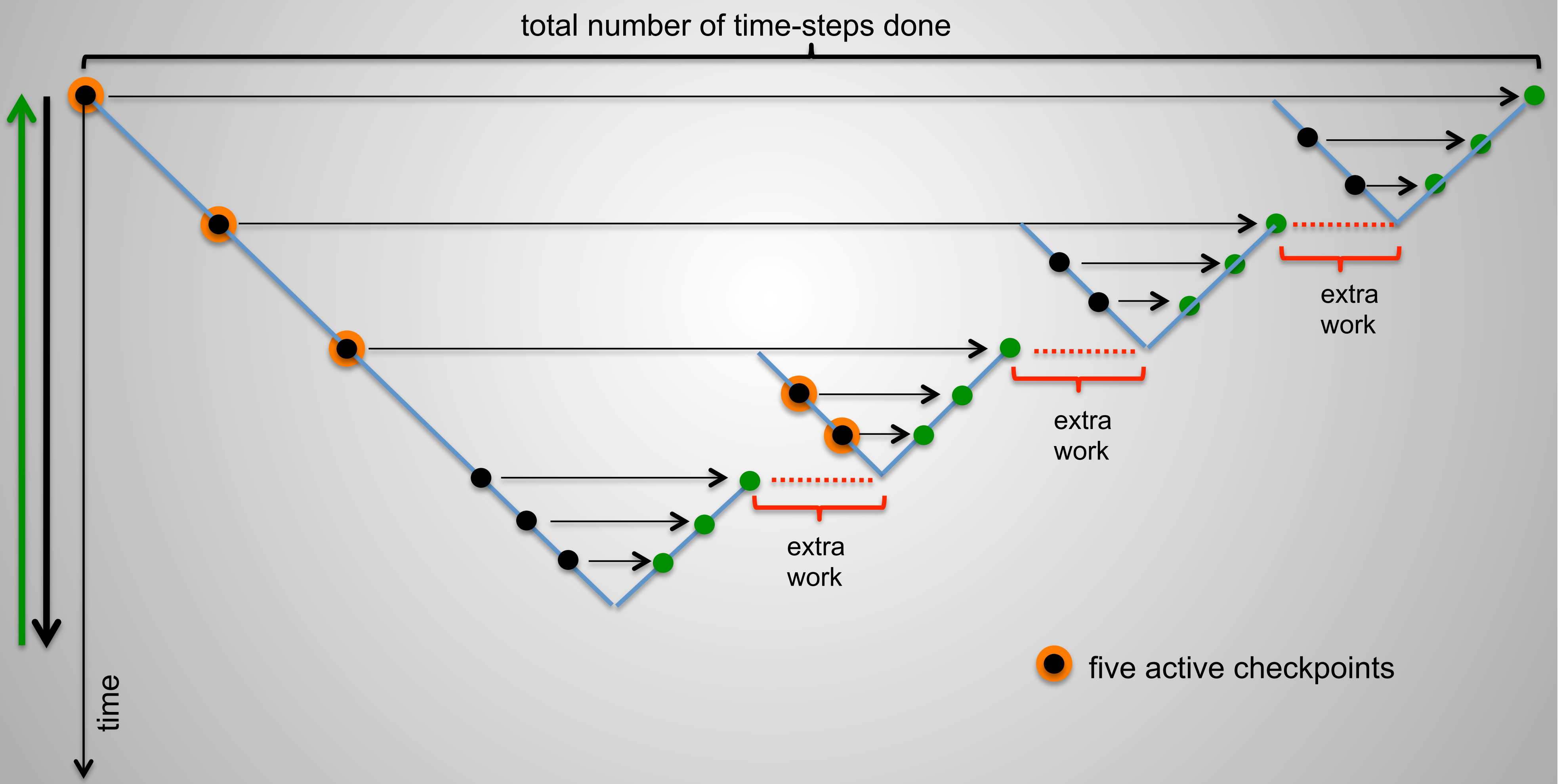

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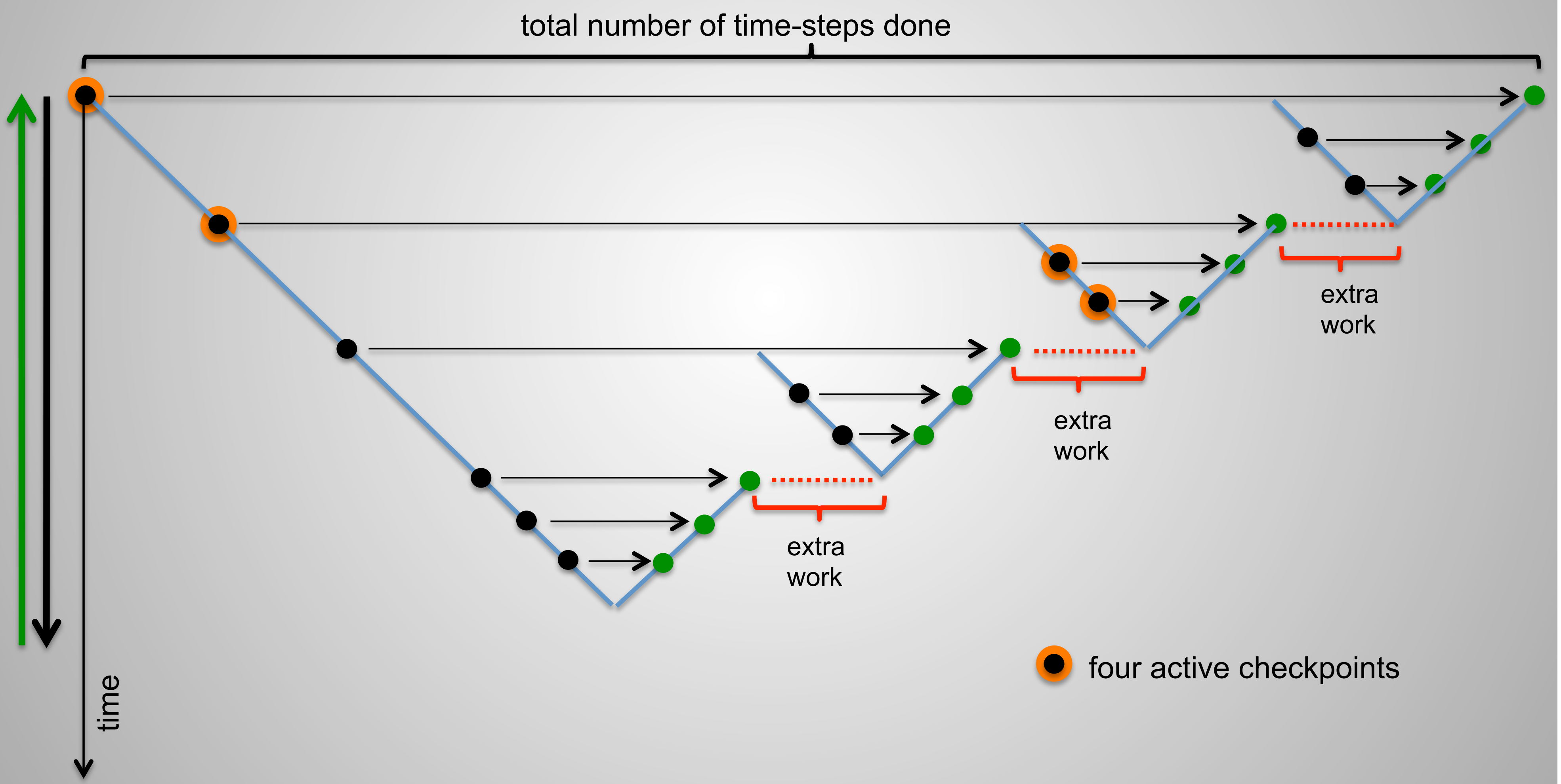

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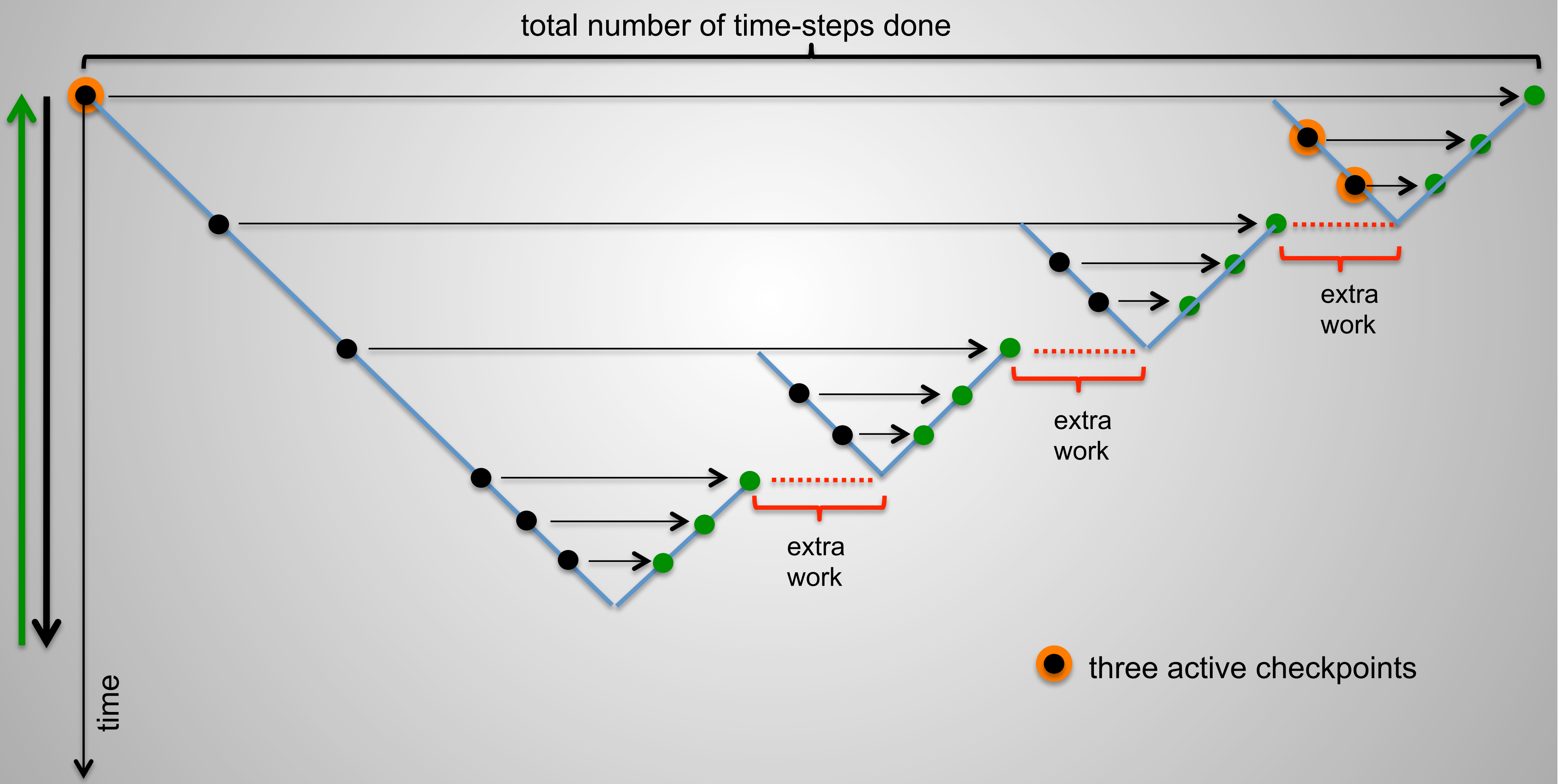

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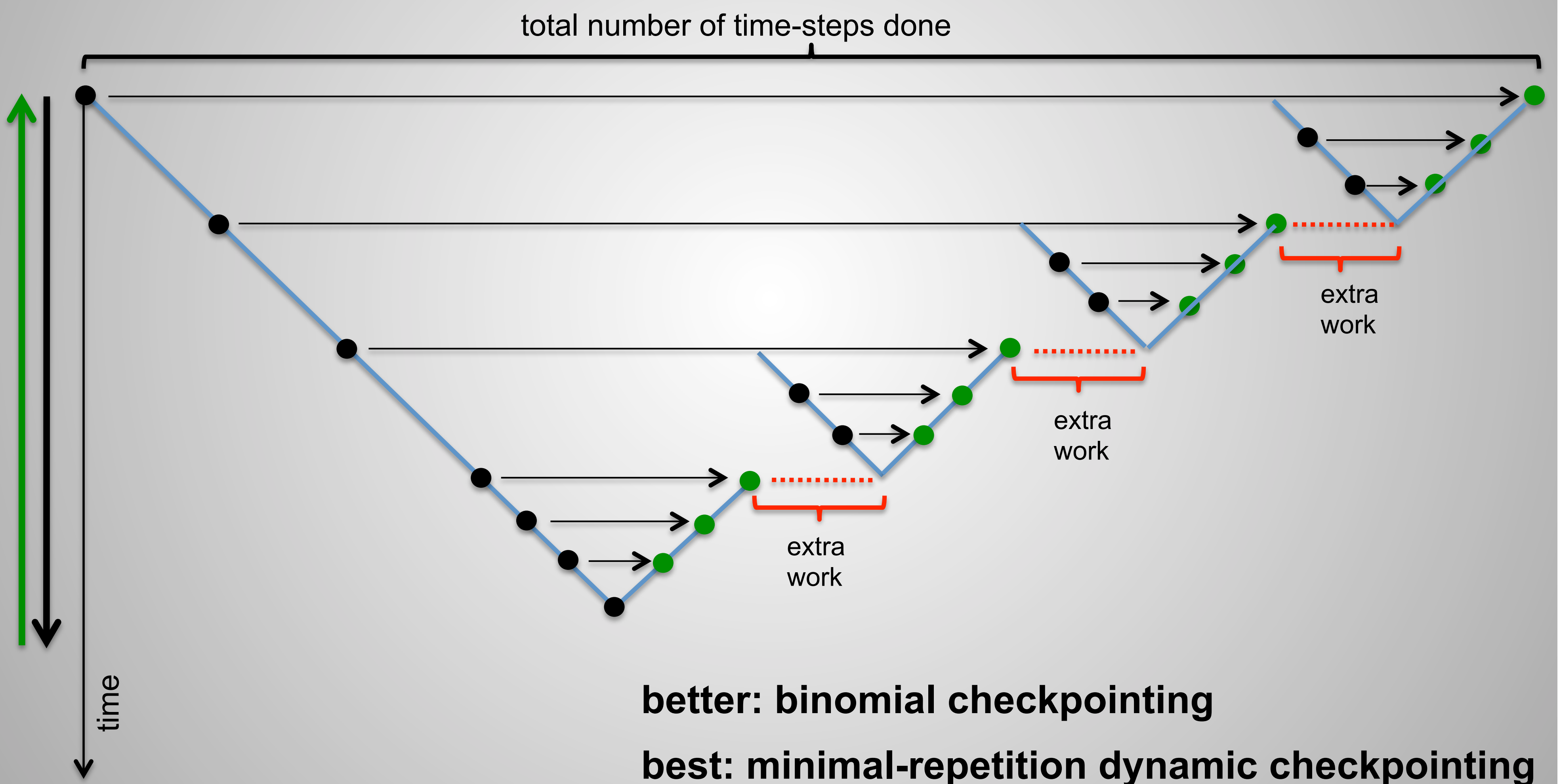


better: binomial checkpointing

best: minimal-repetition dynamic checkpointing

Generalizations

We often have governing equations with auxiliary evolution equations (e.g., for eddy-viscosity), but these auxiliary variables may not be part of the cost objective.

$$q = \begin{pmatrix} \mathbf{u} \\ \nu_t \end{pmatrix} \qquad \frac{d}{dt}\begin{pmatrix} \mathbf{u} \\ \nu_t \end{pmatrix} = \begin{pmatrix} \mathbf{f}(\mathbf{u}, \nu_t) \\ g(\nu_t, \ldots) \end{pmatrix}$$

governing equations

turbulence model

This leads to ***semi-norm*** constraints.

$$\|\mathbf{q}\| \equiv \|\mathbf{u}\|$$

not a true norm

We can have $\|\mathbf{q}\| = 0$ with $\mathbf{q} \neq \mathbf{0}$

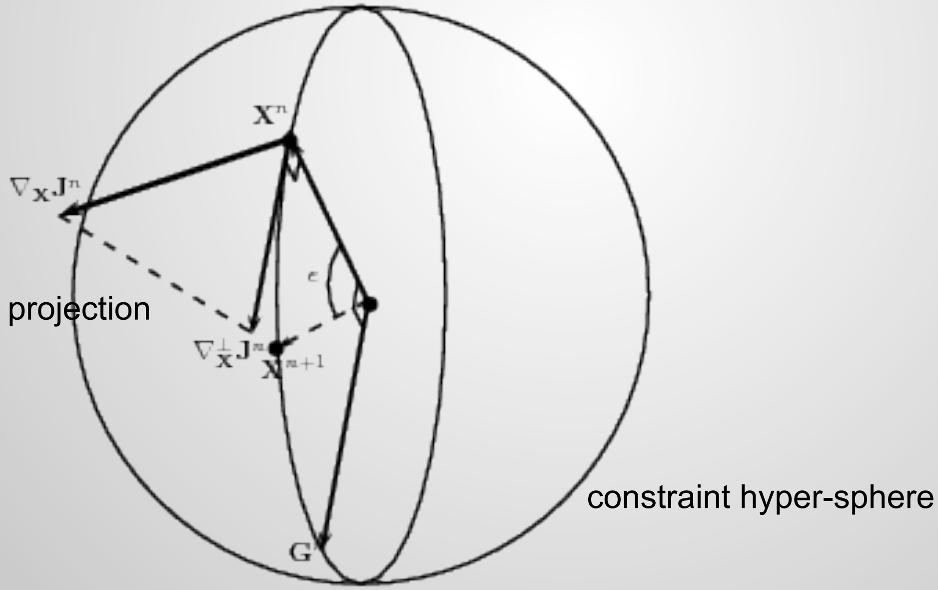
$\longrightarrow$ causes singularities (non-convergence)

Generalizations

We need additional constraints to avoid singularities.

⟶ constrained variational approach (penality terms)

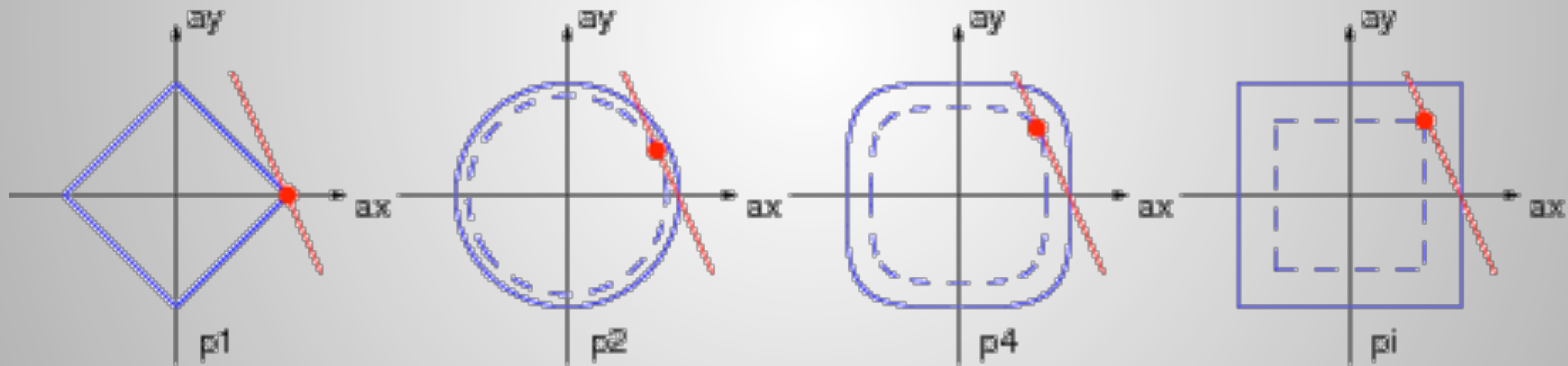
⟶ optimization on hyper-spheres



Generalizations

Is the 2-norm appropriate for all applications ? Can we consider a worst-case scenario ?

introduce a p-norm $$\|\mathbf{a}\|_p = (|a_x|^p + |a_y|^p)^{1/p}$$

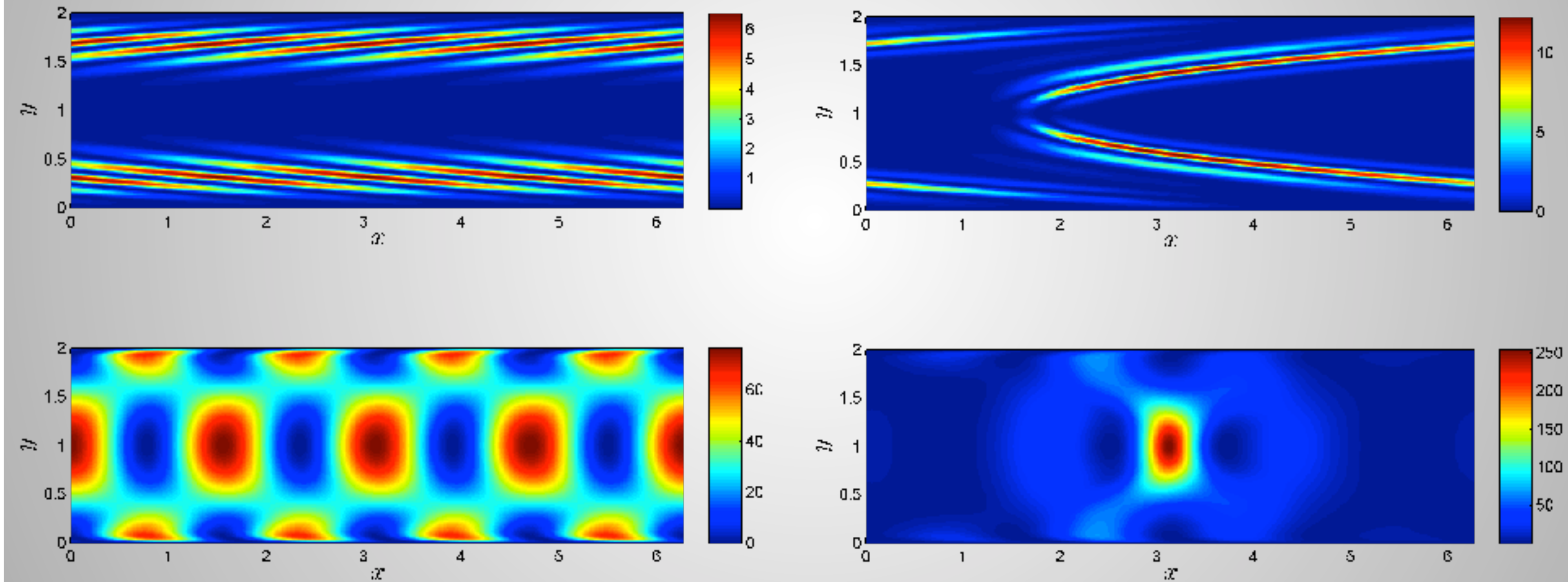


$p = 1$ $p = 2$ $p = 4$ $p = \infty$

dependence on chosen norm

Generalizations

localization of the optimal structures, symmetry breaking *(work in progress)*



$p = 2$ $\qquad$ $p = \infty$

Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

for most industrial applications we cannot assume the existence of homogeneous directions that can be treated by a Fourier transform

rather, the eigenfunction will depend on more than one inhomogeneous coordinate direction

Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

$$q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}$$ ← state vector → $$q = \begin{pmatrix} q_{1,1} \\ q_{1,2} \\ \vdots \\ q_{N,N} \end{pmatrix}$$

$$L \in \mathbb{C}^{N \times N}$$ ← stability matrix → $$L \in \mathbb{C}^{N^2 \times N^2}$$

$$\sim N^3$$ ← operation count → $$\sim N^6$$

one inhomogeneous direction

two inhomogeneous directions

Generalizations

multiple inhomogeneous directions/complex geometry ➡ global mode analysis

➡ direct eigenvalue algorithms quickly become prohibitively expensive

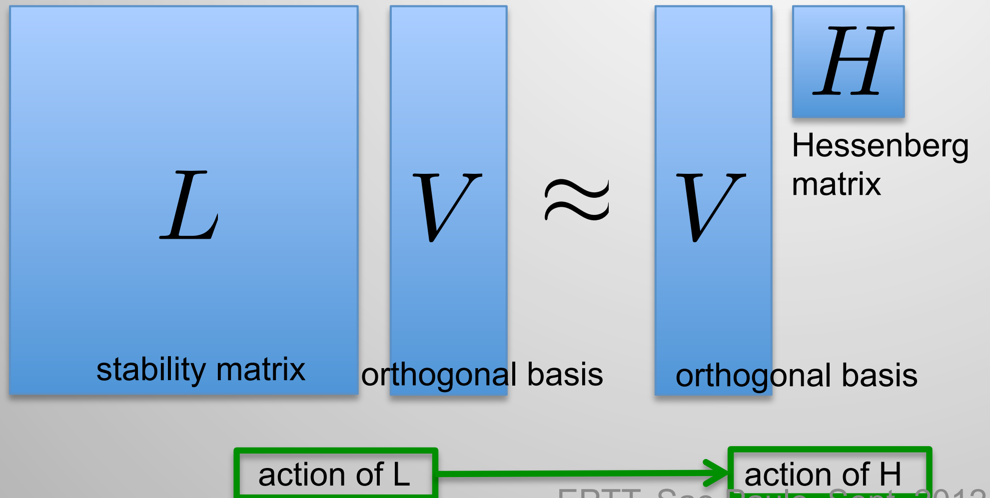
➡ iterative eigenvalue algorithms (Arnoldi technique) have to be used

Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

Arnoldi algorithm

action of the linear operator L is expressed within an orthonormal basis V

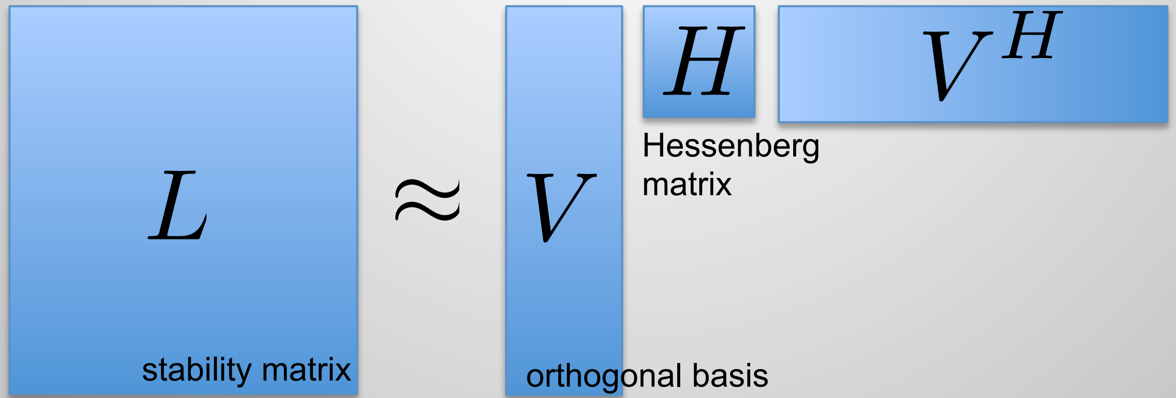


Generalizations

multiple inhomogeneous directions/complex geometry $\Rightarrow$ global mode analysis

Arnoldi algorithm

represent the (large) stability matrix by a low-rank approximation based on an orthogonal basis



Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

$$q_k = L\ q_{k-1}$$

$$\text{for} \quad j = 1 : k-1$$

$$H_{j,k-1} = \langle q_j , q_k \rangle$$

$$q_k = q_k - H_{j,k-1}\ q_j$$

$$\text{end}$$

$$H_{k,k-1} = \|q_k\|$$

$$q_k = q_k / H_{k,k-1}$$

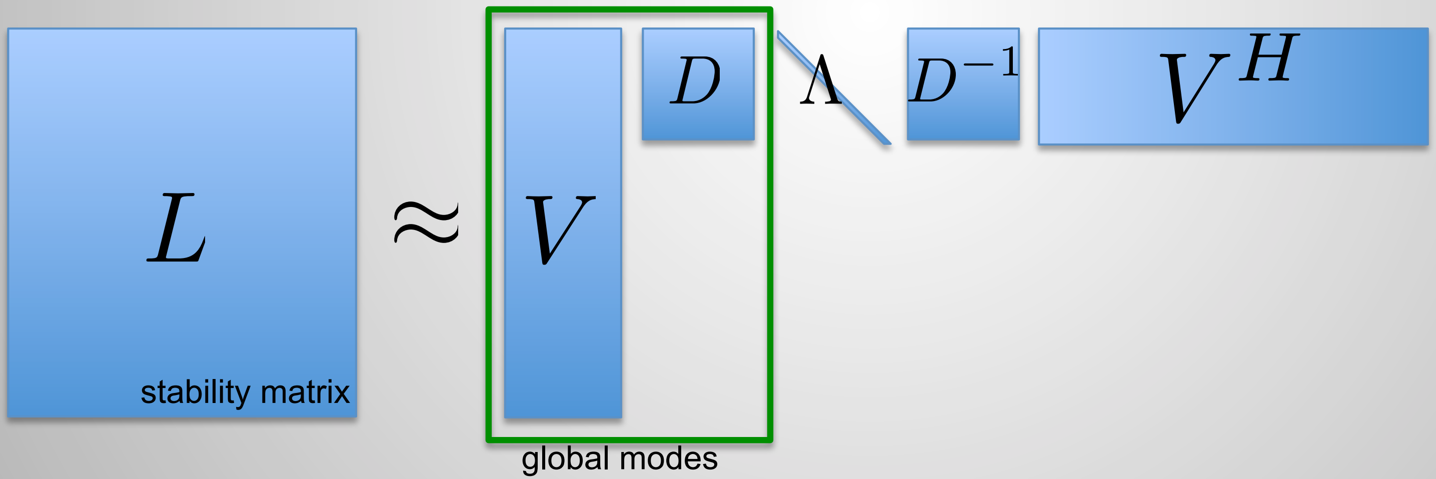
only multiplications by L are necessary

$$\Rightarrow \quad \mathrm{eig}\{L\} \approx \mathrm{eig}\{H\}$$

Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

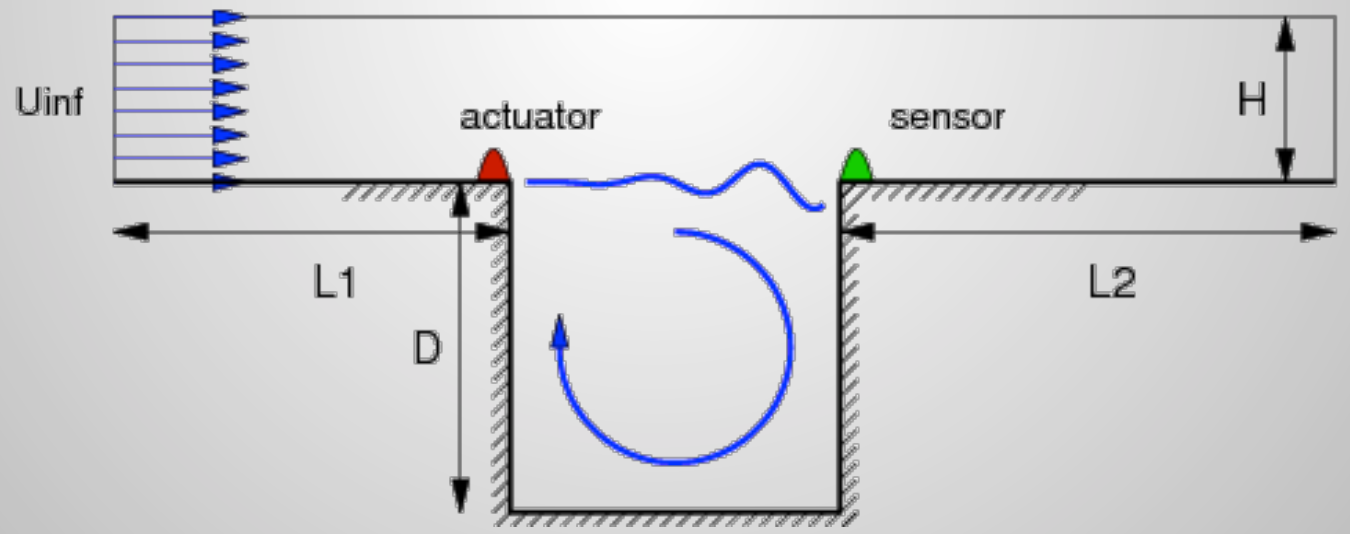
computing global modes by diagonalizing $H = D\Lambda D^{-1}$



Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

Examples of global modes: open cavity flow (two-dimensional)



Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

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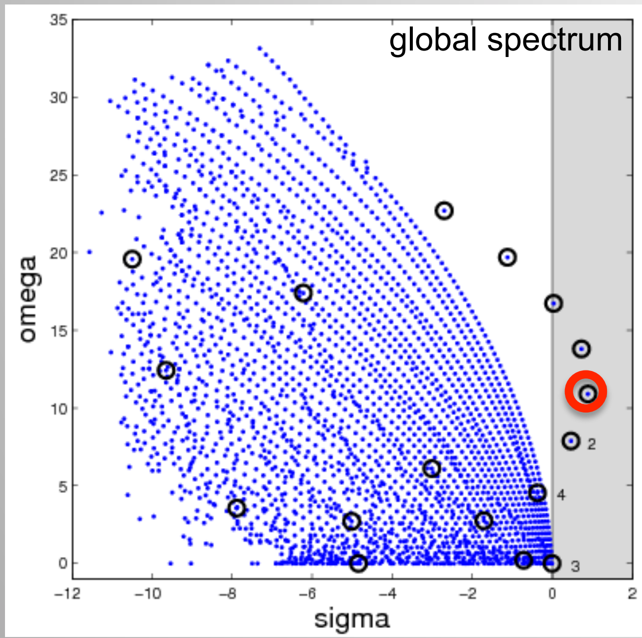
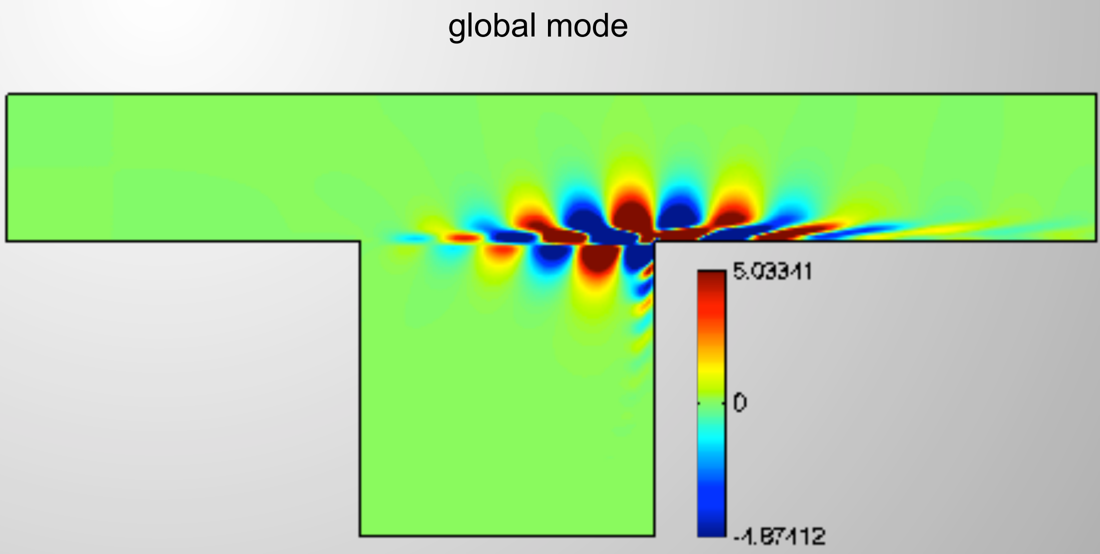
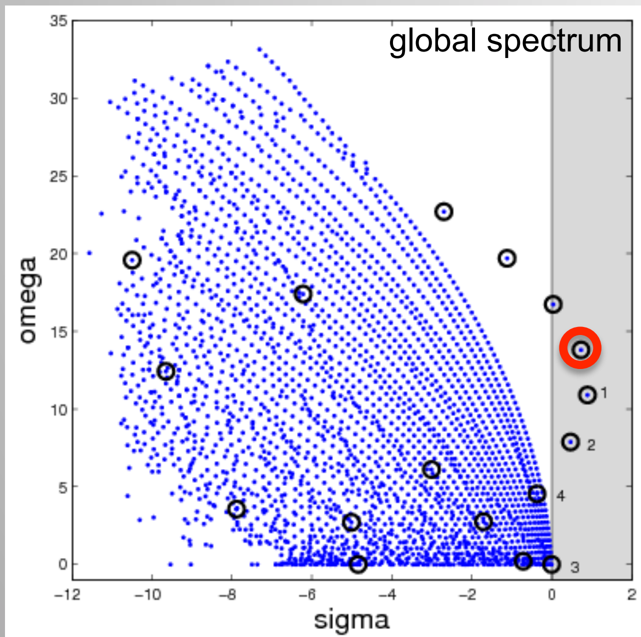


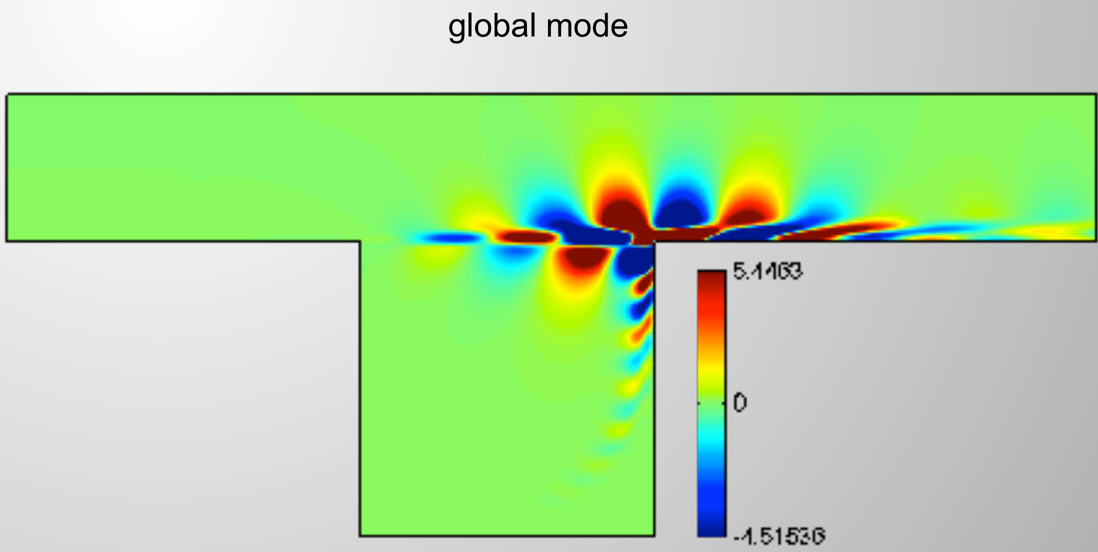


Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

Examples of global modes: open cavity flow (two-dimensional)

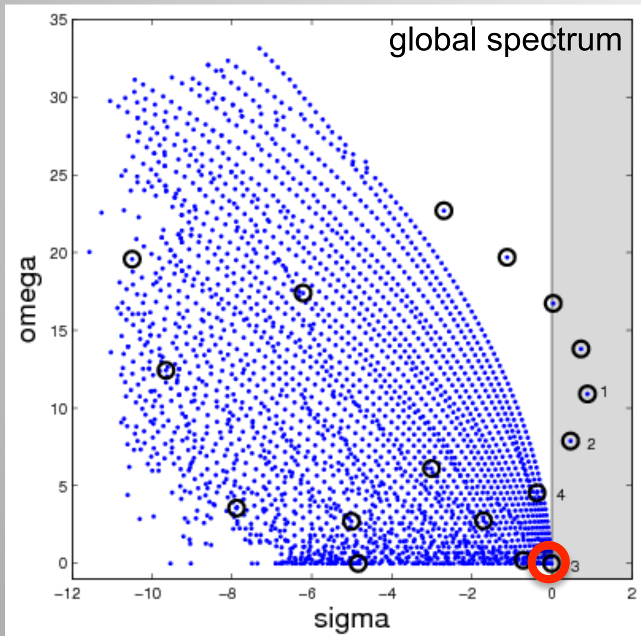


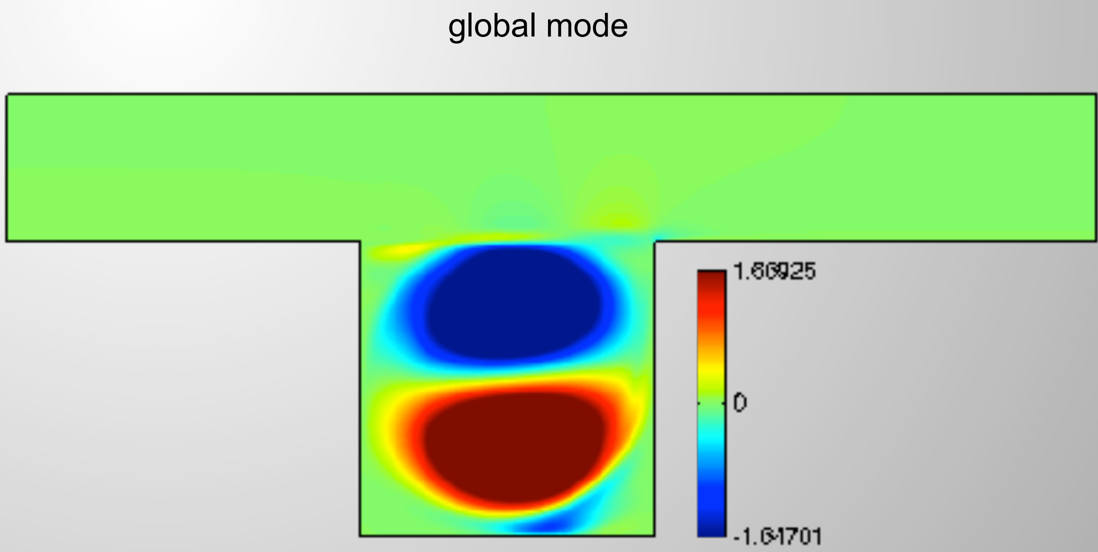


Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

Examples of global modes: open cavity flow (two-dimensional)

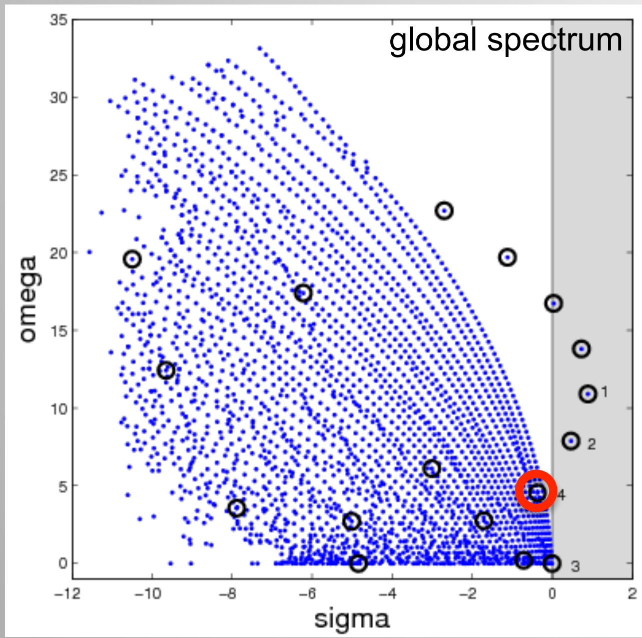


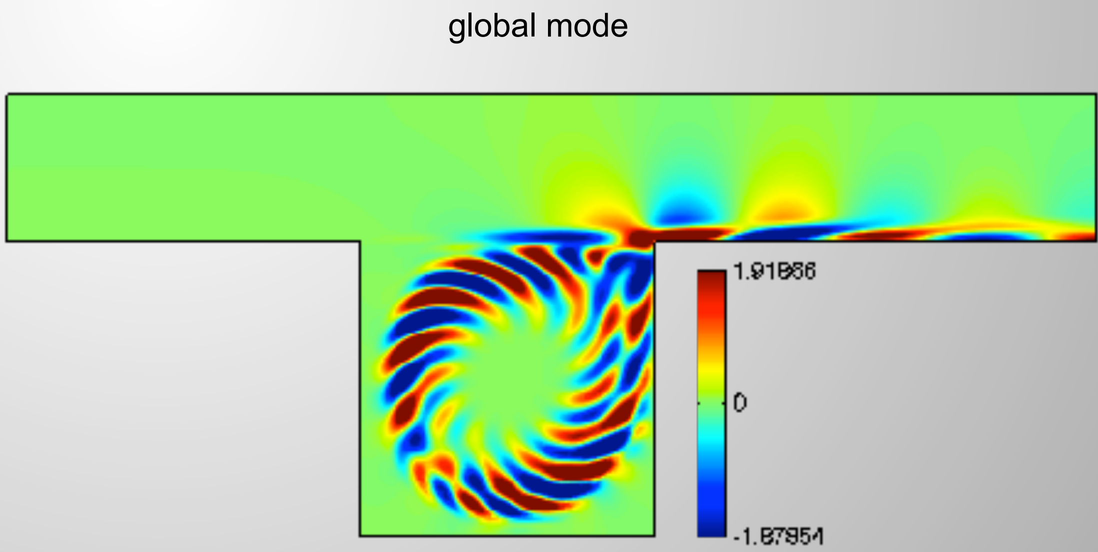


Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

Examples of global modes: open cavity flow (two-dimensional)

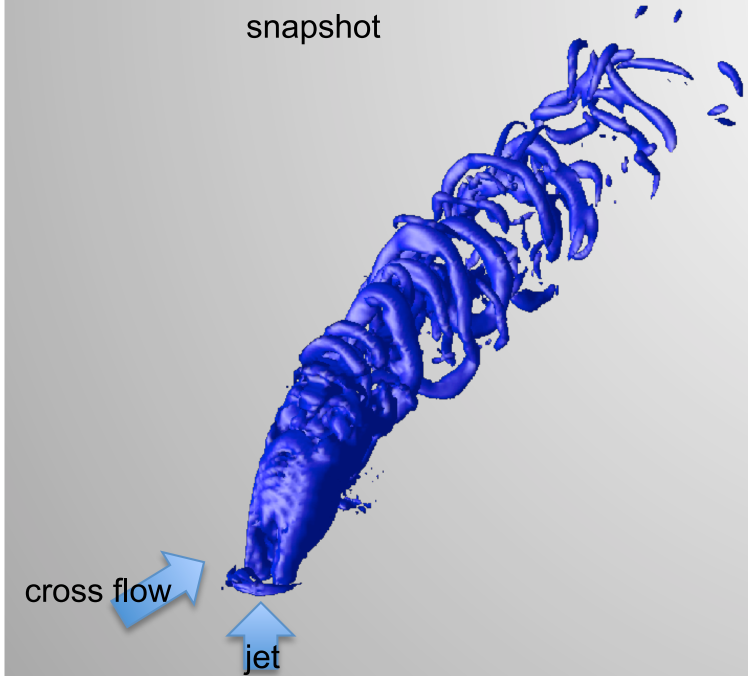




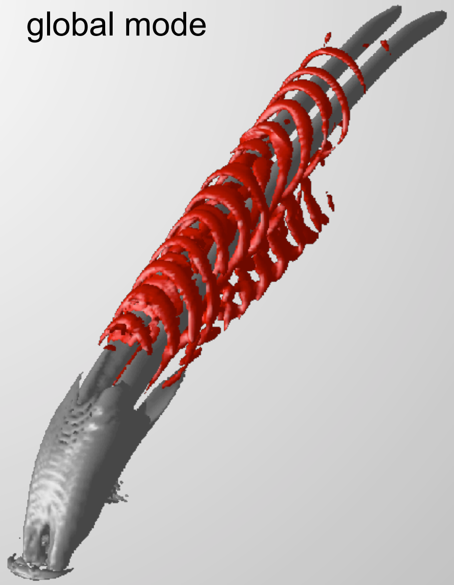
Generalizations

multiple inhomogeneous directions/complex geometry → global mode analysis

Examples of global modes: jet in cross flow (three-dimensional)



Arnoldi algorithm



Arnoldi algorithm (a Krylov subspace technique) to compute the Hessenberg matrix H

```
q_k = A * q_{k-1};
for j=1 to k-1
    h_{j,k-1} = (q_j,q_k);
    q_k = q_k - h_{j,k-1}*q_j;
end
h_{k,k-1} = norm(q_k);
q_k = q_k/h_{k,k-1};
```

ARPACK

Jacobian-free framework

$$\mathrm{A}\mathbf{q}_k \approx \frac{\boxed{\mathcal{F}(\mathrm{Q}_0 + \varepsilon \mathbf{q}_k)} - \mathcal{F}(\mathrm{Q}_0)}{\varepsilon}$$

DNS

$$\mathrm{A}\mathbf{q}_k \approx \mathrm{Imag}\{\boxed{\mathcal{F}(\mathrm{Q}_0 + i\mathbf{q}_k)}\}$$

Summary

- A variational approach for fluid stability problems provides a flexible and effective framework that allows the treatment time-invariant, time-periodic, time-dependent, linear and nonlinear problems.

- Adjoint variables can be interpreted as carriers of gradient/ sensitivity information to external driving terms.

- Weighted (scalar) products of direct and adjoint variables yield structural sensitivity information and can be used for complex flow optimizations or the influence analysis of particular terms in the governing equations.

- Semi-norm constraints and p-norm extensions add even more flexibility.