The relation between the mean flow profile and the topology of turbulent fluctuations in a turbulent channel flow

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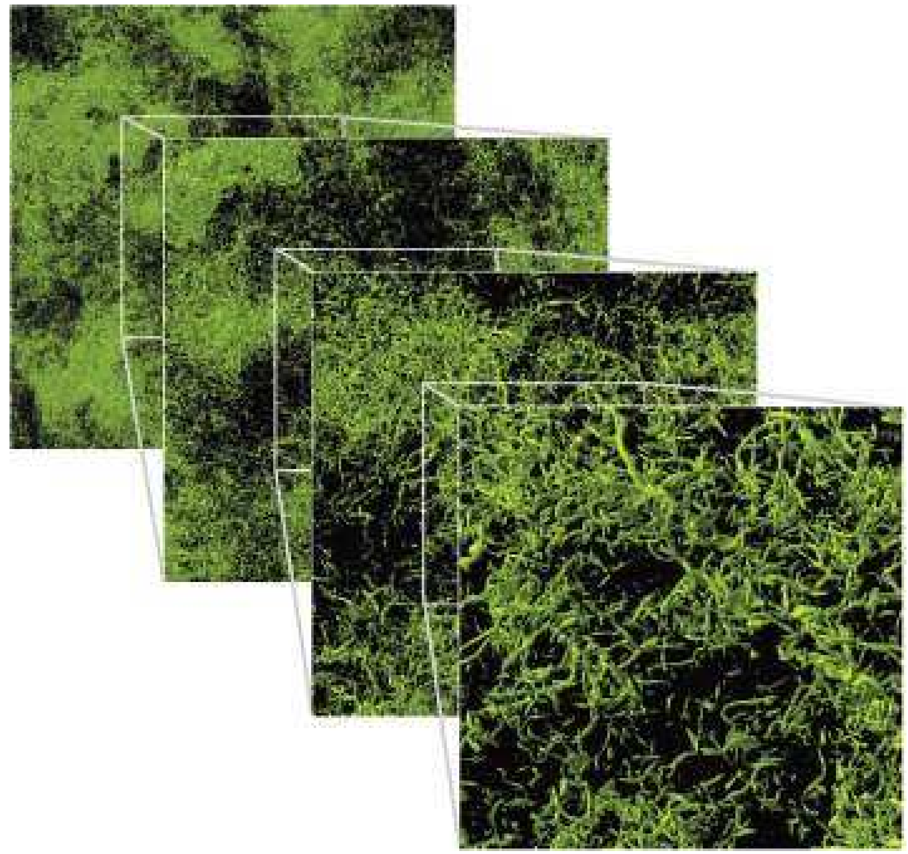
Work done with V. Dallas, S. Goto and N. Mazellier.

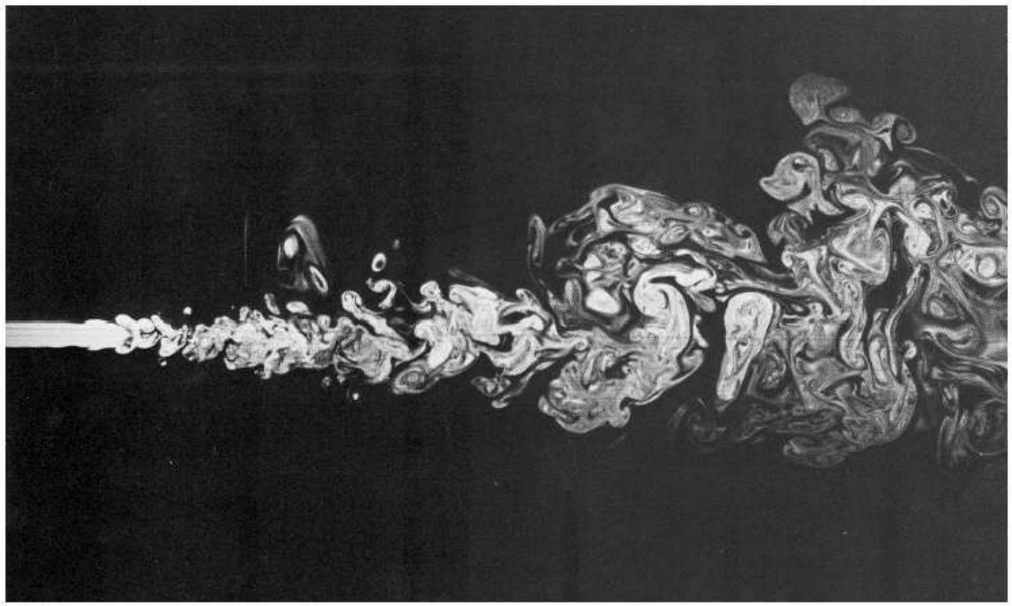
see http://www3.imperial.ac.uk/tmfc

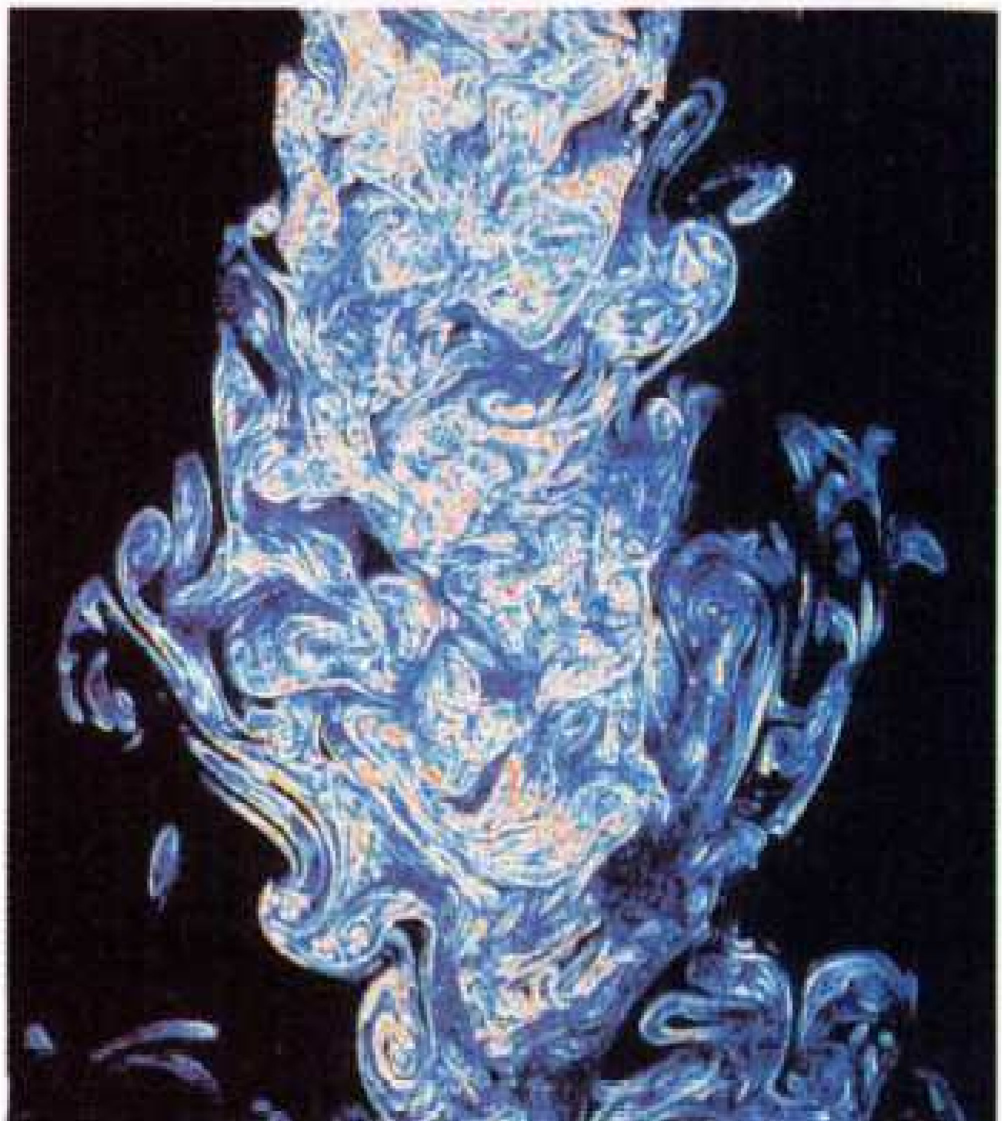
Start with flows away from walls

Periodic turbulence in the computer



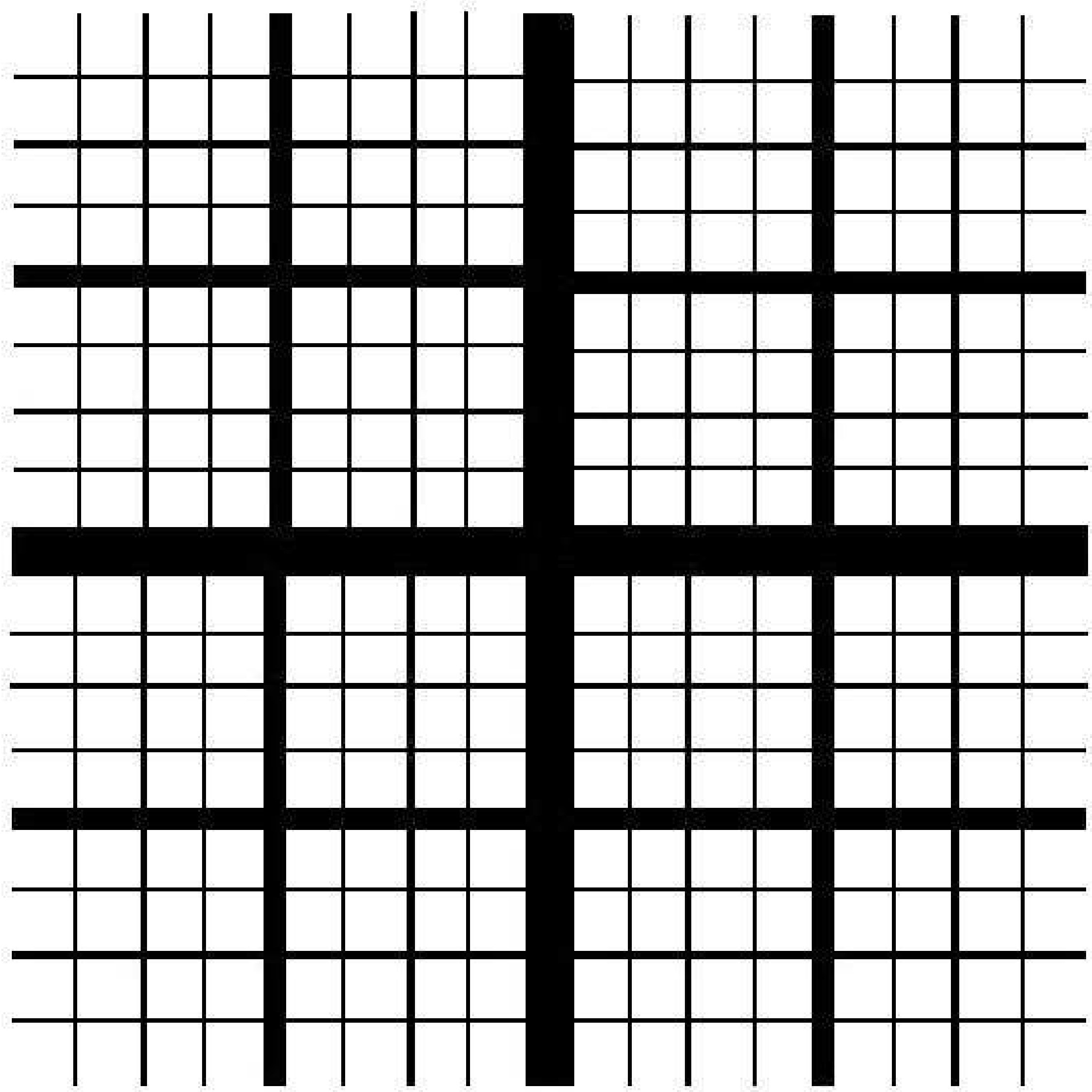
Jets





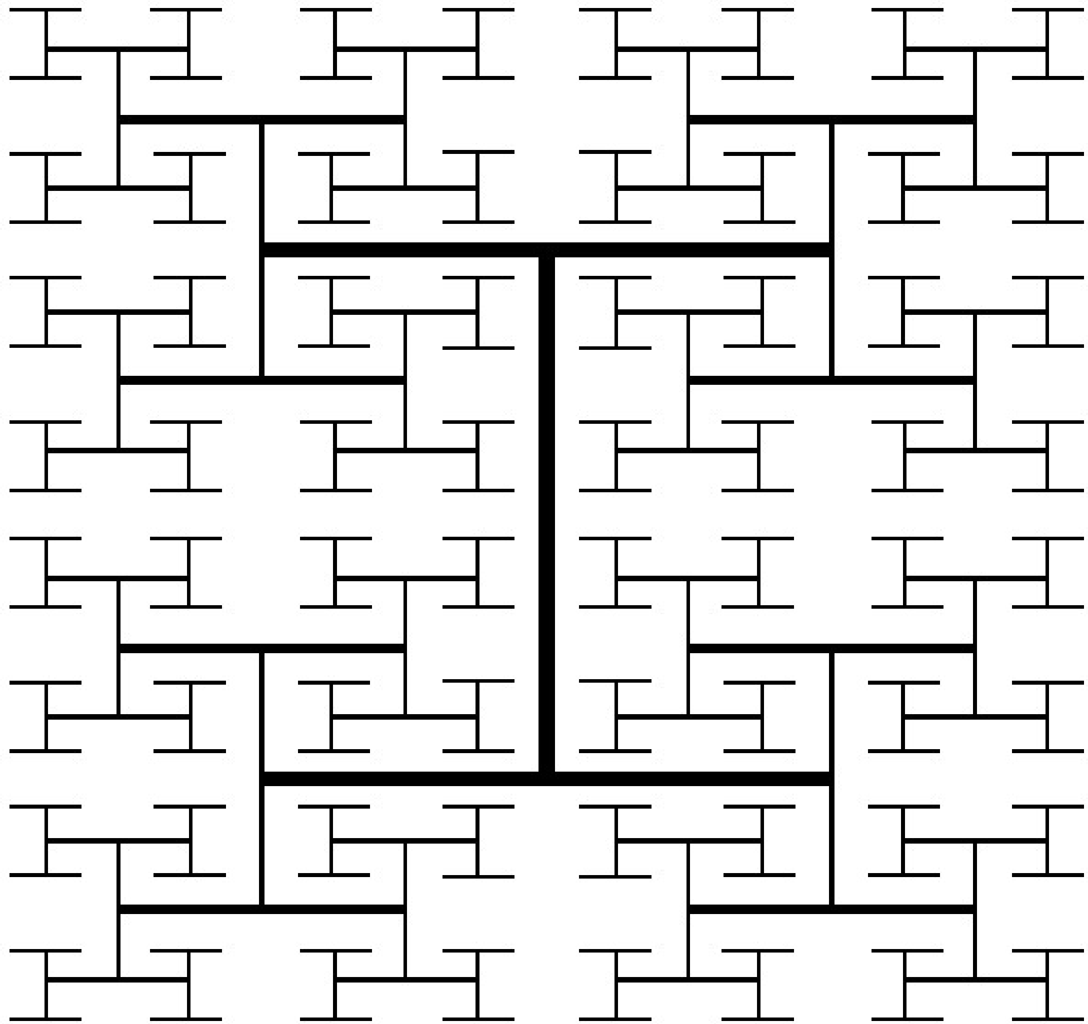
Grid turbulence in the wind tunnel

Regular grids and fractal cross grids



Grid turbulence in the wind tunnel

Fractal I grids



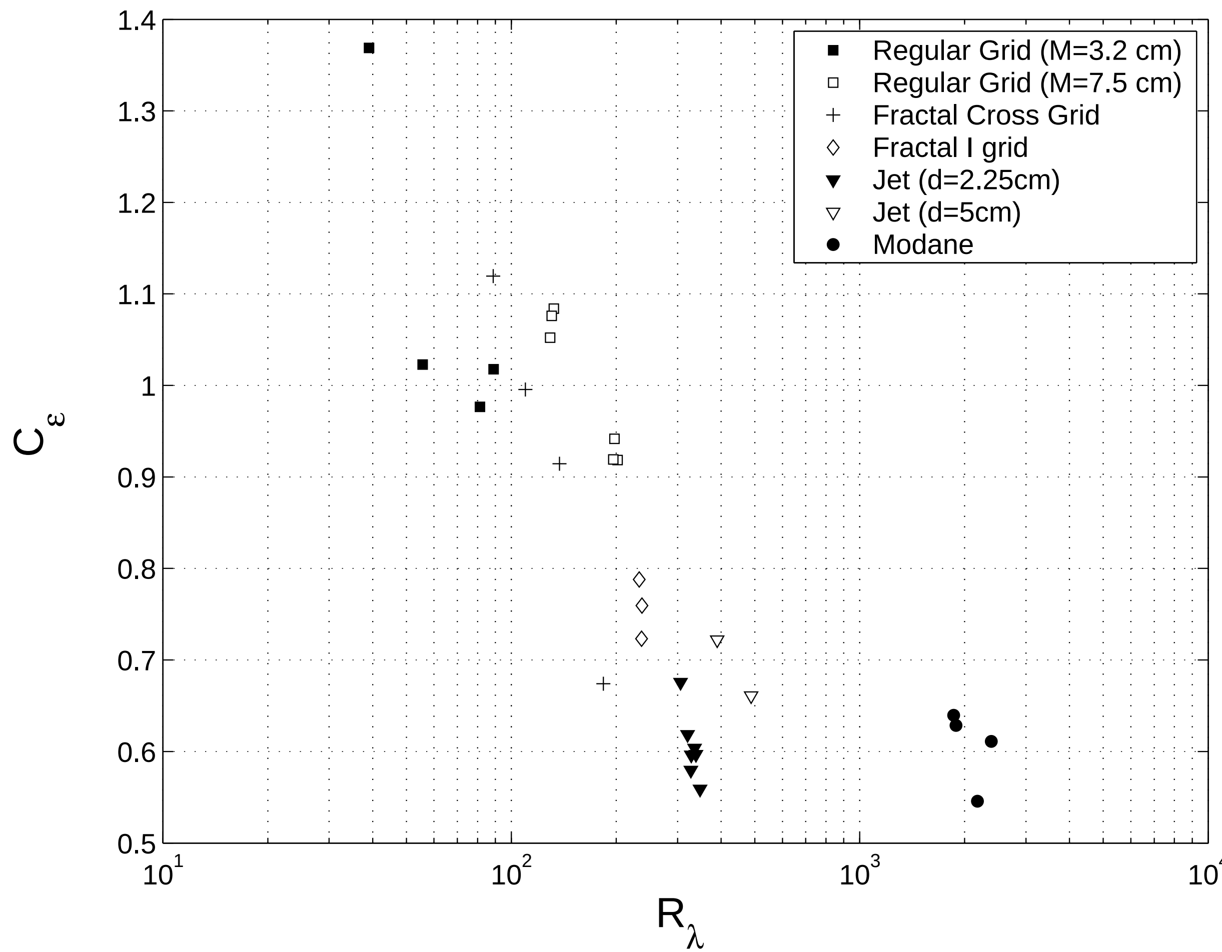
G.I. Taylor (1935): isotropic turbulence

$$\epsilon = 15\nu < (\frac{\partial u}{\partial x})^2 >= 15\nu u'^2/\lambda^2$$

$$\epsilon = C_\epsilon u'^3/L$$

where C_ϵ is indep of Re_λ as $Re_\lambda \to \infty$

Laboratory experiments and numerical simulations support the view that C_ϵ is independent of Re_λ in the limit $Re_\lambda \to \infty$ but suggest that C_ϵ is not independent of flow conditions, e.g. see Batchelor 1953, Sreenivasan 1984 and 1998, Kaneda *et al* 2003, Pearson *et al* 2004, Burattini, Lavoie & Antonia 2005.



30 data sets from 7 different turbulent flows

9 data sets from CL of 2 round air jets

Nozzle diameter	exit velocity	turb intensity	Re_λ
$d = 2.25cm$	50m/s	26%	380
$d = 5cm$	18m/s & 30m/s	28% & 27%	390 & 490

streamwise distance from nozzle	L
$50d$, $60d$, $70d$, $80d$, $90d$, $100d$, $110d$	4.9cm to 10.5cm
$50d$	12cm and 10.4cm

17 data sets from centerline of 4

wind tunnel grid-generated turbulent flows

	Section size, length	Mesh	solidi
classical grid	$75cm \times 75cm$, 4m	$M = 7.5cm$	34%
classical grid	$46cm \times 46cm$, 5m	$M = 3.2cm$	34%
fractal cross grid	$91cm \times 91cm$, 5.4m	$M_{eff} = 5.7cm$	21%
fractal I grid	$46cm \times 46cm$, 5m	$M_{eff} = 3.55cm$	25%

U_∞	streamwise x	u'/U
9 and 16m/s	$35M$, $38M$ and $42M$	3.3%
2.5, 5, 10 and 15.5m/s	$40M$	2.5%
6, 8, 12 and 16m/s	$75M_{eff}$	2.7%
10m/s	$65M_{eff}$, $72M_{eff}$ and $83M_{eff}$	7%

Continued

Re_λ	L
130 and 180	5.9 to 6.8cm
40, 56, 81 and 89	2.4cm
89, 110, 137 and 184	5.7cm
237	6.3cm

Also, 4 data sets from "chunk" turbulence in wind tunnel S1 at Modane ($24m$ diameter) with mean inlet velocities 19.9, 20, 20.8 and 20.6m/s; 7% turbulence intensities and $Re_\lambda = 1890, 1860, 2180$ and 2380 respectively. L from 1.64 to $2.13m$.

Rice 1944

In 1944, Rice proved that the average distance $\bar{l}$ between consecutive zero-crossings of a statistically stationary zero-mean stochastic function $u(x)$ is equal to the inverse of $< |\frac{du}{dx}| > p(u=0)$ if $u(x)$ and $\frac{du}{dx}$ are statistically independent.

If, furthermore, $u(x)$ is statistically gaussian, then $\sqrt{2\pi} < u^2 >^{1/2} p(u=0) = 1$; in which case

$$\bar{l} = \sqrt{2\pi} < u^2 >^{1/2} / < |\frac{du}{dx}| >$$

Finally, if $\frac{du}{dx}$ is also statistically gaussian, then $< |\frac{du}{dx}| >= \sqrt{\frac{2}{\pi}} < (\frac{du}{dx})^2 >^{1/2}$; in which case

$$\bar{l} = \pi < u^2 >^{1/2} / < (\frac{du}{dx})^2 >^{1/2}$$

Rice 1944 and Liepmann 1949, 1952

$$\bar{l} = \pi < u^2 >^{1/2} / < (\frac{du}{dx})^2 >^{1/2}$$

Sreenivasan, Prabhu & Narasimha (1983) demonstrated that this relation holds for many different turbulence signals in many different turbulent flows (longitudinal velocity fluctuations in boundary layers and a wake, wall shear stress in a channel and temperature derivatives in a heated boundary layer) and suggested, as a result, that the assumption of gaussianity may, in fact, not be necessary.

If $u(x)$ is the longitudinal velocity fluctuation component, then $< u^2 >^{1/2} / < (\frac{du}{dx})^2 >^{1/2} = u' / < (\frac{du}{dx})^2 >^{1/2}$ is the Taylor microscale λ.

The average distance $\bar{l}$ between consecutive zero-crossings of $u(x)$ is such that $\bar{l} = \pi\lambda$.

Liepmann 1949, 1952

$$\bar{l} = \pi\lambda$$

where $\bar{l}$ is the average distance between consecutive zero-crossings of $u(x)$.

Realising that Rice's 1944 theorem implies that $\bar{l} \propto \lambda$ holds in turbulence even if $\bar{l} = \pi\lambda$ doesn't quite, Liepmann used the constant C defined by $C < |\frac{du}{dx}| >= \sqrt{\frac{2}{\pi}} < (\frac{du}{dx})^2 >^{1/2}$.

The result of Rice and Liepmann is

$$\bar{l} = C\pi\lambda$$

For gaussian du/dx, $C = 1$. $C \neq 1$ measures deviations from gaussianity.

Number density of zero-crossings

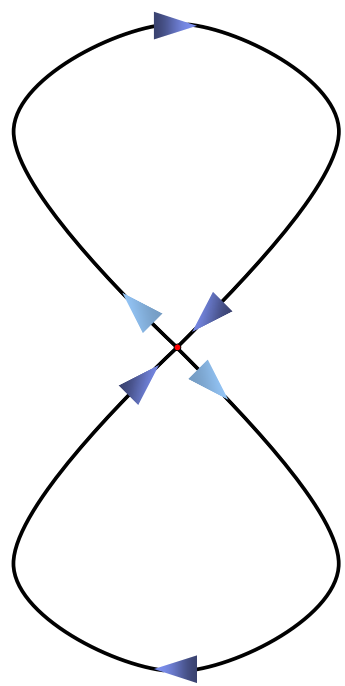
Sreenivasan and colleagues (see Ann. Rev.Fluid Mech. 1991) and Davilicos (2003, PRL 91(14), 144501) demonstrated that the number density n_s of zero-crossings of the longitudinal velocity fluctuation component $u(x)$ is a power-law function of L/η_c where $2\pi/\eta_c$ is the filter wavenumber of a low-pass filter applied on $u(x)$. Specifically, they found that

$$n_s(L/\eta_c) = \frac{C_s'}{L}(L/\eta_c)^{2/3}$$

in terms of a dimensionless constant C_s'.

Low-pass filtering operation: illustration

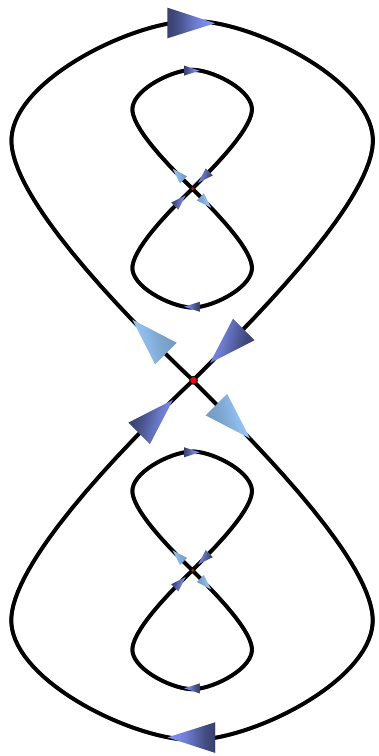
E.g. multiscale streamline structure in $d = 2$ (closed streamlines because of conservation of mass).



$n_s = 1$ and $L/\eta_c = 1$

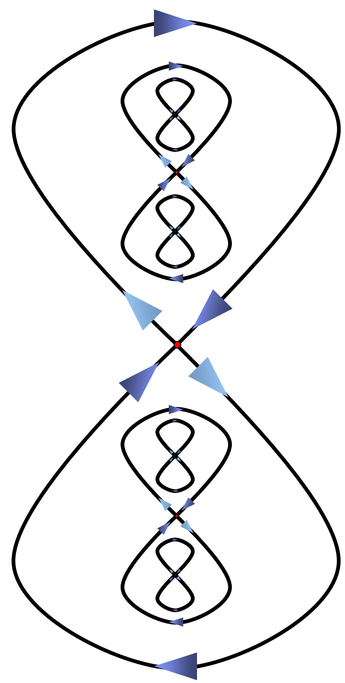
Low-pass filtering operation: illustration

$n_s = 3$ and $L/\eta_c \approx 3$



Low-pass filtering operation: illustration

$n_s = 7$ and $L/\eta_c \approx 9$



$n_s \sim (L/\eta_c)^D$ with $D \approx 0.75$

Number density of zero-crossings

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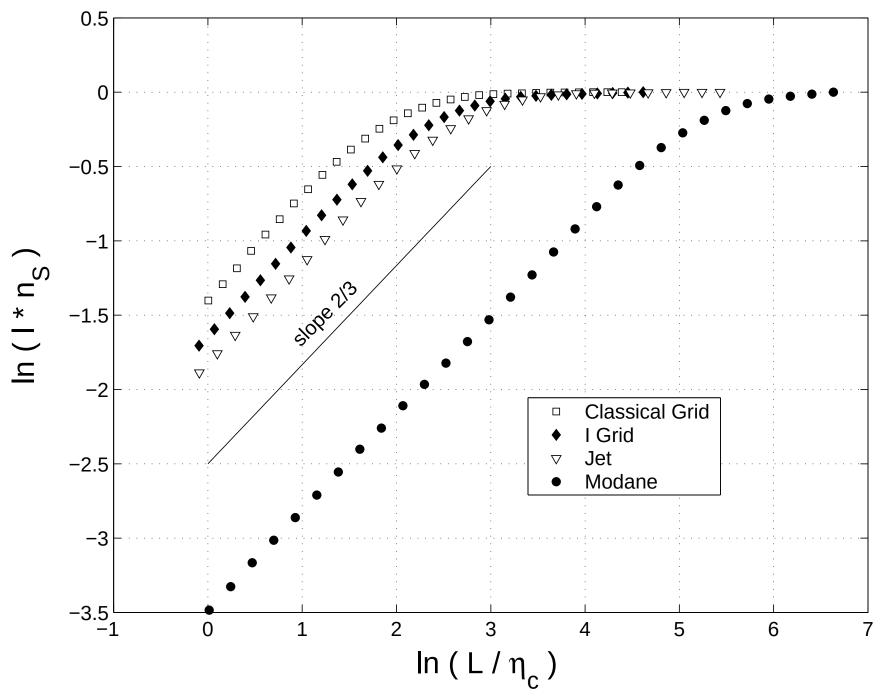
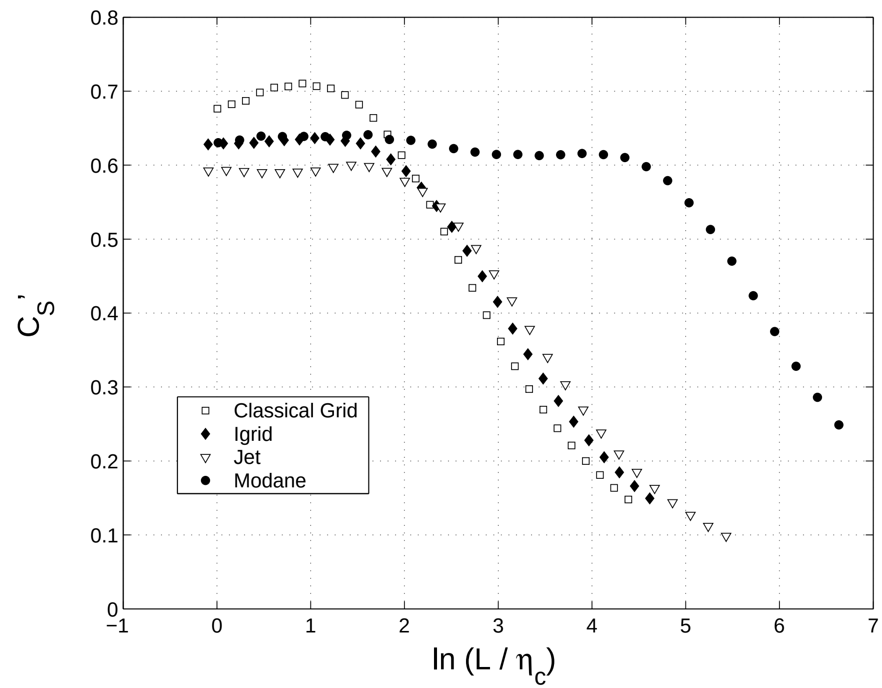
$$n_s(L/\eta_c) = \frac{C_s'}{L}(L/\eta_c)^{2/3}$$

in terms of a dimensionless constant C_s'.

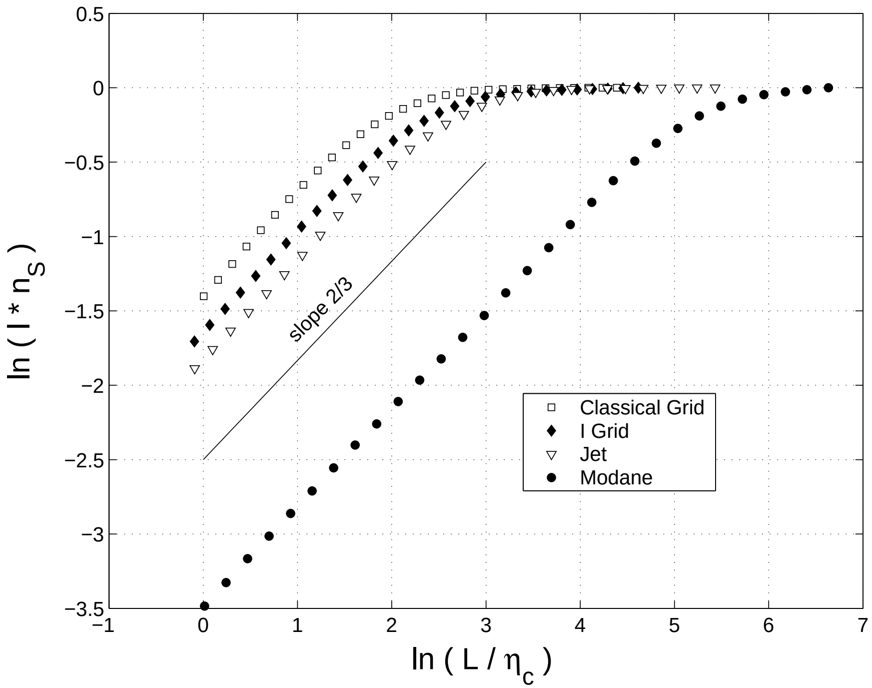
Confirmation

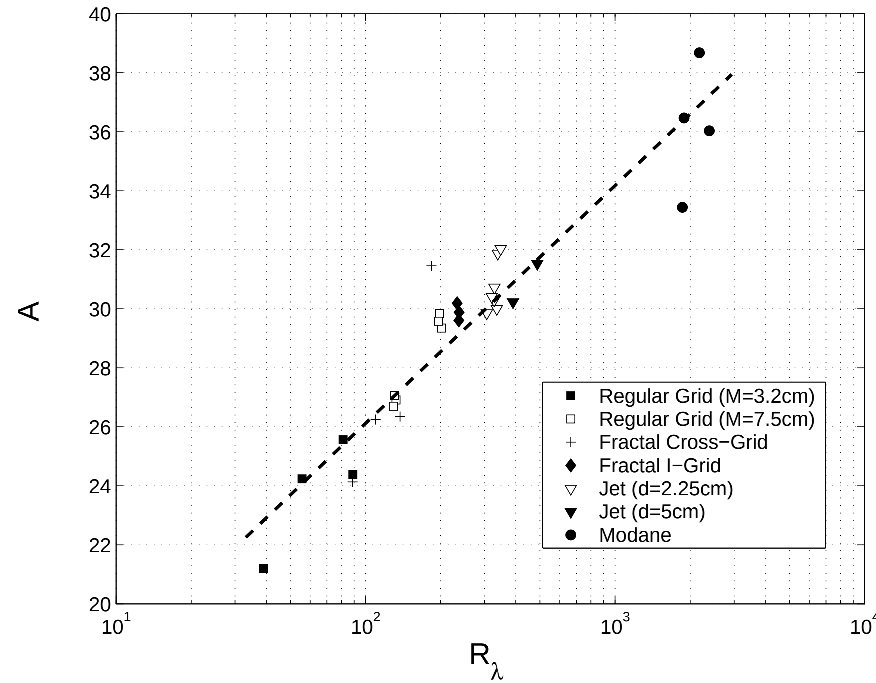




C_s' is a dimensional constant characterising the largest eddies of the turbulence. Indeed, the value of C_s' can be obtained unaltered after low-pass filtering a turbulence data set irrespective of the filter size η_c as long as η_c is between η_* and L.

Inner cutoff length-scale η_*

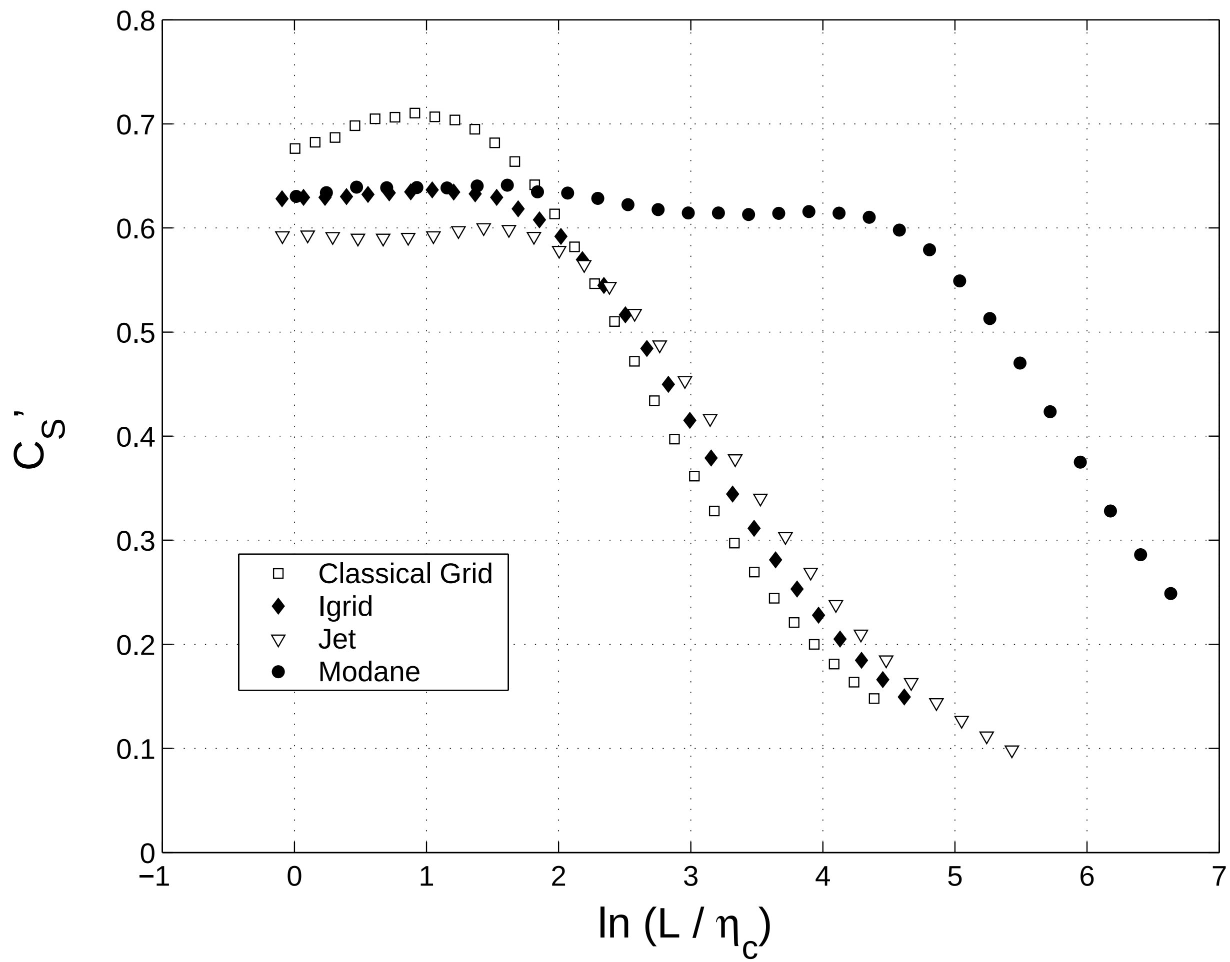




$n_s(L/\eta_c) = \frac{C_s'}{L}(L/\eta_c)^{2/3}$ valid for $\eta_* \leq \eta_c \leq L$

$$A \equiv \eta_*/\eta = 8.2 + 8.9 \log Re_\lambda$$

C_s' is a large-scale constant



C_s' differs from flow to flow; for example, it is significantly larger for classical grid turbulence than for jet turbulence.

Bring everything together

$$\epsilon = 15\nu u'^2/\lambda^2 = C_\epsilon u'^3/L$$

$$\bar{l} = C\pi\lambda$$

$$\bar{l} = n_s^{-1}(L/\eta_*)$$

$$n_s(L/\eta_*) = \frac{C_s'}{L}(L/\eta_*)^{2/3}$$

$$\eta_* = A\eta = A(\nu^3/\epsilon)^{1/4}$$

IMPLY

$$C_\epsilon = K(\frac{CC_s'}{A^{2/3}})^3$$

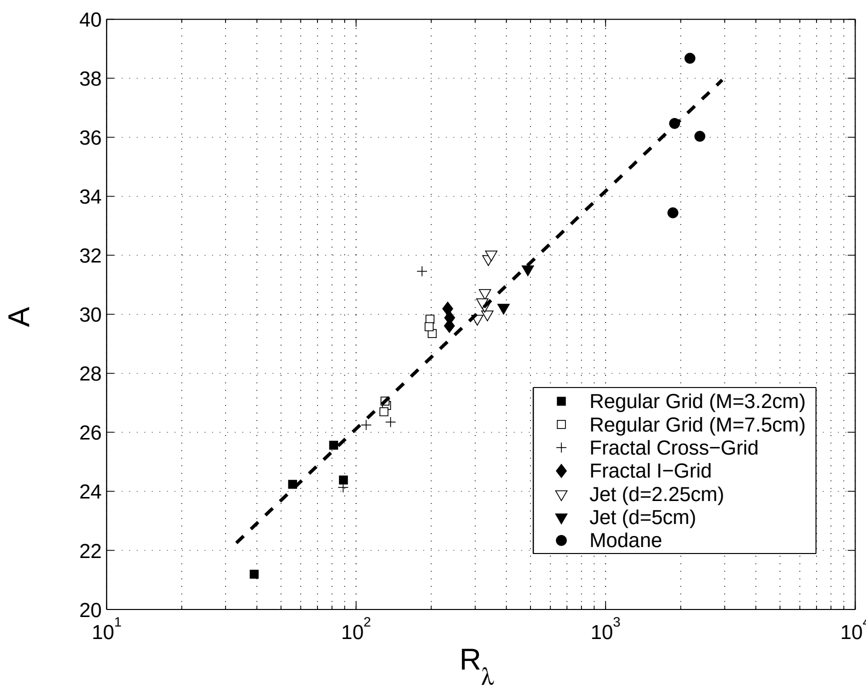
with $K = (15\pi^2)^{3/2} \approx 1801.3$

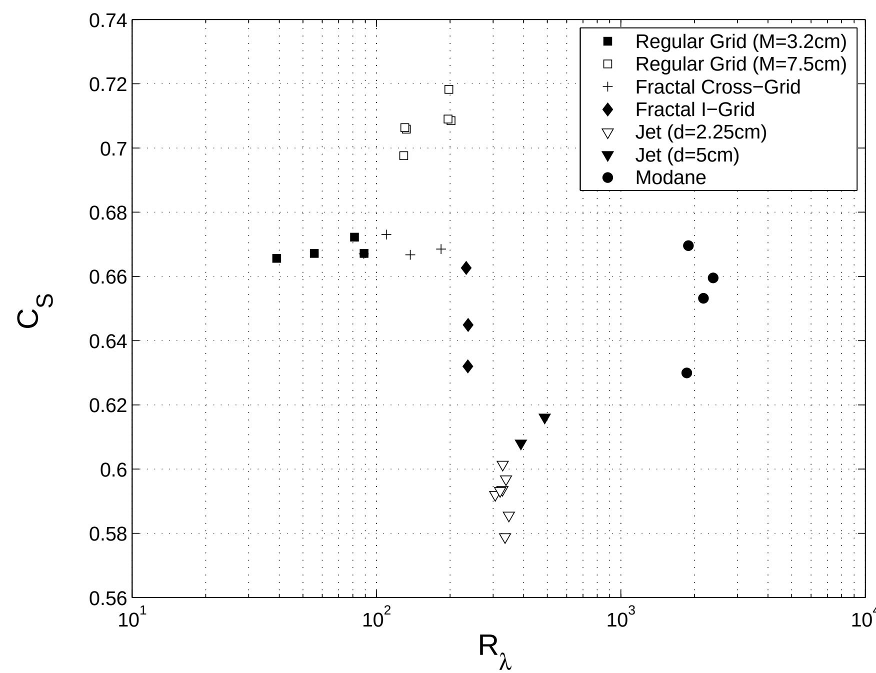
This relation suggests that the dimensionless (small-scale) dissipation rate C_ϵ is directly and strongly dependent on the large-scale dimensionless number C_s'. This large-scale number is something like a number of large-scale eddies. Modify this number and you modify the turbulence dissipation rate. This may be the reason why the high Reynolds number values of C_ϵ obtained in various laboratory and numerical experiments seem to differ from turbulent flow to turbulent flow.

Calculate C_s'

from

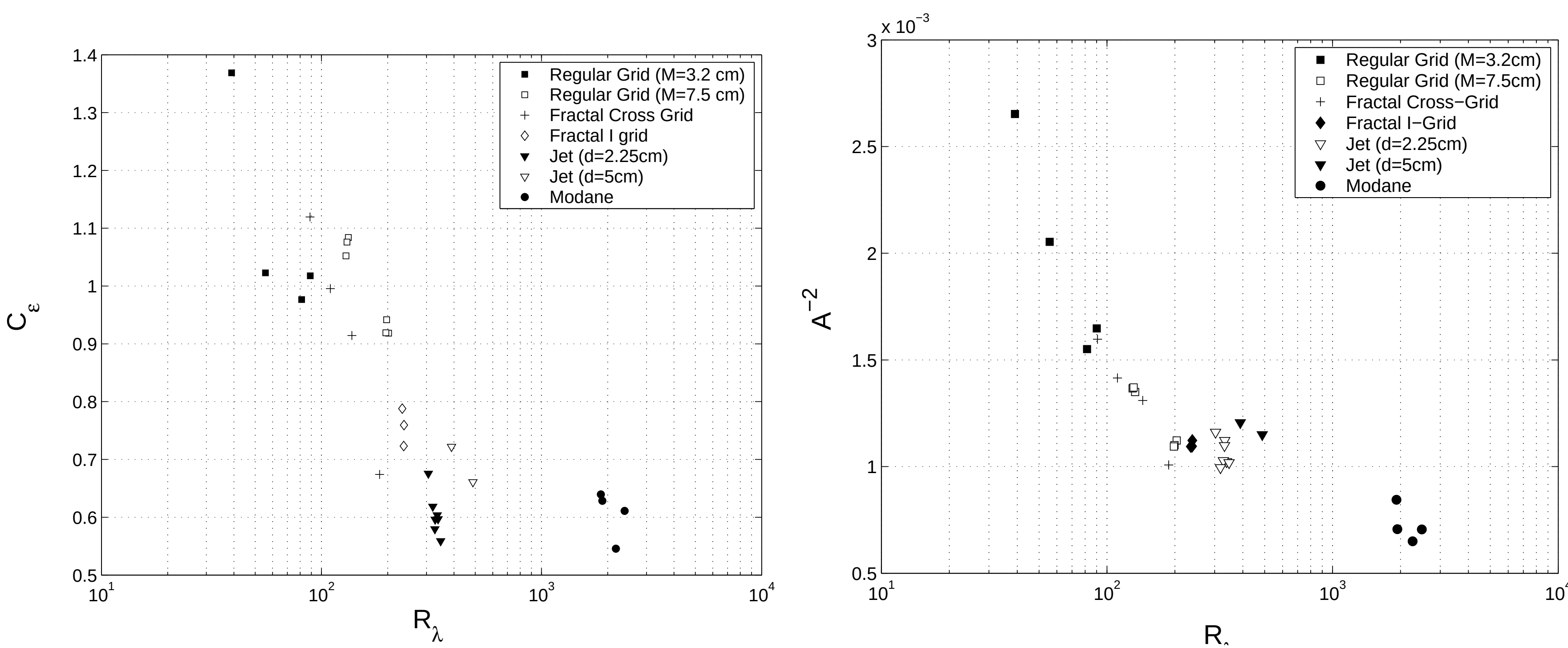
$$C_s' = n_s(L/\eta_*)L(L/\eta_*)^{-2/3}$$





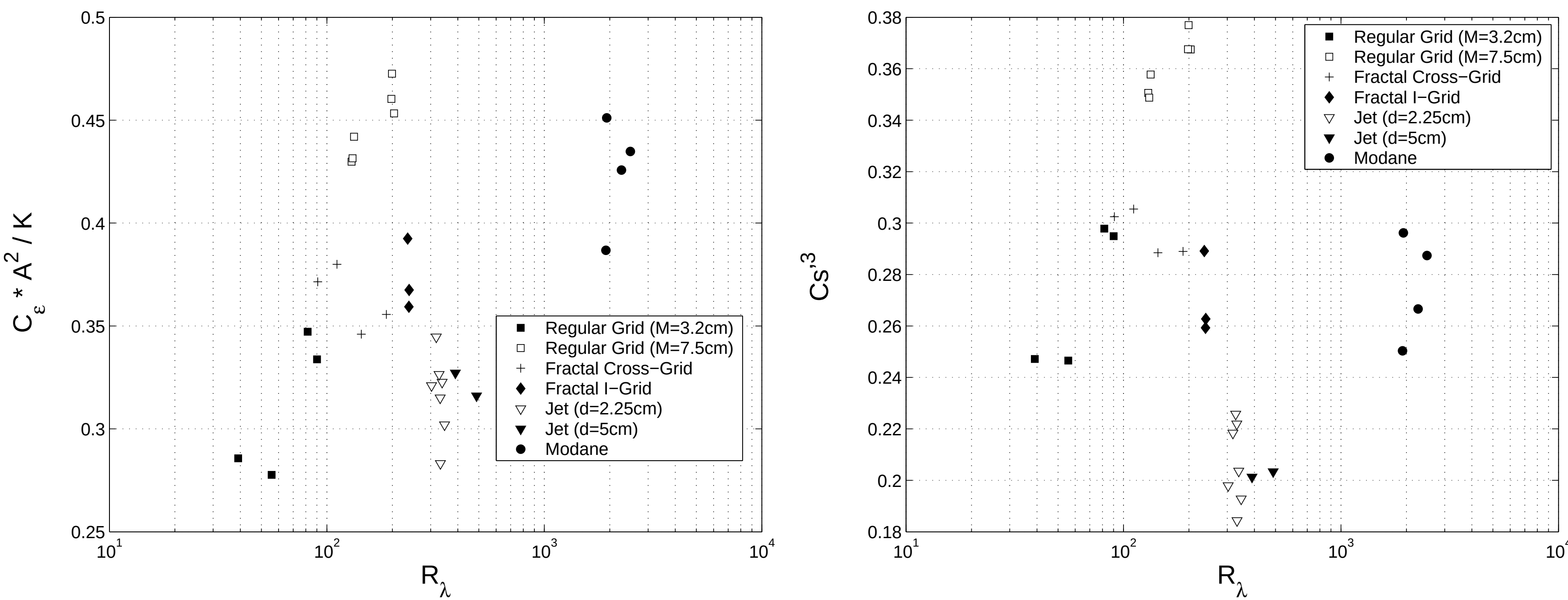
The monotonic increase of $A \equiv \eta_* / \eta$

with Reynolds number accounts for much of the Re_λ-dependence of C_ϵ (as it equals $K \frac{(CC_s')^3}{A^2}$)



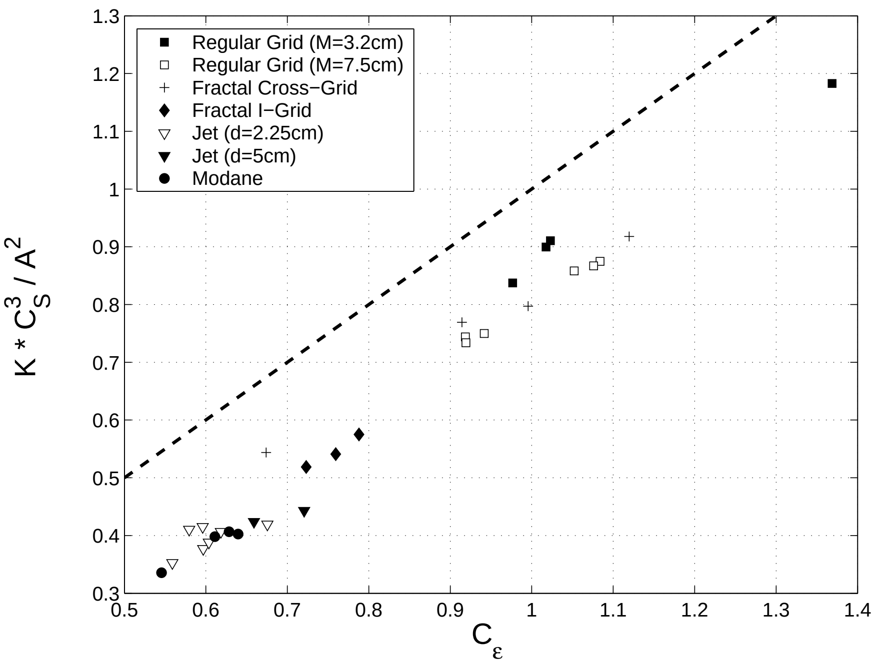
Once much of the Re_λ-dependence

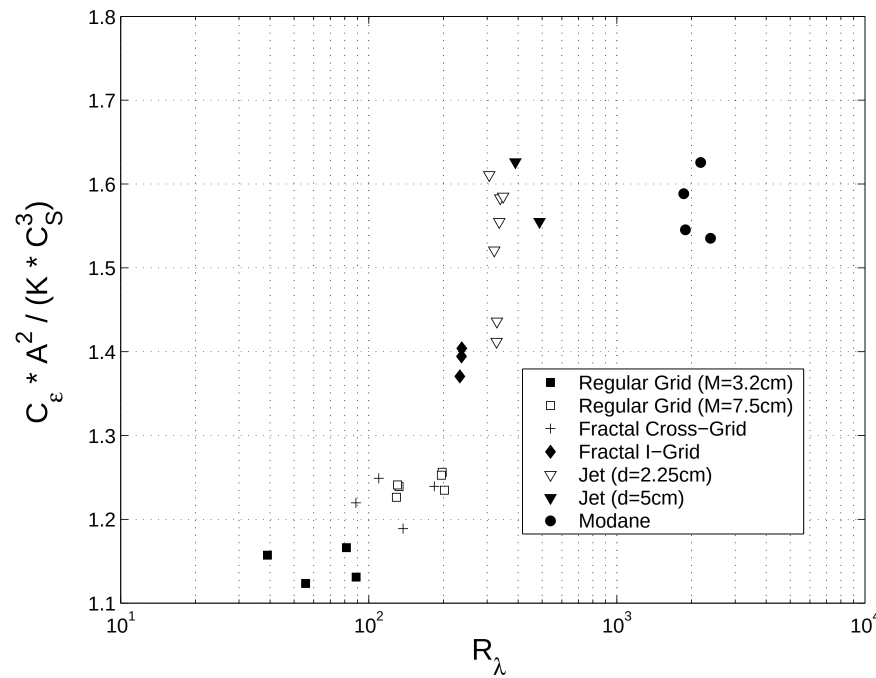
is removed, $C_\epsilon A^2/K$ can be compared to $C_s'^3$



Qualitative agreement with $C_\epsilon A^2/K = C_s'^3$ which is $C_\epsilon = K(\frac{CC_s'}{A^{2/3}})^3$ with $C=1$ (gaussian stats)

Not quite $C_\epsilon = K(\frac{C_s'}{A^{2/3}})^3$

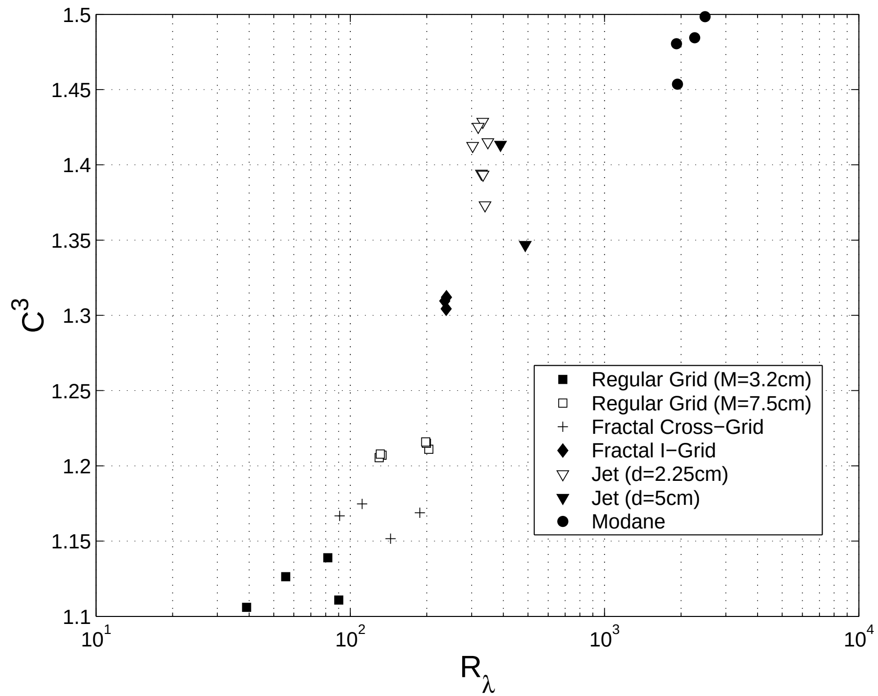


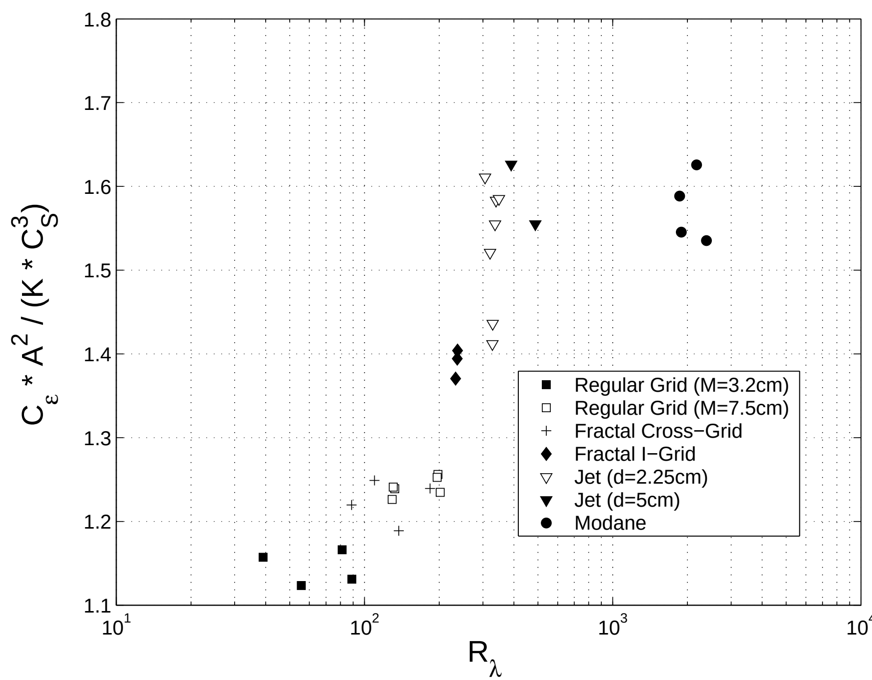


More like $C_\epsilon = K(\frac{CC_s'}{A^{2/3}})^3 + something$ or, better,

$C_\epsilon = K(\frac{CC_s'}{A^{2/3}})^3$ with $C = C(\log Re_\lambda) \neq 1$

Get C from $C < |\frac{du}{dx}| >= \sqrt{\frac{2}{\pi}} < (\frac{du}{dx})^2 >^{1/2}$



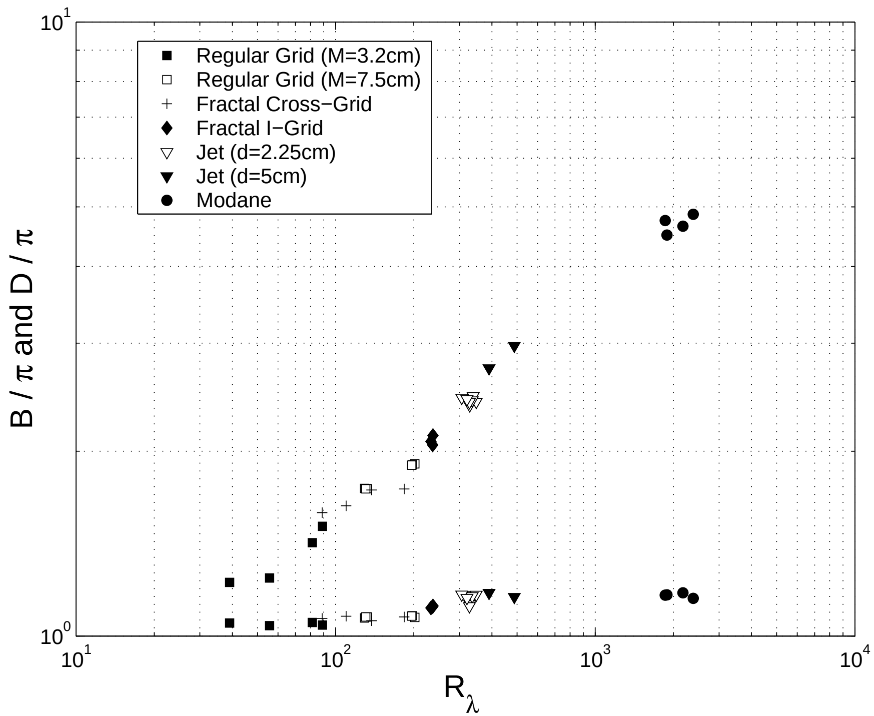


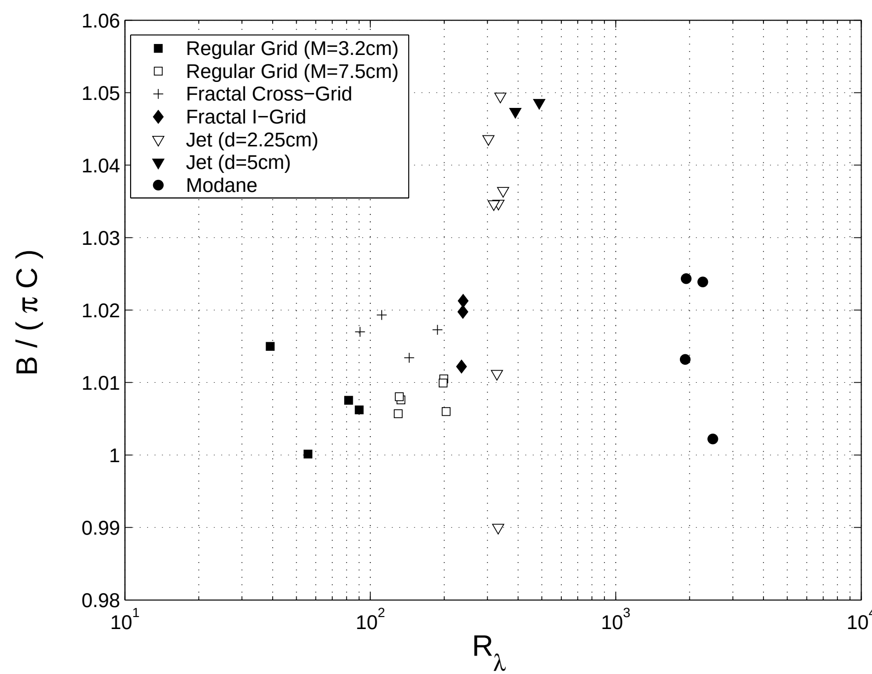
Note 1: C^3 qualitatively matches $C_\epsilon A^2/(K{C'_s}^3)$

Note 2: $C^3 \approx (0.87 + 0.11 \log Re_\lambda)^2$ is an acceptable fit

A check and a bonus

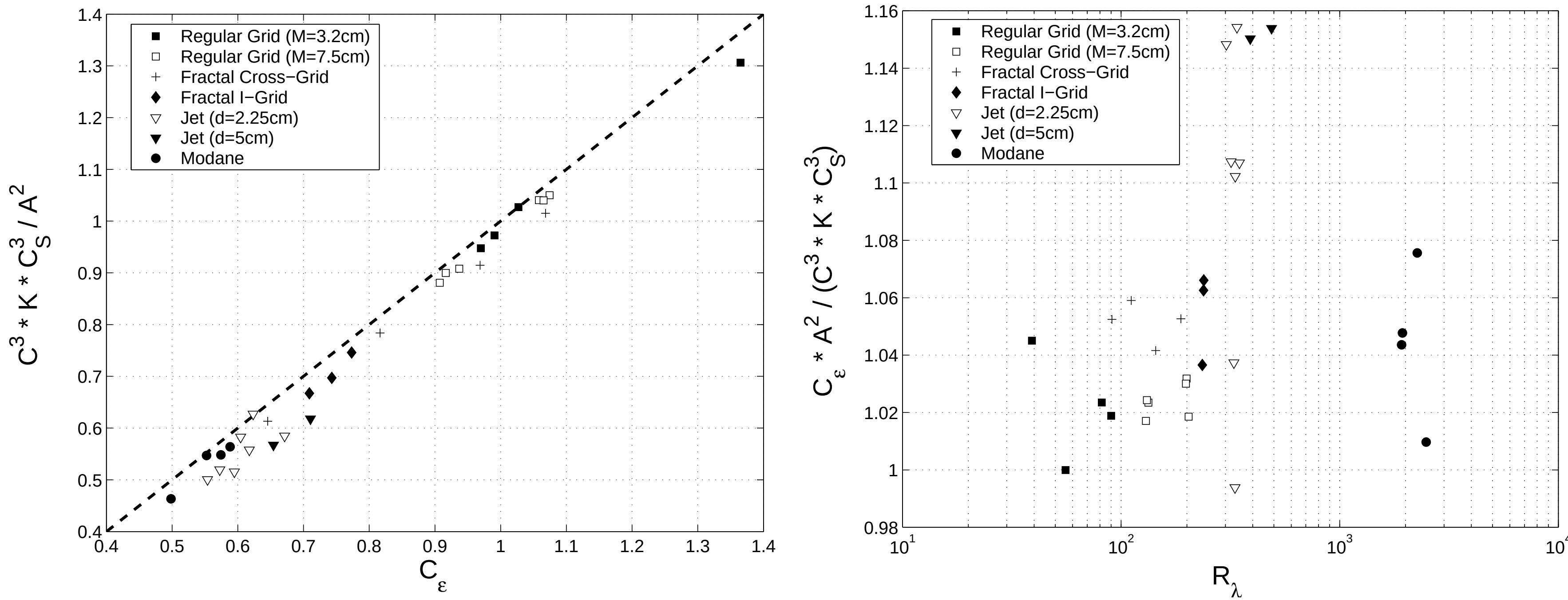
B **calculated from** $\bar{l} = B\lambda$ **and** C **calculated from**
$C < |\frac{du}{dx}| >= \sqrt{\frac{2}{\pi}} < (\frac{du}{dx})^2 >^{1/2}$ **so as to check that** $B = C\pi$





A bonus: define the voids length-scale $\lambda_v \equiv < (l - \bar{l})^2 >^{1/2} = D\lambda$
and find $D \sim Re_\lambda^{1/3}$**.**

$$C_\epsilon = K\left(\frac{CC'_s}{A^{2/3}}\right)^3$$



Deviations remain, but they are all below 16% of $\frac{C_\epsilon}{K(\frac{CC'_s}{A^{2/3}})^3} = 1$. Most of these deviations are $\frac{C_\epsilon}{K(\frac{CC'_s}{A^{2/3}})^3}$ slightly larger than 1 and may be due to varying degrees of anisotropy, slight large-scale non-gaussianity, slight large-small-scale statistical dependence and errors in η_*.

$C_\epsilon \sim C_s'^3$

Most of the non-universal behaviour of C_ϵ is accounted for by the universal strong dependence of C_ϵ on C_s', i.e. $C_\epsilon \sim C_s'^3$, and the non-universality of $C_s'^3$ which characterises, in some sense, the number of large-scale eddies, i.e. the topography of the large scales of the turbulence.

The Reynolds number dependence of C_ϵ

results mostly from its dependence on the slow growth (with Reynolds number) of the range of viscous scales of the turbulence, i.e. on $A = \eta_*/\eta \approx 8.2 + 8.9 \log Re_\lambda$.
However, a smaller part also results from the slow increase (with Reynolds number) of the non-gaussianity of the small scales, i.e. $C^3 \approx (0.87 + 0.11 \log Re_\lambda)^2$. This type of fit has been chosen such that the Reynolds number dependence of C_ϵ, which is all in C^3/A^2 because of $C_\epsilon = K(\frac{CC_s'}{A^{2/3}})^3$, tends to a constant as $Re_\lambda \to \infty$.
C^3/A^2 tends to $\approx 1.5 \times 10^{-4}$ as $\log Re_\lambda \to \infty$. Hence,

$$C_\epsilon \approx 10^{-4} 1.5 (15\pi^2)^{3/2} C_s'^3 \approx 0.275 C_s'^3$$

in the limit $\log Re_\lambda \gg 1$

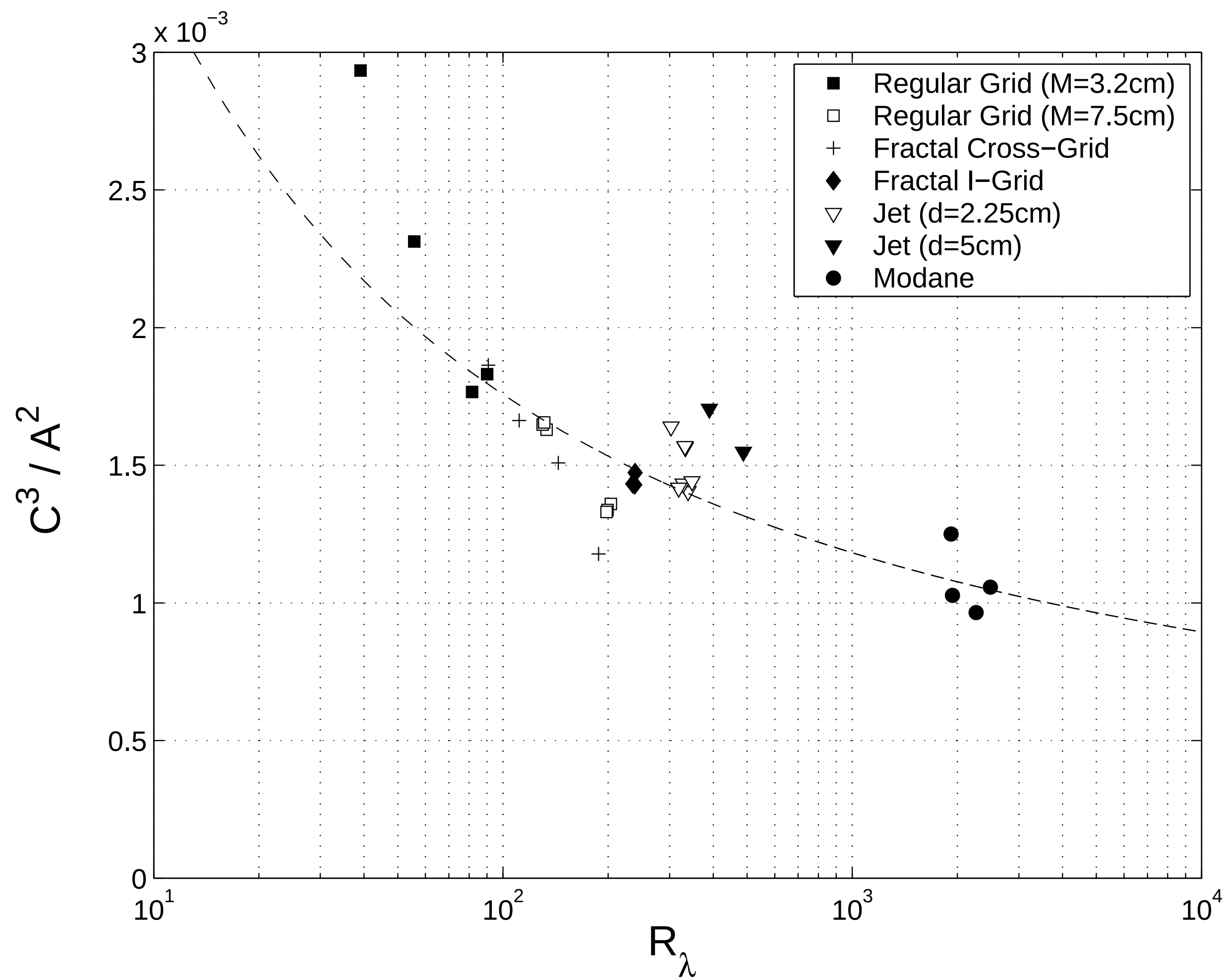
$C_\epsilon \approx 0.275 C_s'^3$ for $\log Re_\lambda \gg 1$

Most of the Reynolds number dependence of C_ϵ at small to moderate values of $\log Re_\lambda$ comes from $A = \eta_*/\eta$, but the non-gaussianity of du/dx catches up as $\log Re_\lambda$ increases bringing in its own Reynolds number dependence which eventually compensates that of A.

In that limit,

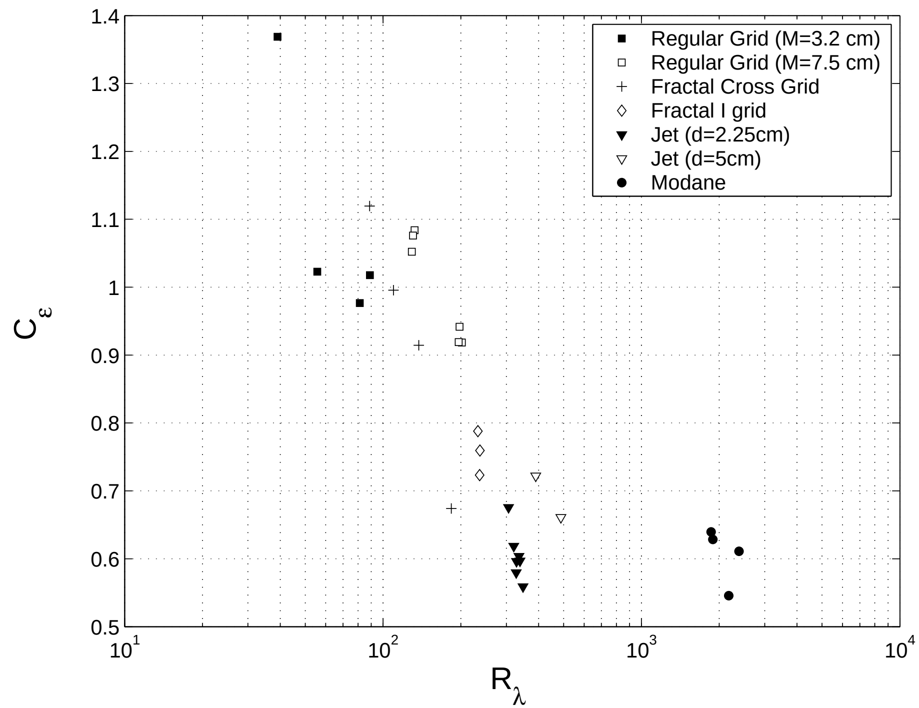
$$C_\epsilon \approx 0.275 C_s'^3$$

The limit $\log Re_\lambda \rightarrow \infty$



The dotted curve is $C^3/A^2 \approx (\frac{0.87+0.11\log Re_\lambda}{8.2+8.9\log Re_\lambda})^2 \rightarrow \approx 1.5 10^{-4}$.

C_ϵ in the limit $\log Re_\lambda \to \infty$



0.5 or 0.6 may not be the real asymptotic value of C_ϵ. The real asymptotic value may be between 0.07 and 0.09, i.e. an order of magnitude smaller because of the very slow asymptotics in $\log Re_\lambda$. Need Re_λ up to at least 10^9 to see this!

CONCLUSIONS SO FAR

The dimensionless dissipation rate constant C_ϵ of homogeneous isotropic fluid turbulence is such that

$$C_\epsilon = f(\log Re_\lambda) C_s'^3$$

where $f(\log Re_\lambda)$ is a dimensionless function of $\log Re_\lambda$ which appears to tend to 0.275 (by extrapolation!) in the limit where $\log Re_\lambda \gg 1$ (as opposed to just $Re_\lambda \gg 1$).

The dimensionless number C_s' reflects the number of “large-scale eddies” and is therefore non-universal.

Most of the non-universal asymptotic values of C_ϵ stem, therefore, from its universal dependence on C_s' and can be calculated from it!

CONCLUSIONS SO FAR

The Reynolds number dependence of C_ϵ at values of $\log Re_\lambda$ close to and not much larger than 1 is primarily governed by the slow growth (with Reynolds number) of the range of viscous scales of the turbulence.

The eventual Reynolds number independence of C_ϵ is achieved by an eventual balance between this slow growth and the increasing non-gaussianity of the small-scales.

However this happens in $\log Re_\lambda$ asymptotics and values of Re_λ as high as at least 10^9 are needed to reach the asymptotic constant.

Generalised Rice theorem

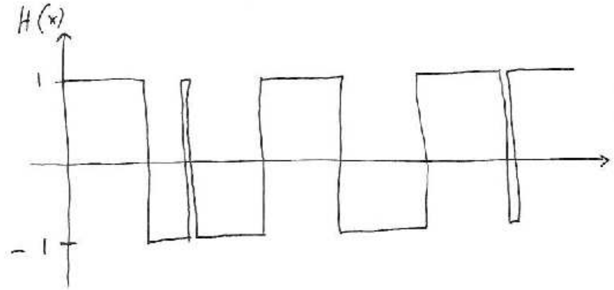
For high Reynolds number HIT

$$\lambda \sim l_s$$

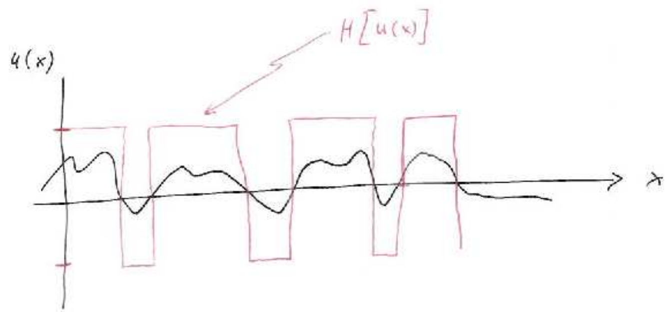
where λ is the Taylor microscale ($\epsilon = 15\nu u'^2/\lambda^2$) and l_s is the average distance between neighboring stagnation points defined as the $-1/3$ power of the number density of stagnation points. Stagnation points are points where the turbulent fluctuation velocity is 0.

Proved in HIT under assumptions of (i) statistical independence between large and small scales and (ii) absence of small-scale intermittency effects.

Rice theorem



$\int \left| \frac{d}{dx} H(x) \right| dx =$ number of "zero-crossings" within range of integration.



$\int \left| \frac{d}{dx} H[u(x)] \right| dx =$ number of zero-crossings of $u(x)$ within range of integration.

$$= \int \delta[u(x)] \left| \frac{du}{dx} \right| dx$$

Rice theorem

$n_0 \equiv$ number density of zero-crossings of $u(x)$

$$n_0 = \left\langle \delta[u(x)] \left| \frac{du}{dx} \right| \right\rangle$$

$$= \int P(u, u')\, \delta(u)\, |u'|\, du\, du'$$

IF $P(u, u') = p(u)\, p'(u')$

Then

$$\boxed{n_0 = p(u=0) \left\langle \left| \frac{du}{dx} \right| \right\rangle}$$

Generalised Rice theorem

From zero-crossings to <u>stagnation points</u>

$$\underline{u} = (u, v, w) = (0, 0, 0)$$

Generalise as follows:

number of stagnation points within volume V is

$$\int dV \, \left|\underline{\nabla} H[u(\underline{x})]\right| \left|\underline{\nabla} H[v(\underline{x})]\right| \left|\underline{\nabla} H[w(\underline{x})]\right|$$

$$= \int dV \, \delta[u(\underline{x})] \, \delta[v(\underline{x})] \, \delta[w(\underline{x})] \, |\underline{\nabla} u| \, |\underline{\nabla} v| \, |\underline{\nabla} w|$$

Hence, the number density of stagnation points is given by

$$n_s = \langle \delta(u) \, \delta(v) \, \delta(w) \, |\underline{\nabla} u| |\underline{\nabla} v| |\underline{\nabla} w| \rangle$$

Generalised Rice theorem

IF $\underline{P}(\underline{u}, \underline{\nabla} \underline{u}) = p_\ell(\underline{u}) \; p_s(\underline{\nabla} \underline{u})$

THEN $$\boxed{n_s = p_\ell(\underline{u} = 0) \; \langle |\underline{\nabla} u| \, |\underline{\nabla} v| \, |\underline{\nabla} w| \rangle}$$

Assuming that $p_s(\underline{\nabla} \underline{u})$ scales with

$$\sigma \equiv \langle \left(\frac{\partial}{\partial x} u\right)^2 \rangle^{1/2}$$

and that $p_\ell(\underline{u})$ scales with $u' = \langle u^2 \rangle^{1/2}$

we get

$$n_s \sim \frac{\sigma^3}{u'^3}$$

$$\Rightarrow \quad \ell_s \equiv n_s^{-1/3} = B \frac{u'}{\sigma} = B \lambda$$

where B is a dimensionless constant.

Number density of stagnation points

The number density of stagnation points in homogeneous isotropic turbulence scales as [PRL **91**, 144501 (2003)]

$$n_s(L/\eta_c) = \frac{C_s}{L^3}(L/\eta_c)^2$$

Compare this with the number density of zero-crossings of the longitudinal velocity component

$$n_s(L/\eta_c) = \frac{C_s'}{L}(L/\eta_c)^{2/3}$$

$$C_\epsilon = 15^{3/2}\frac{C_s}{A^2B^3}$$

FOLLOWS FROM

$$\epsilon = 15\nu u'^2/\lambda^2 = C_\epsilon u'^3/L$$

$$\lambda = Bl_s$$

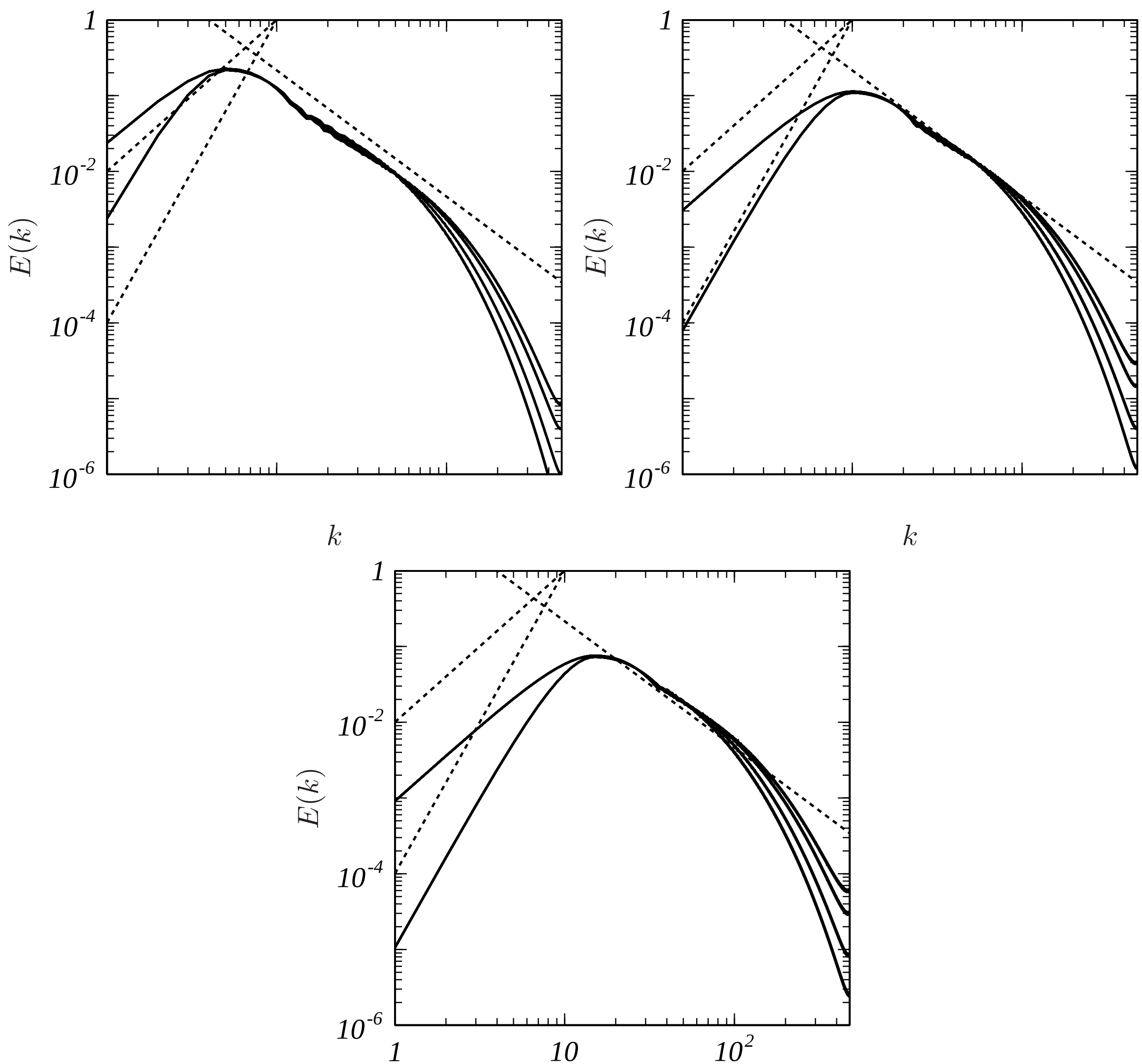
$$l_s = n_s^{-3}(L/\eta_*)$$

$$n_s(L/\eta_*) = \frac{C_s}{L^3}(L/\eta_*)^2$$

$$\eta_* = A\eta = A(\nu^3/\epsilon)^{1/4}$$

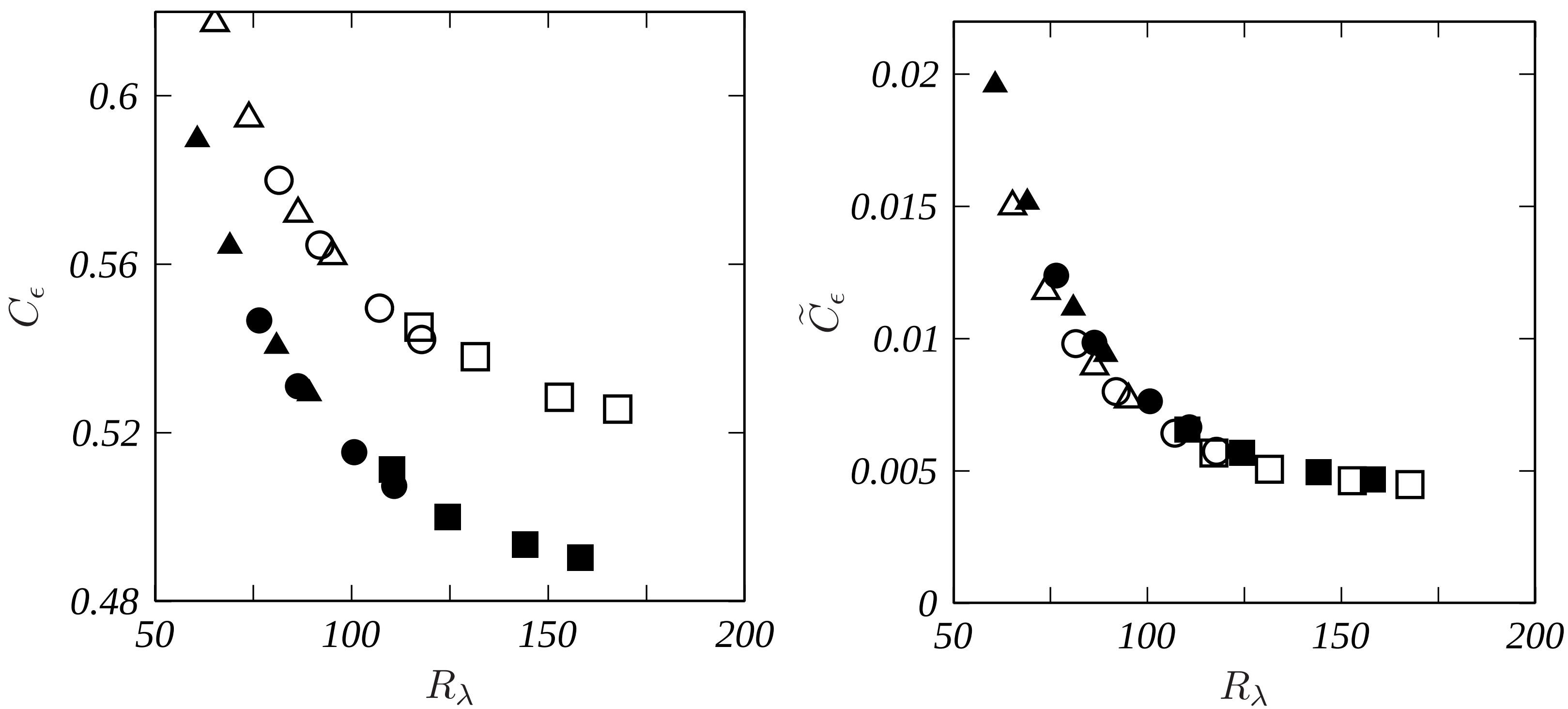
$E(k)$ **for 24 DNS of HIT:** $60 \leq Re_\lambda \leq 170$

$E(k) \sim k^p$, $p = 2$, 4 at $k \ll k_{peak}$: 4 values of ν and 3 of k_{peak}



C_ϵ and $\tilde{C}_\epsilon \equiv C_\epsilon B^3 / C_s$

C_ϵ is a function of p but $\tilde{C}_\epsilon$ is not.



These DNS results support
$C_\epsilon(p, Re_\lambda) \sim C_s(p)/A^2(Re_\lambda)B^3(Re_\lambda)$.

(see Goto & V PoF **21**, (2009) for more details)

CONCLUSION

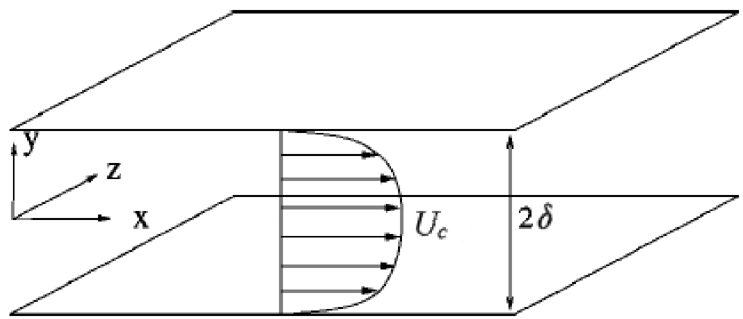
$$C_\epsilon = f(Re_\lambda) C_s$$

where C_s is the number of large-scale stagnation points

and $f(Re_\lambda) \sim A^{-2}(Re_\lambda) B^{-3}(Re_\lambda)$ is determined by the opposing (perhaps balancing) effects of (i) the slow growth (with Reynolds number) of the range of viscous scales of the turbulence – represented by $A(Re_\lambda)$, and (ii) the increasing non-gaussianity of the small-scales (small-scale intermittency) – represented by $B(Re_\lambda)$.

DNS of turbulent channel flow

$Re_\tau \equiv \frac{u_\tau \delta}{\nu} \approx 110$ to 400: too small?...perhaps not for everything...

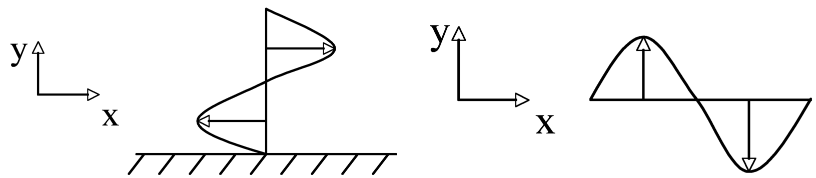


We use a code developed at the University of Poitiers by S. Laizet & E. Lamballais: 6th order compact finite difference scheme; fractional step method for incompressible N.S. using 3-stage 3d order R-K scheme; Poisson pressure equation solved in Fourier space (with staggered grid and FFT on non-uniform grid).

DNS of turbulent channel flow

Periodic boundary conditions except at the walls, where
(i) either $\mathbf{u} = 0$;
or the boundary conditions and near wall forcings are borrowed from numerical studies of flow control schemes aimed at drag reduction, i.e.
(ii) either $\mathbf{u} = 0$ with forcing $\mathbf{f} = (-A\sin(2\pi y/\Lambda)H(\Lambda - y), 0, 0)$ added to N.S. (Xu, Dong, Maxey & Karniadakis JFM 2007) with $A = 0.16U_c^2/\delta \approx 2u_\tau^2/\delta_\nu$ and $\Lambda = 11\delta_\nu$;
(iii) or $\mathbf{u}(x,t) = (0, a\cos(\alpha(x - ct)), 0)$ (Min, Kang, Meyer & Kim JFM 2006) with $a/U_c = 0.05$, $\alpha = 0.5/\delta$ and $c = -2U_c$;
(iv) or $\mathbf{u} = 0$ at walls and $v(x, y_d, z, t)$ replaced by $-v(x, y_d, z, t)$ where $y_d = 10\delta_\nu$ at every time step (Choi, Moin & Kim JFM 1994).
Notation: $\delta_\nu \equiv \nu/u_\tau$ and U_c is the mean centre-line velocity.
We set the bulk velocity U_b equal to 2/3 by varying the mean pressure gradient accordingly at all times in all simulations.

DNS of turbulent channel flow



(a) Xu et al (b) Min et al

Case	Forcing	Re_c	Re_τ	L_x	L_z	$N_x \times N_y \times N_z$
A	No	4250	178.99	$4\pi\delta$	$4\pi\delta/3$	$200 \times 129 \times 200$
A1	Xu *et. al.*	4250	114.40	$4\pi\delta$	$4\pi\delta/3$	$200 \times 129 \times 200$
A2	Min *et. al.*	4250	222.27	$4\pi\delta$	$4\pi\delta/3$	$200 \times 129 \times 200$
A3	Choi *et. al.*	4250	141.63	$4\pi\delta$	$4\pi\delta/3$	$200 \times 129 \times 200$
B	No	2400	109.48	$4\pi\delta$	$2\pi\delta$	$100 \times 65 \times 100$
C	No	10400	390.99	$2\pi\delta$	$\pi\delta$	$256 \times 257 \times 256$

$\nu \frac{d}{dy}\overline{U} - < uv > = u_\tau^2(1 - y/\delta)$ for all y in all cases except with Xu et al forcing where it holds for $y > \Lambda$ and $2\delta - y > \Lambda$.

When $Re_\tau = \delta/\delta_\nu \gg 1$

one might expect an intermediate region $\delta_\nu \ll y \ll \delta$ where production balances dissipation locally (Townsend 1961), i.e.

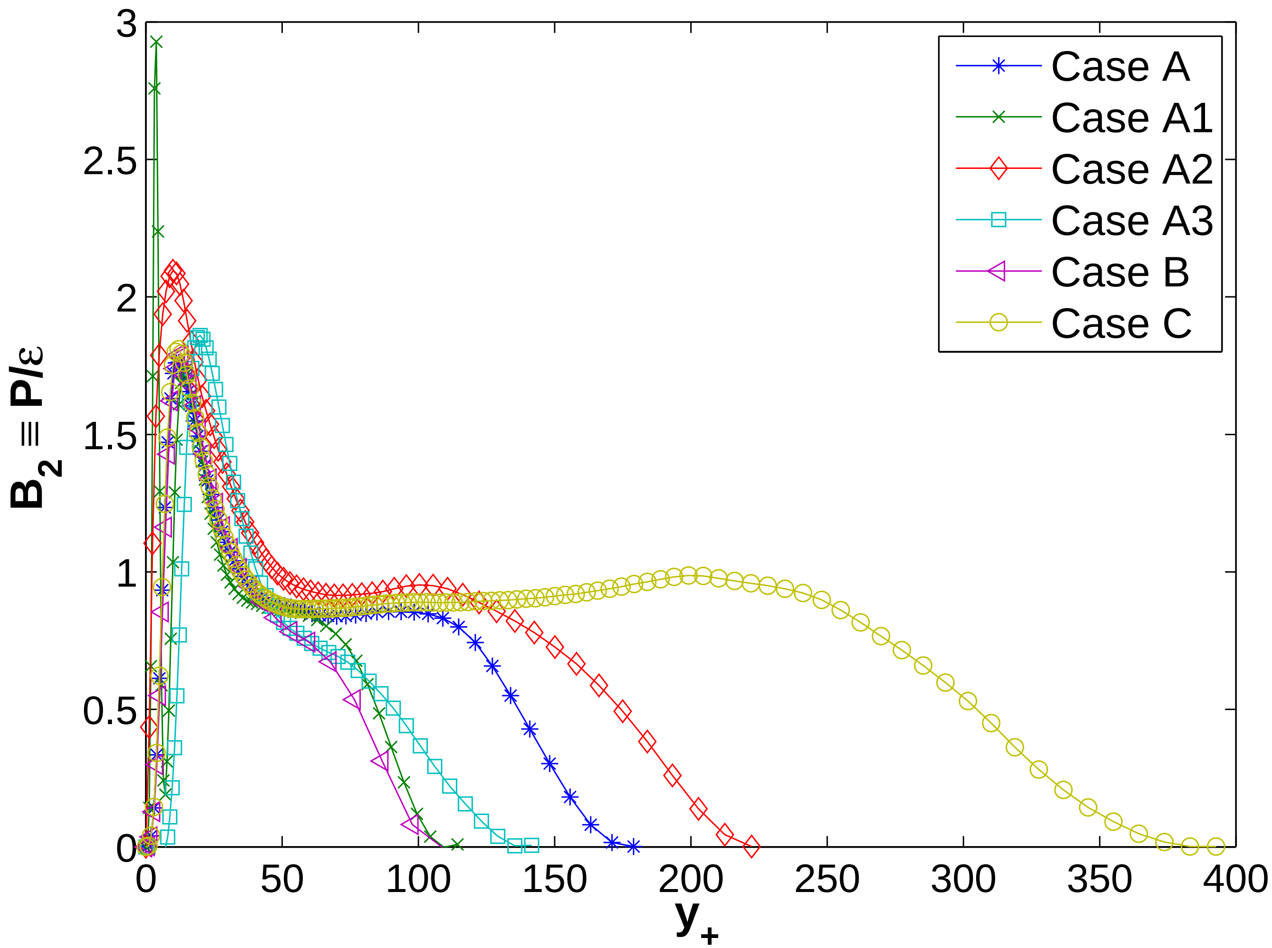
$$- < uv > \frac{d}{dy}\overline{U} \approx \epsilon.$$

$\nu \frac{d}{dy}\overline{U} - < uv > = u_\tau^2(1 - y/\delta)$ implies $- < uv > \approx u_\tau^2$ in this intermediate region as $\delta/y \to \infty$ and $\delta_\nu/y \equiv 1/y_+ \to 0$. (Assuming that, in this limit, $\frac{d}{d \ln y_+}\overline{U}_+$ does not increase faster than y_+^p with $p \geq 1$.)

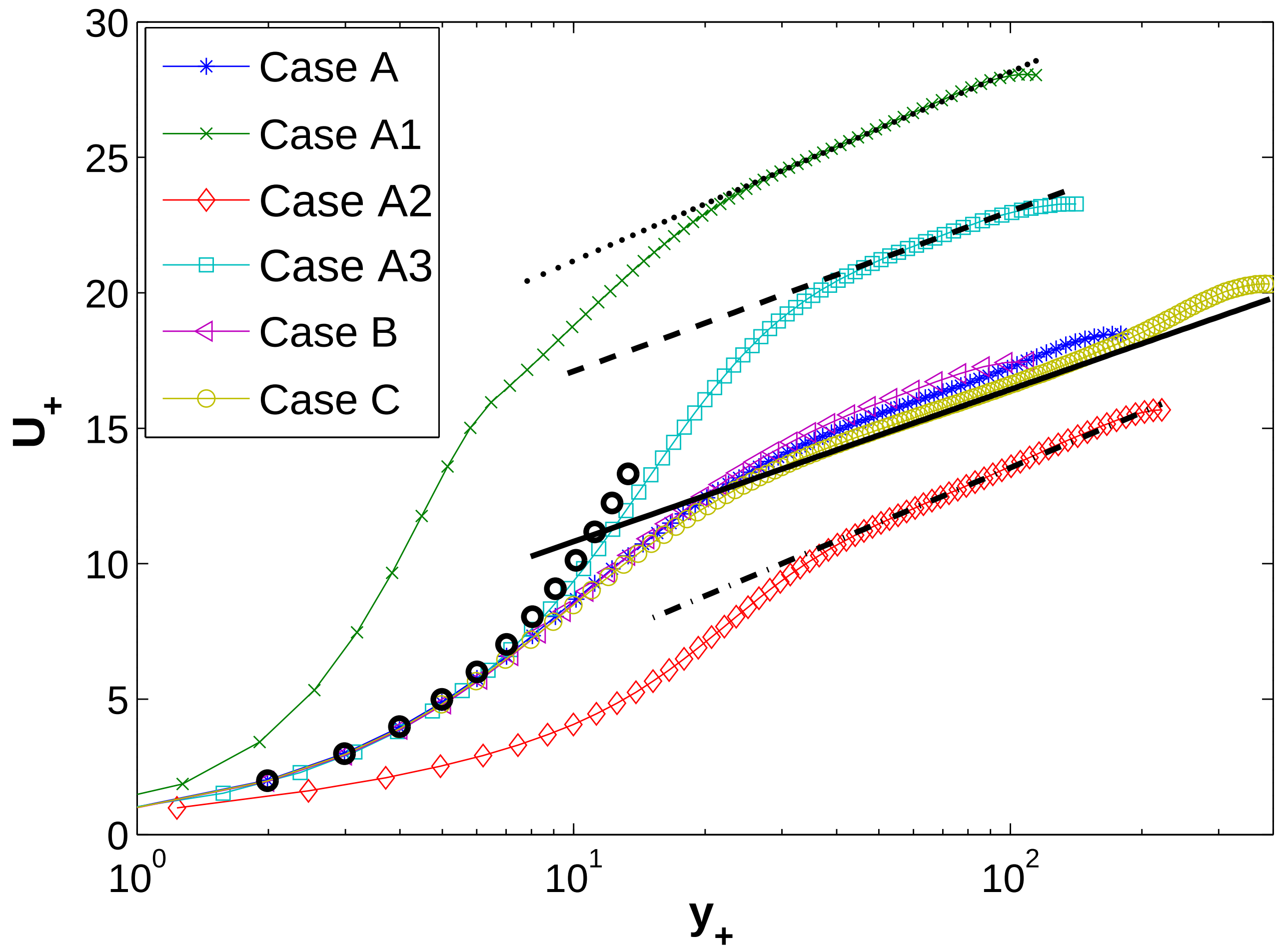
It then follows that, in this equilibrium intermediate region,

$$\epsilon \approx \frac{u_\tau^3}{\kappa y} \text{ IFF } \frac{d}{dy}\overline{U} \approx \frac{u_\tau}{\kappa y}$$

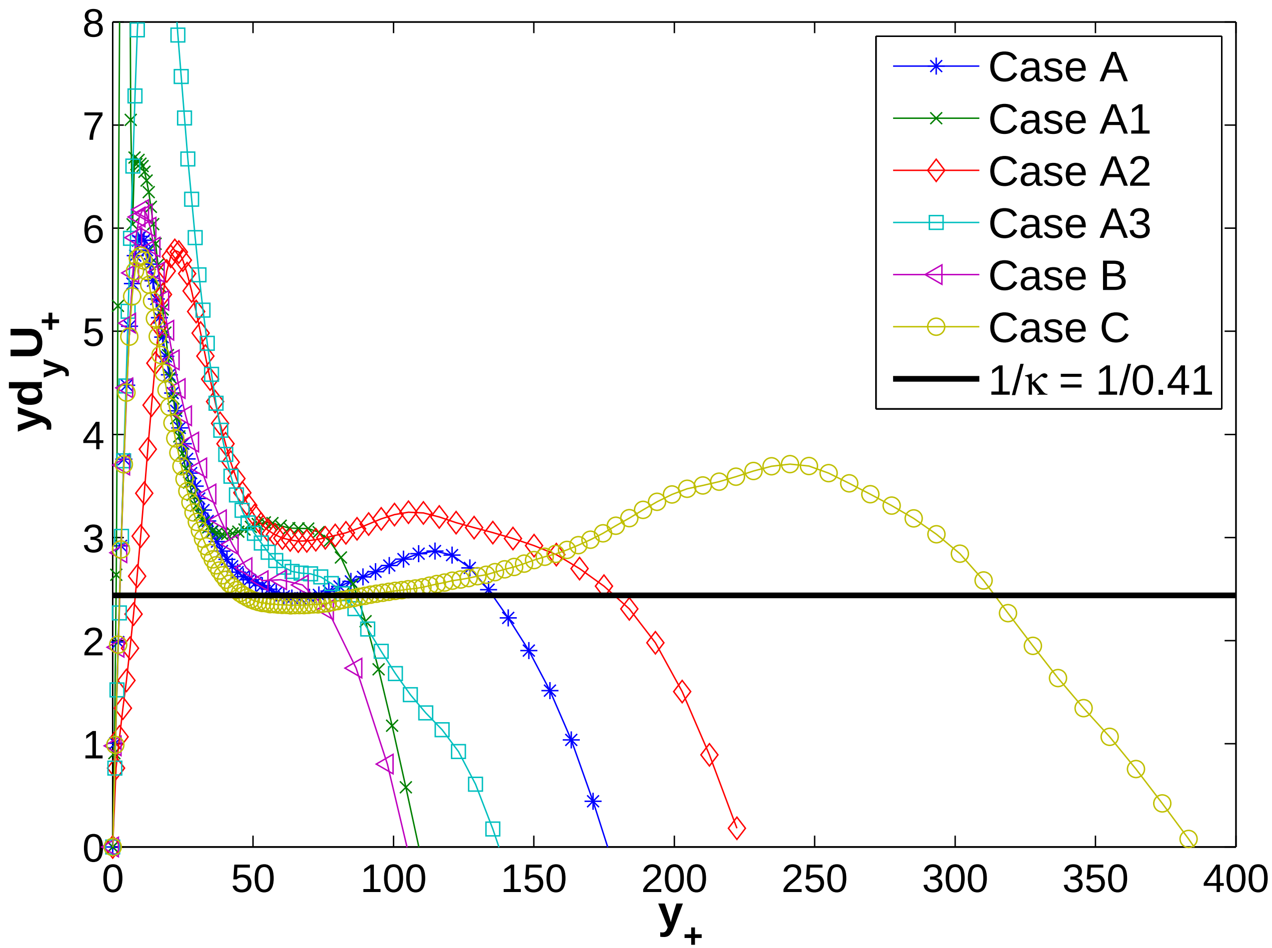
$P \equiv - < uv > \frac{d}{dy}\overline{U}$ **balances** ϵ



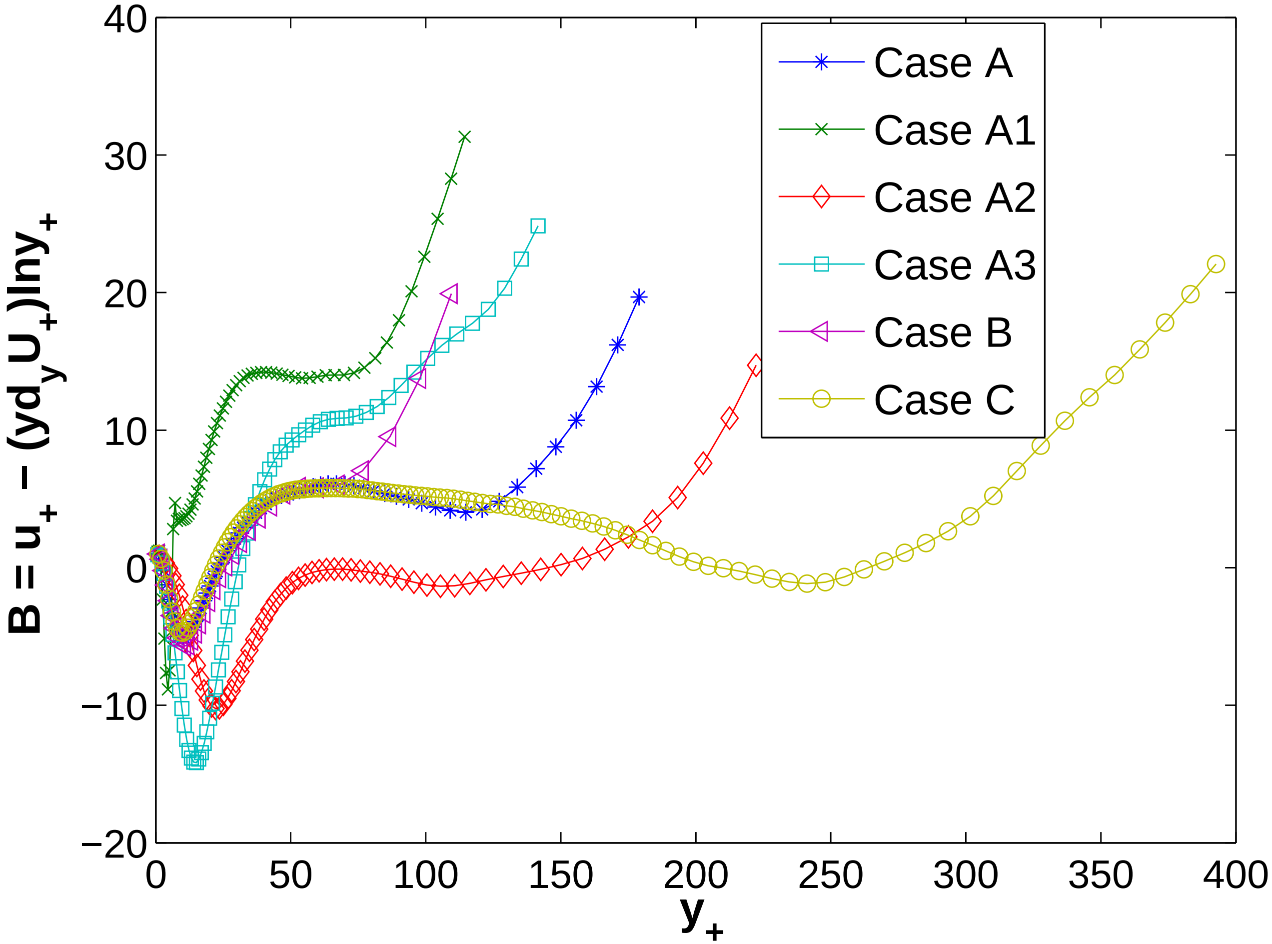
Mean flow profiles



Karman “constant”



$$B = \overline{U}_+ - \frac{\ln y_+}{\kappa}$$



Look at $\epsilon \approx \frac{u_\tau^3}{\kappa y}$ instead of $\frac{d}{dy}\overline{U} \approx \frac{u_\tau}{\kappa y}$

Generalised Rice theorem for high Reynolds number HIT (Mazellier & V 2008 PoF **20**, 014102; Goto & V 2009 PoF **21**, 035104):

$$\lambda \sim l_s$$

where λ is the Taylor microscale ($\epsilon = 15\nu u'^2/\lambda^2$) and l_s is the average distance between neighboring stagnation points defined as the $-1/3$ power of the number density of stagnation points. Stagnation points are points where the turbulent fluctuation velocity is 0.

Proved in HIT under assumptions of (i) statistical independence between large and small scales and (ii) absence of small-scale intermittency effects.

$\lambda \sim l_s$ in TCF?

Does $\lambda \sim l_s$ hold in the intermediate equilibrium region of Turbulent Channel Flow (TCF) in the sense that

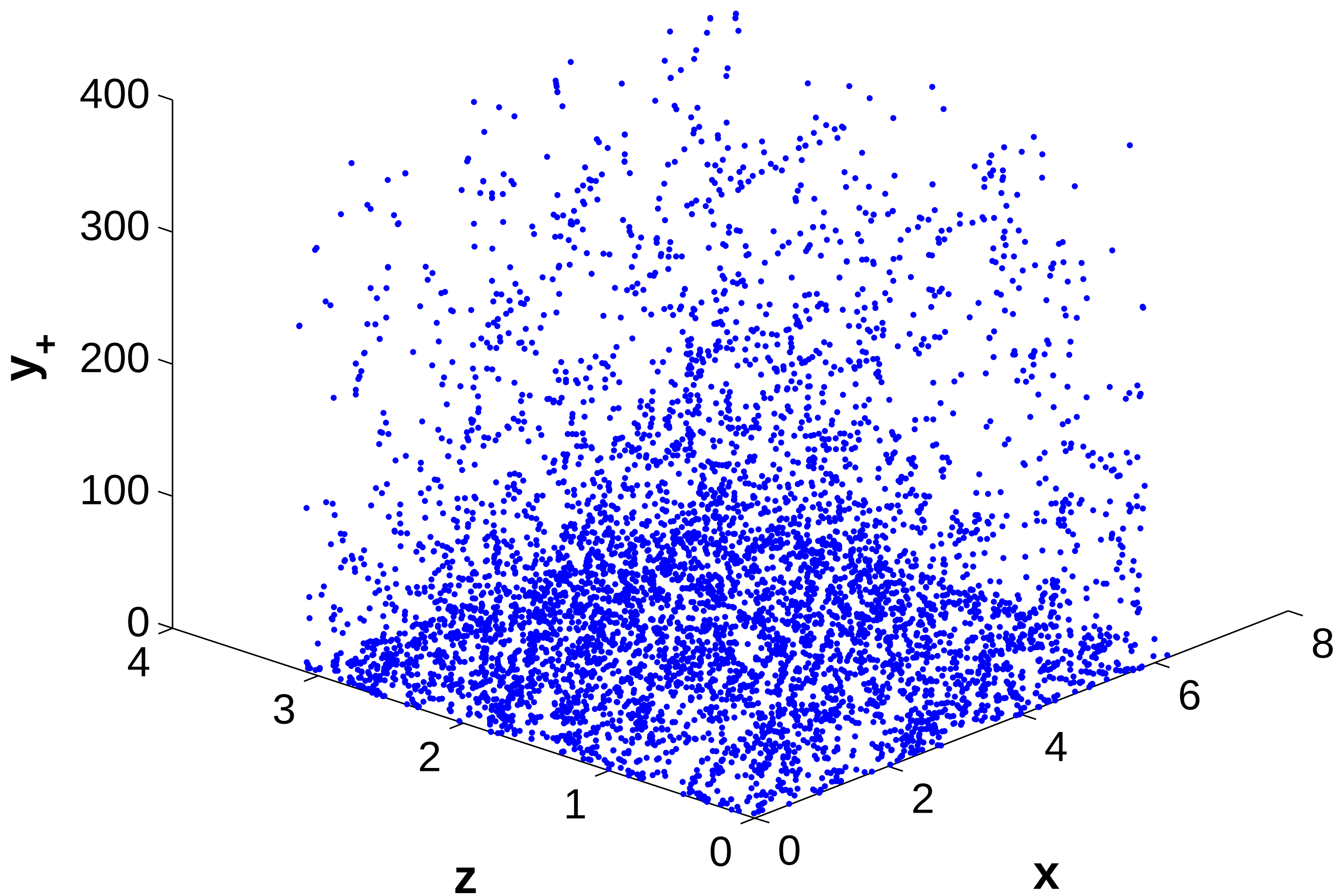
$$\lambda = B_1 l_s$$

where

B_1 is independent of y and Reynolds number for Reynolds number $>> 1$?

Stagnation points of fluctuating velocities

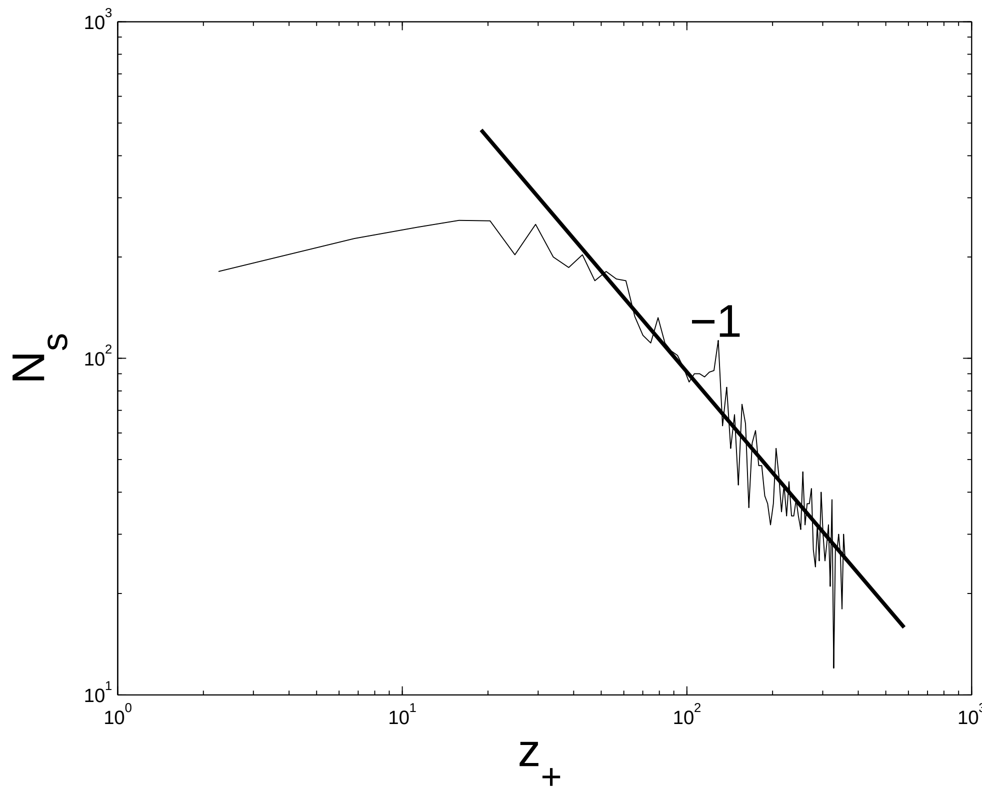
Points where $\mathbf{u}' \equiv \mathbf{u} - \overline{U}\mathbf{e_x} = 0$



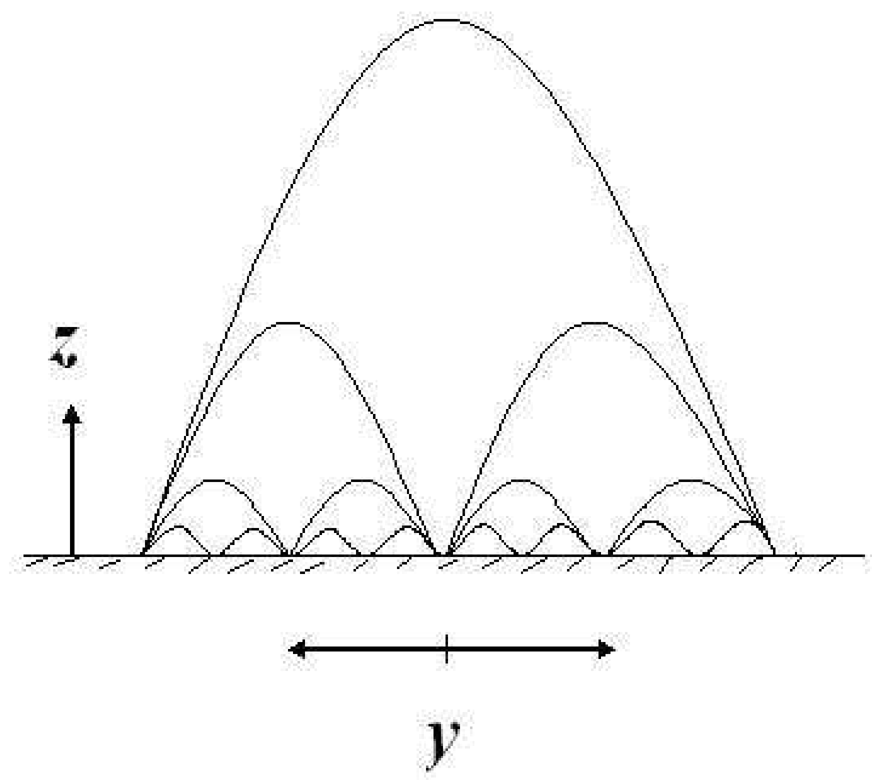
Number of stagnation points

$N_s \equiv$ total number of points where all components of the velocity fluctuations around the local mean are zero in a thin slab of dimensions $L_x \times L_z \times \delta_y$ (δ_y small, $\delta_y \sim \delta_\nu$) parallel to the channel's $y = 0$ wall.

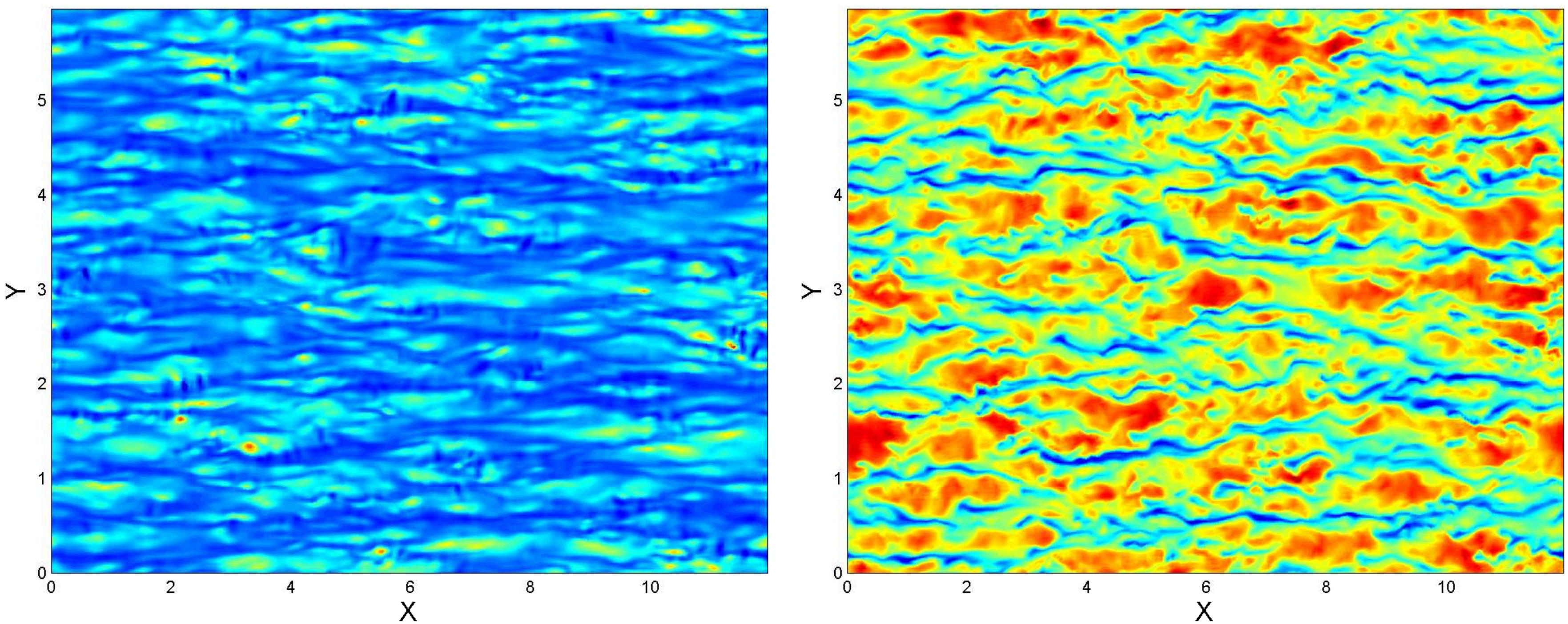
$$Observation : N_s = N_s(y_+) \sim y_+^{-1}.$$



Townsend-Perry schematic picture

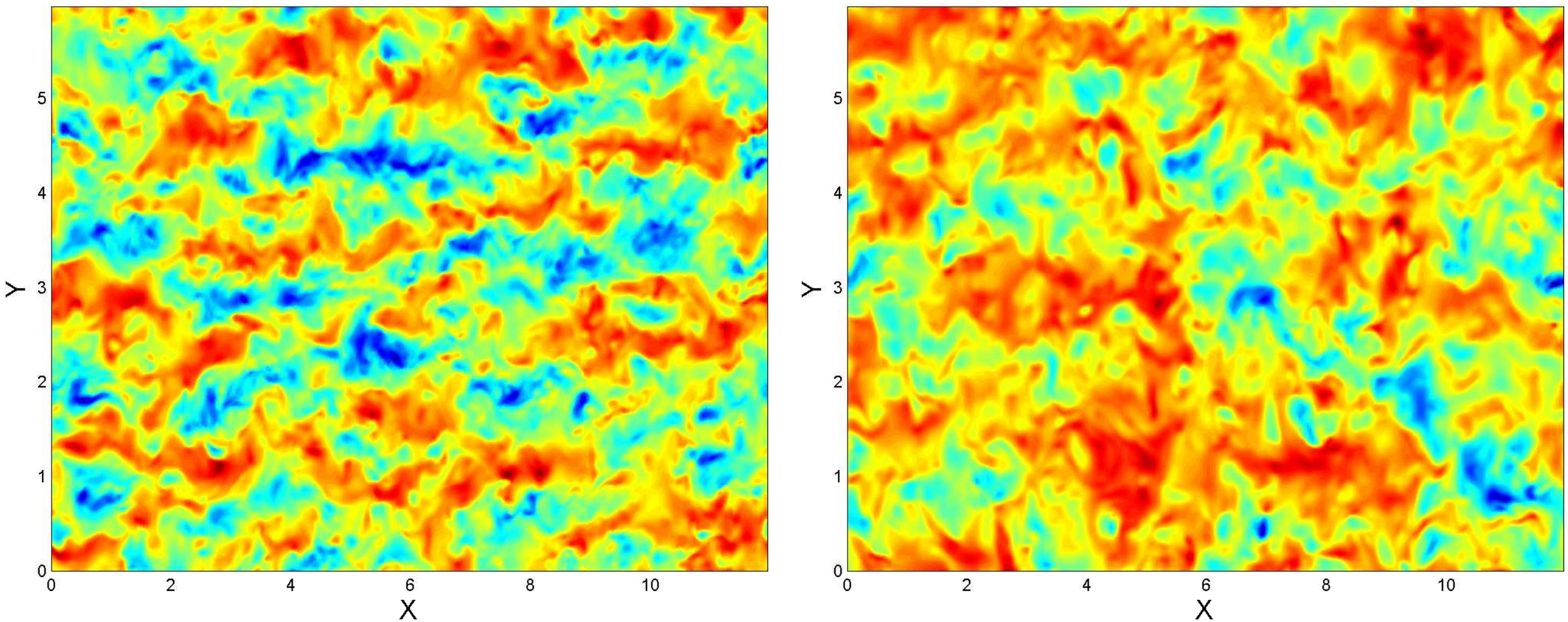


Streaks



lengths and spacings $\sim z$

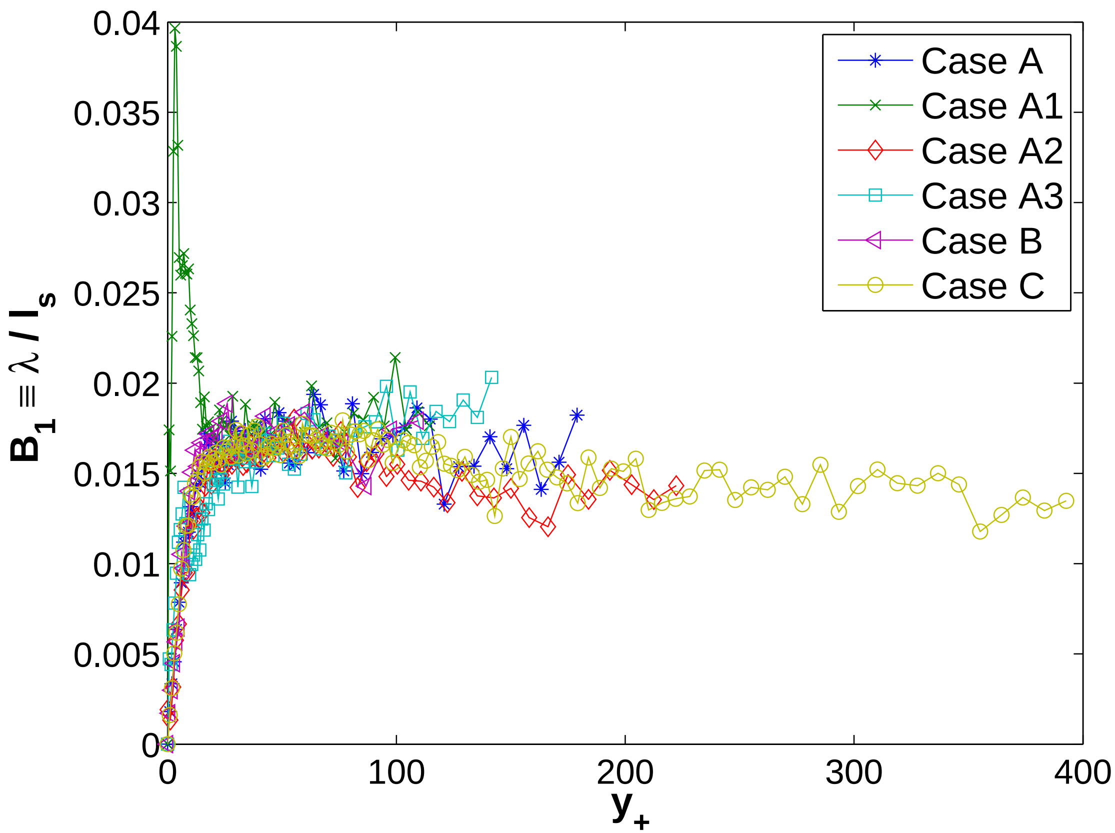
Streaks



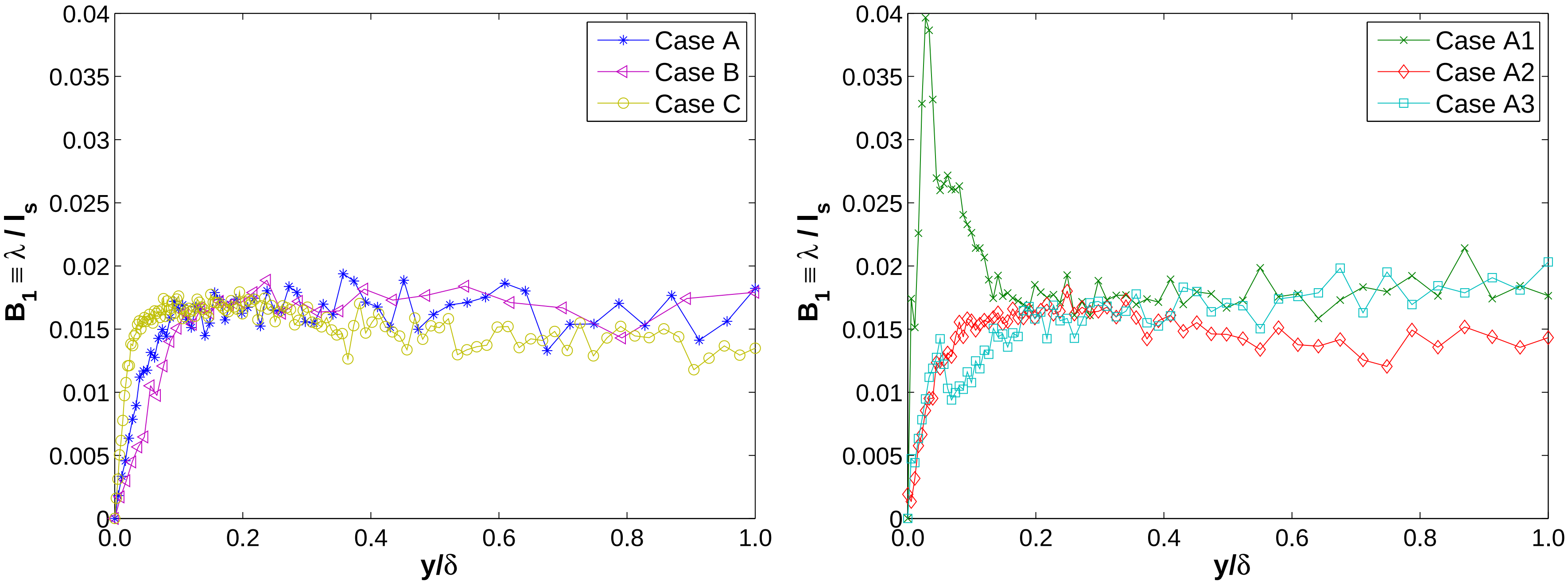
lengths and spacings $\sim z$

$B_1 \equiv \lambda / l_s$

Calculate λ from $\epsilon = 2\nu < s_{ij}s_{ij} >= \frac{\nu}{3}\frac{2E}{\lambda^2}$ where $E \equiv \frac{1}{2} < |\mathbf{u}'|^2 >$
and
calculate l_s from $l_s \equiv \sqrt{\frac{L_x L_y}{N_s}}$.



$B_1 \equiv \lambda / l_s$



Concentrate on dissipation

$\epsilon = \frac{\nu}{3}\frac{2E}{\lambda^2} = \frac{\nu}{3}\frac{2E}{B_1^2 l_s^2} = \frac{\nu}{3}\frac{2E}{B_1^2 L_x L_z} N_s = \frac{\nu}{3}\frac{2E}{B_1^2}\delta_\nu n_s$
where the number density $n_s \equiv N_s/(L_x L_z \delta_\nu)$.

Combine with $- < uv > \frac{d}{dy}\overline{U} = B_2 \epsilon$ and $\frac{d}{dy}\overline{U} = \frac{u_\tau}{\kappa y}$ as well as $C \equiv -\frac{2E}{3<uv>}$ (classical claims: κ and C are constants ($\kappa \approx 0.4$, $C \approx 2$) as $Re_\tau \to \infty$)

It then follows that

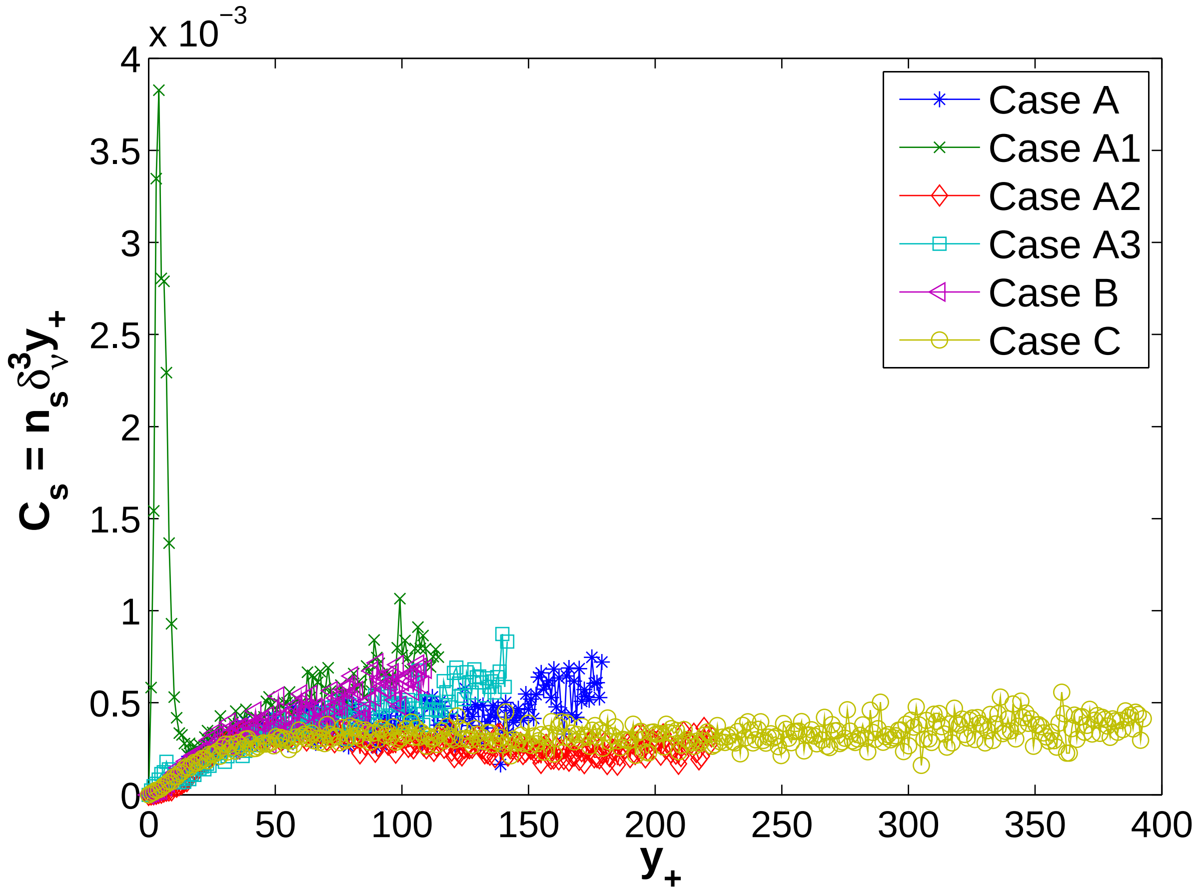
$$n_s = \frac{C_s}{\delta_\nu^3} y_+^{-1}$$

where

$$C_s = \frac{B_1^2}{\kappa B_2 C}$$

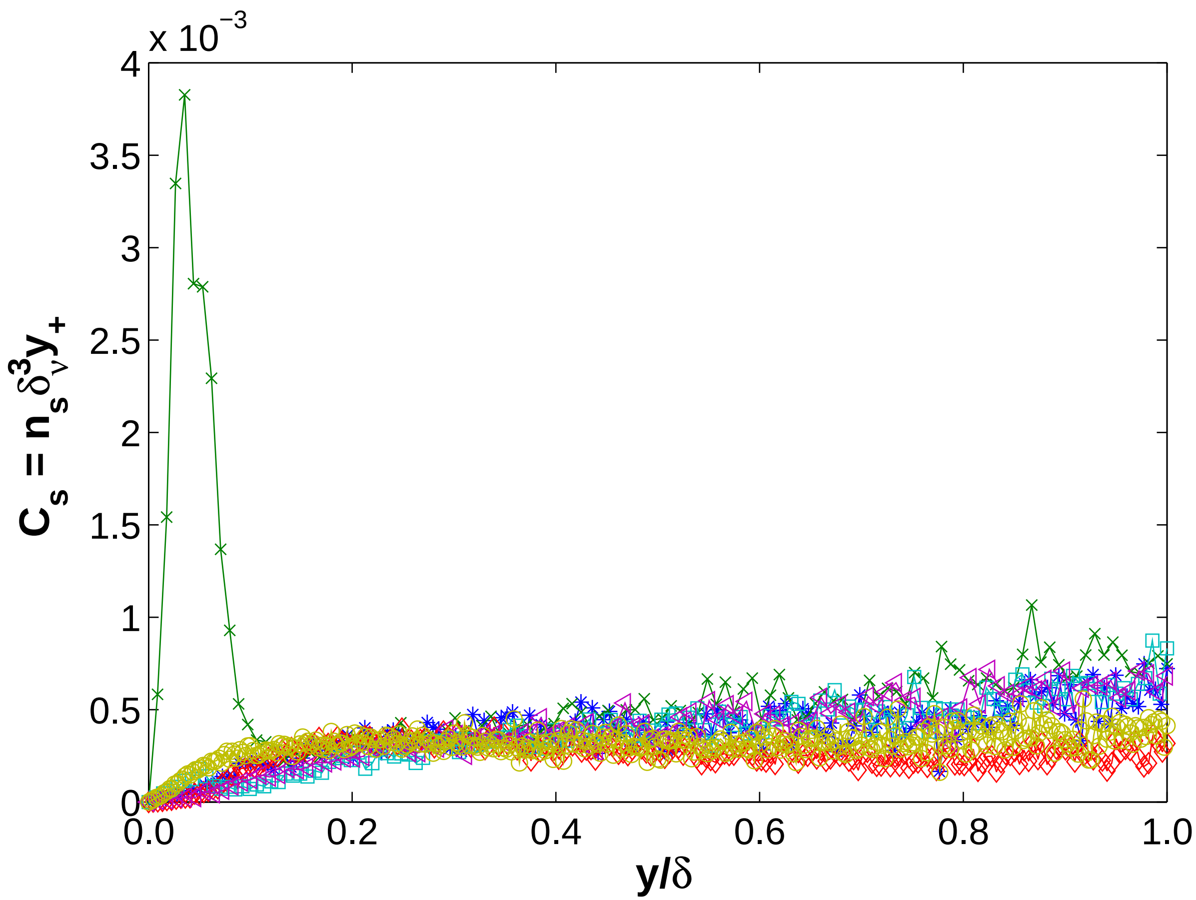
$n_s = \frac{C_s}{\delta_\nu^3} y_+^{-1}$ with C_s about constant

even though κ and C are not constant.



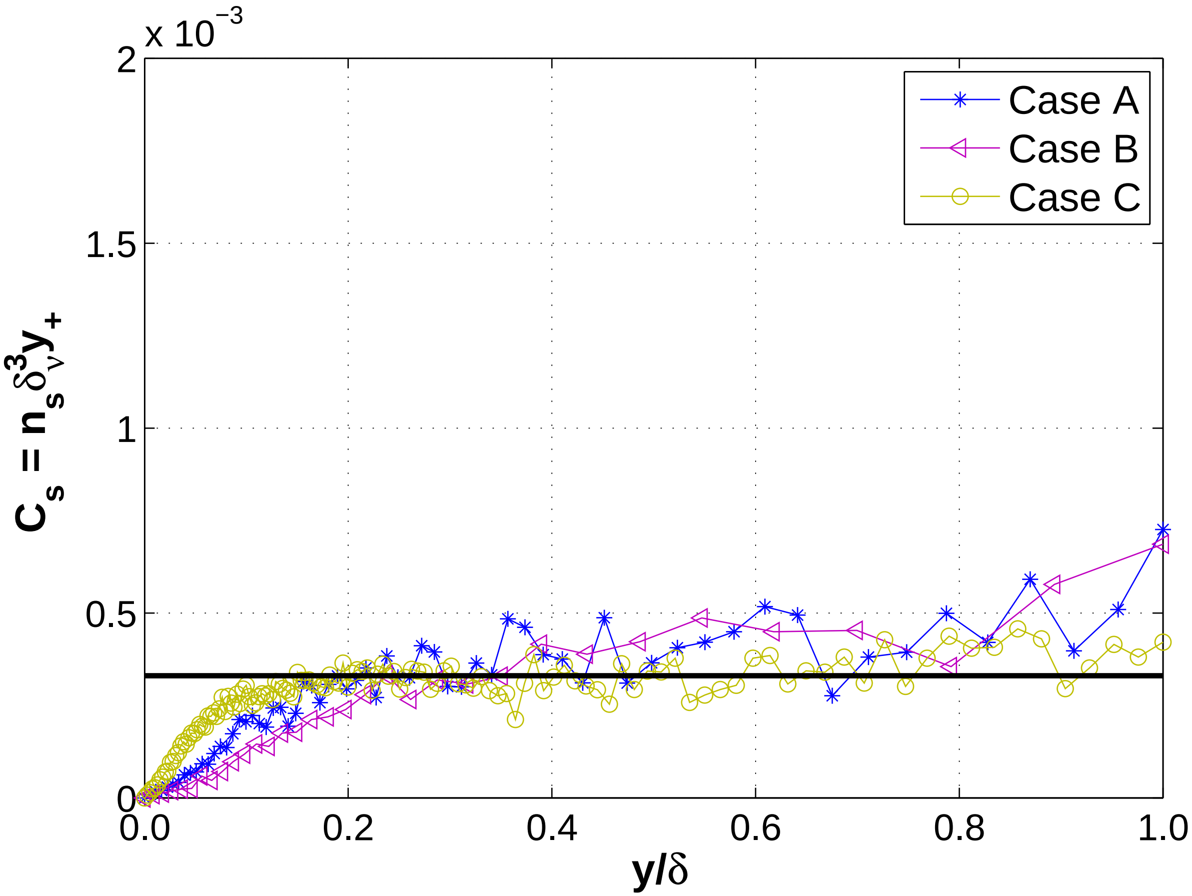
$n_s = \frac{C_s}{\delta_\nu^3} y_+^{-1}$ with C_s about constant

even though κ and C are not constant.

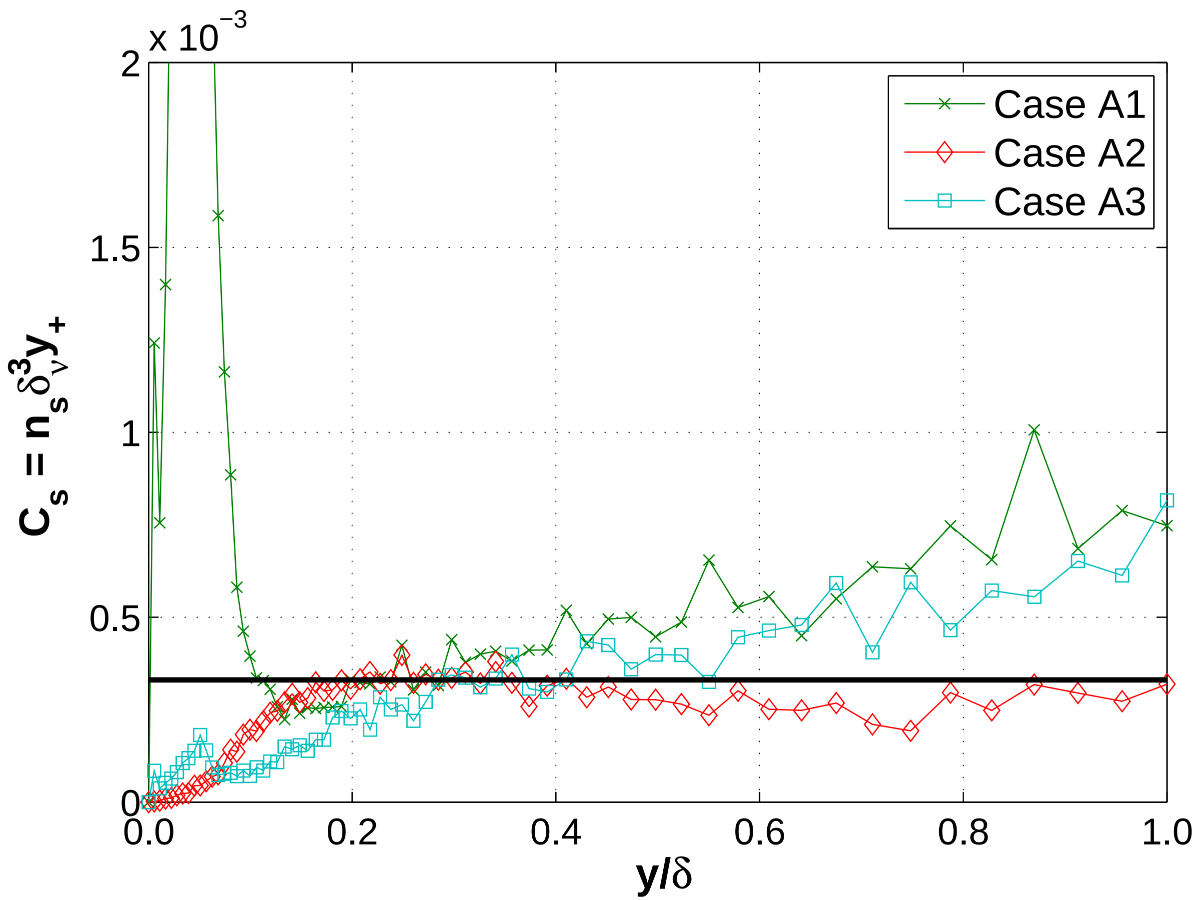


$n_s = \frac{C_s}{\delta_\nu^3} y_+^{-1}$ with C_s about constant

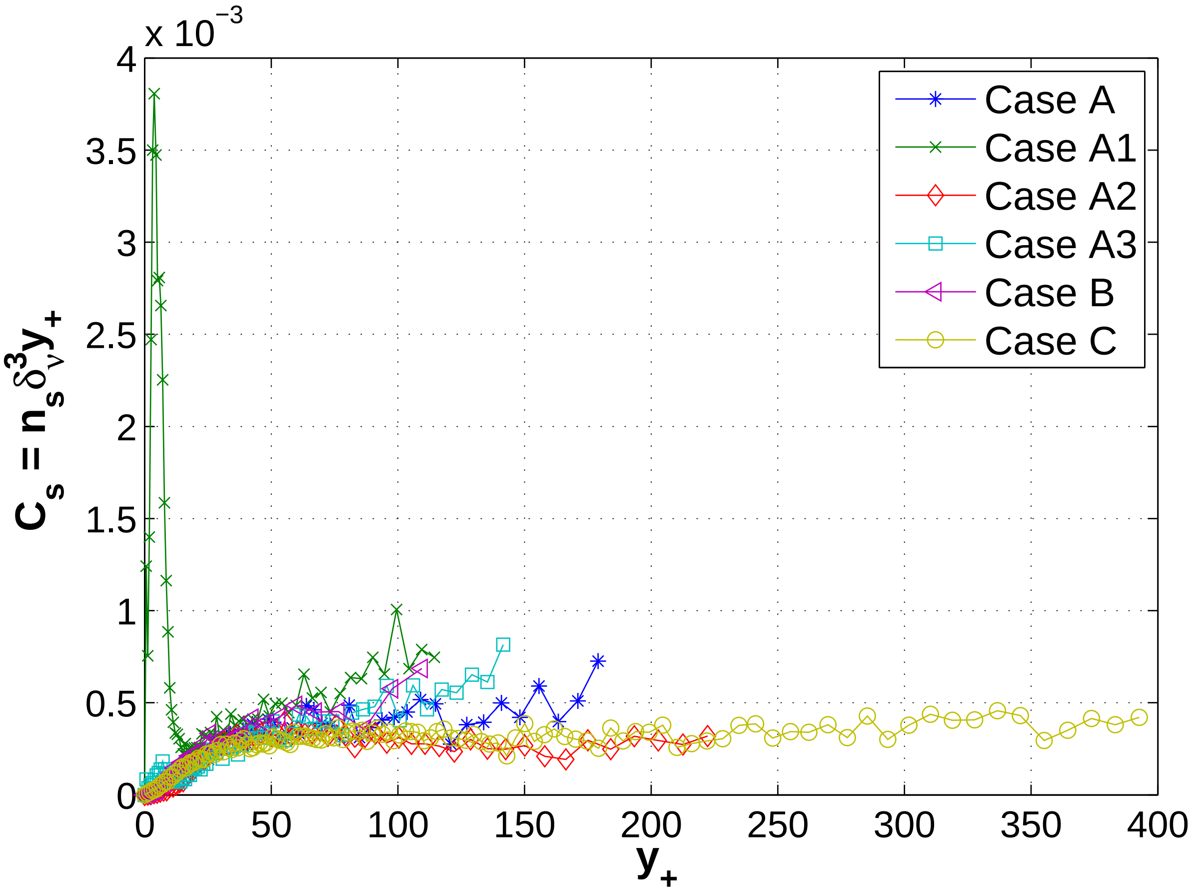
in the unforced cases...



but not quite in the forced cases

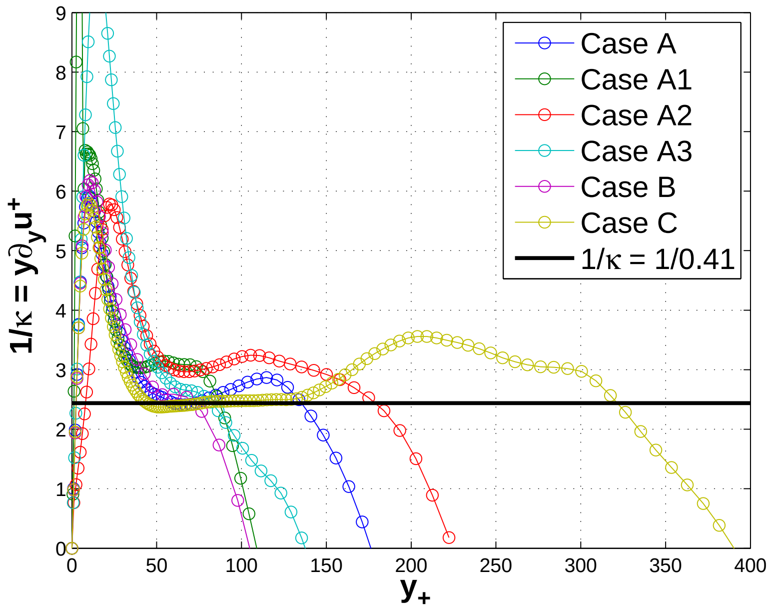


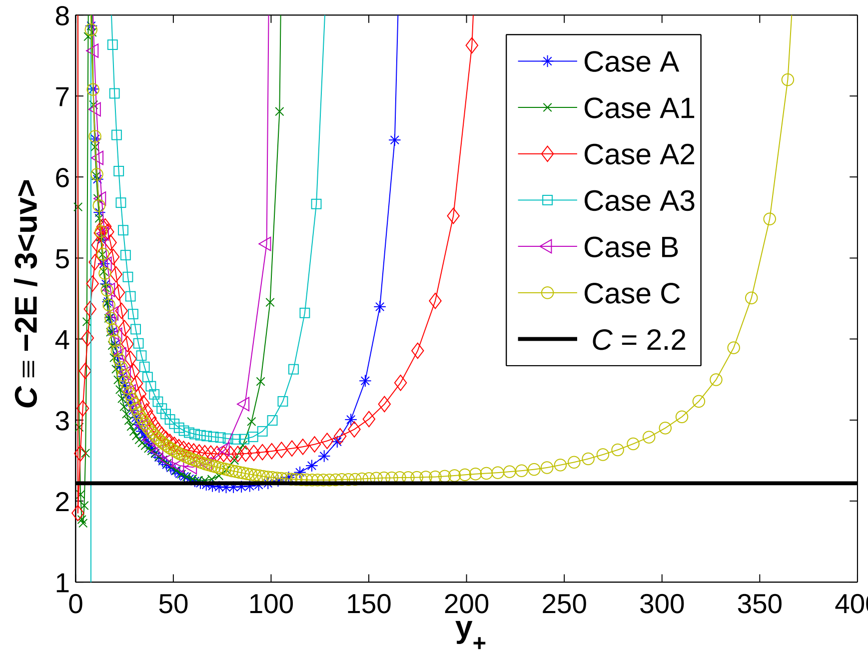
Recap



x 10^−3
4
3.5
3
2.5
2
1.5
1
0.5
0
0
50
100
150
200
250
300
350
400
$C_s = n_s \delta_\nu^3 y_+$
y_+
Case A
Case A1
Case A2
Case A3
Case B
Case C

κ and $C \equiv -\dfrac{2E}{3<uv>}$





Meaning of $B_1 \equiv \lambda / l_s$ constant

The eddy turnover time τ is given by $\epsilon = E/\tau$.

Hence, from $\epsilon = \frac{2\nu}{3}\frac{E}{\lambda^2}$, $3\lambda^2 = 2\nu\tau$.

Therefore, B_1 constant means $l_s^2/\nu \sim \tau$

This means that in the equilibrium layer, the time it takes for viscous diffusion to spread over neighboring stagnation points is the same proportion of the eddy turnover time (i.e. the time it takes to cascade the energy to the smallest scales) at all locations and all Reynolds numbers.

Meaning of C_s constant

$n_s = \frac{C_s}{\delta_\nu^3} y_+^{-1}$ implies $l_s^2 = C_s^{-1} \delta_\nu y$

From $\epsilon = \frac{\nu}{3} \frac{2E}{\lambda^2}$ and $B_1 = \lambda / l_s$ it then follows that

$\epsilon = \frac{2}{3} \frac{E u_\tau}{\kappa_* y}$ with $\kappa_* \equiv B_1^2 / C_s$

instead of $\epsilon = \frac{u_\tau^3}{\kappa y}$

Therefore, C_s constant means $\tau \sim y / u_\tau$
in the equilibrium layer where the constant of proportionality
$3\kappa_*/2$ is the same at all locations and all Reynolds numbers.

Indeed κ_* is related to the stagnation point constants B_1
and C_s and is constant if they are constant.

κ_* IS THE STAGNATION POINT KARMAN COEFFICIENT

Start from B_1 and C_s constants

B_1 constant and C_s constant as $Re_\tau \to \infty$ in the equilibrium layer mean that $\epsilon = \frac{2}{3}\frac{Eu_\tau}{\kappa_* y}$, $\kappa_* = B_1^2/C_s$ in that layer and that limit.

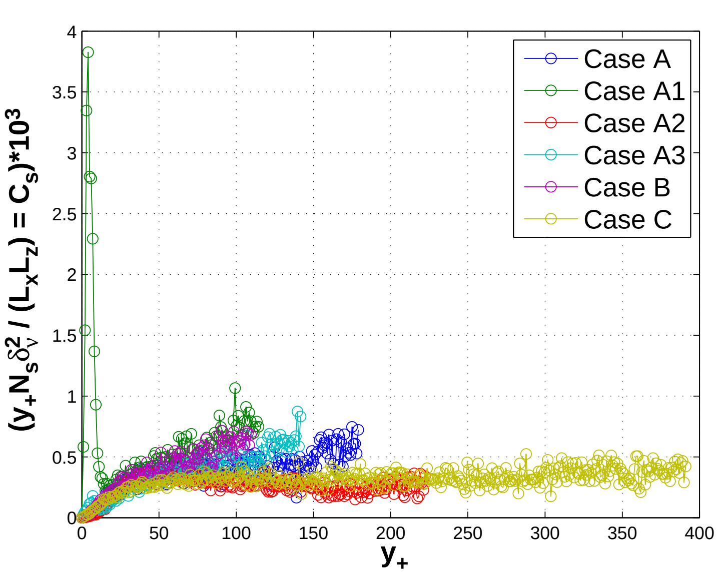
In the equilibrium layer $-<uv> \frac{d}{dy}\overline{U} \approx \epsilon$, in fact $-<uv> \frac{d}{dy}\overline{U} = B_2 \epsilon$ with $B_2 \to 1$ as $Re_\tau \to \infty$.

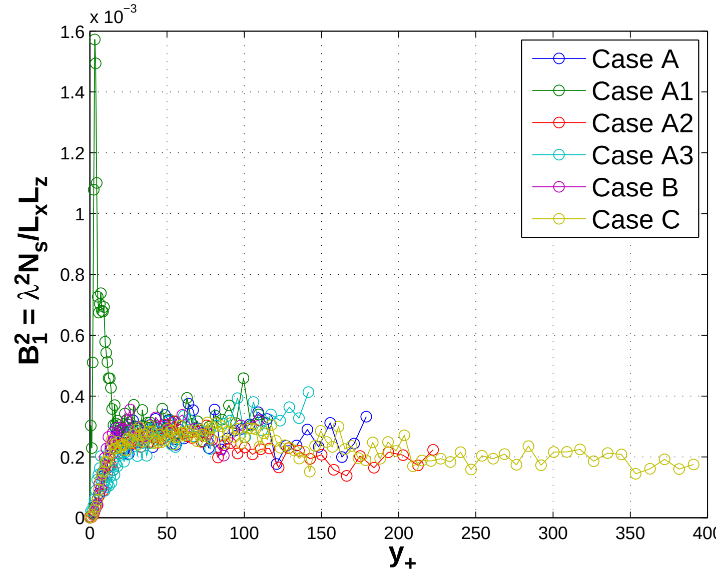
This means that $-<uv> \frac{d}{dy}\overline{U} = B_2 \frac{2}{3}\frac{Eu_\tau}{\kappa_* y}$

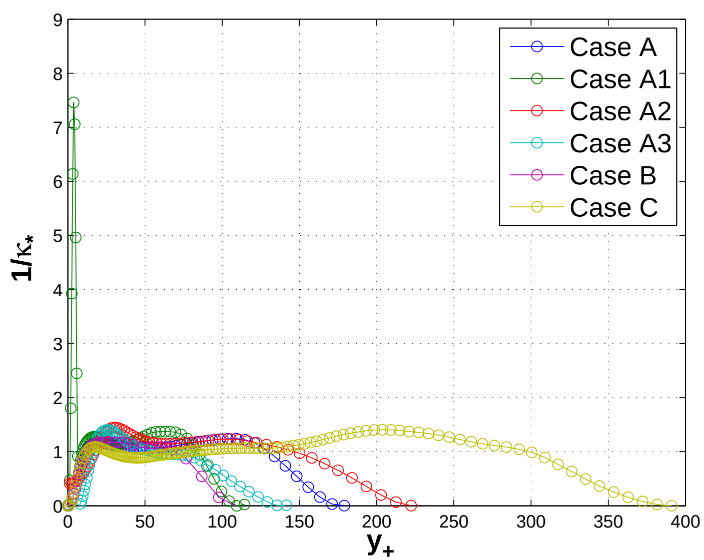
In the equilibrium layer and in the limit $Re_\tau \to \infty$, $-<uv> \to u_\tau^2$ and even $B_2 \to 1$ (Brouwers, PoF 2007): hence

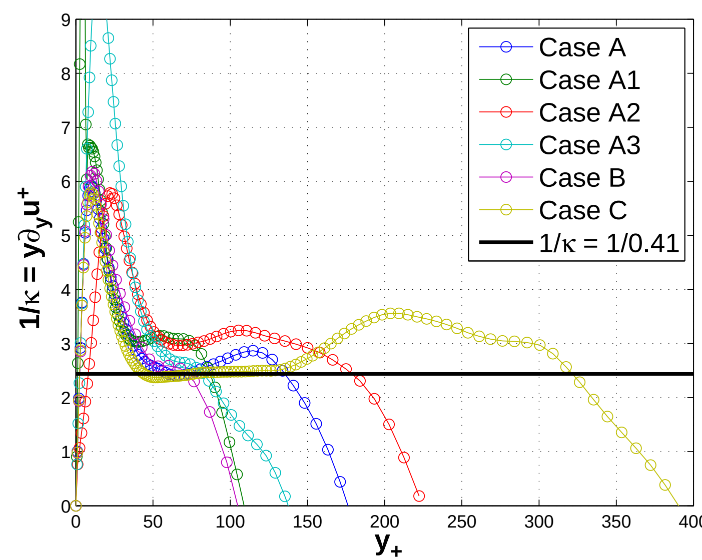
$$\frac{d}{dy}\overline{U} = \frac{2}{3} E_+ \frac{u_\tau}{\kappa_* y} \text{ with } \kappa_* = B_1^2/C_s$$

C_s, B_1^2, $\frac{3}{2}\frac{Py}{Eu_\tau}$ and $1/\kappa$

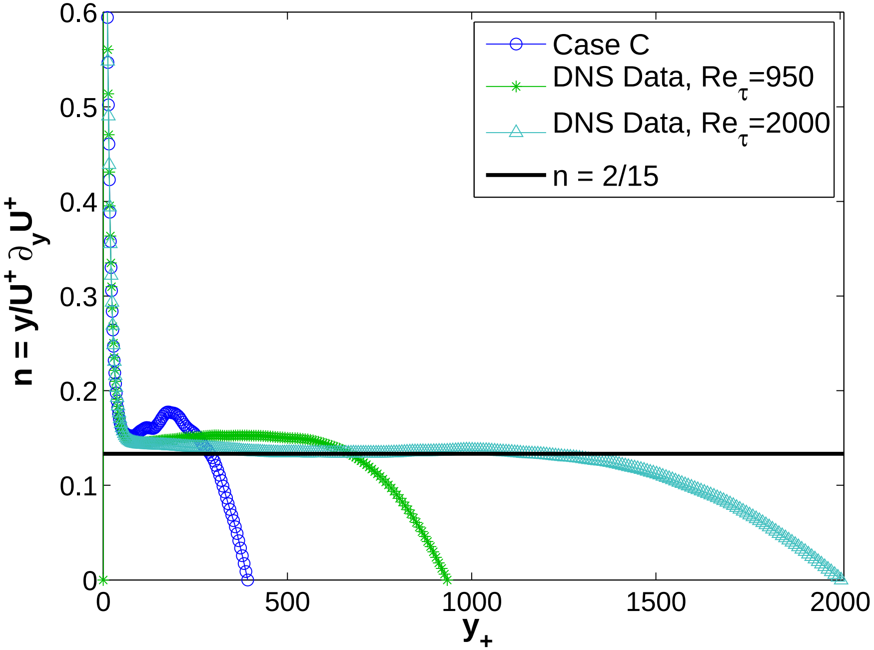


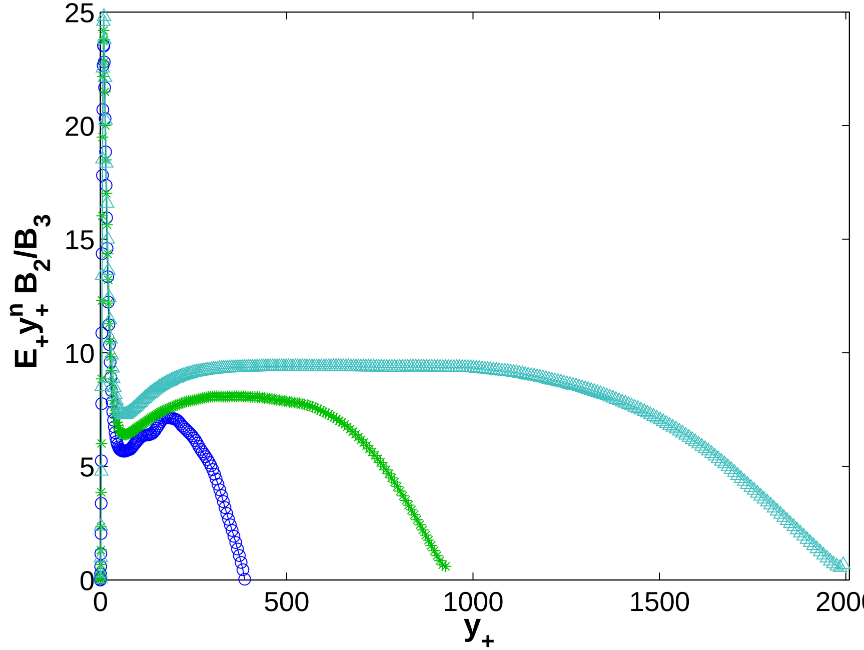






$$\frac{d}{dy}\overline{U} = \frac{u_\tau}{\kappa_s y}(2E_+/3)$$





(results obtained using the DNS data of Hoyas & Jimenez, PoF 2006)

$n = -\frac{y}{\overline{U}}\frac{d}{dy}\overline{U} = 2/15$ vs. y_+ and, effectively, $E_+ y_+^{2/15}$ vs. y_+.

Main conclusions

1. The DNS suggest that $B_1 = \lambda / l_s$ and $C_s = n_s \delta_\nu^3 y_+$ may be constants in the limit $Re_\tau \to \infty$ and in the region $\delta_\nu \ll y < \delta$, their asymptotic values being reached at Re_τ as low as a few hundred.

2. This is equivalent to stating that in the limit $Re_\tau \to \infty$ and in the region $\delta_\nu \ll y < \delta$,
the eddy turnover time τ is proportional to l_s^2/ν and also equals $\frac{3}{2}\kappa_* y/u_\tau$ with $\kappa_* = B_1^2/C_s$.

3. Either of these equivalent statements (1 or 2 above) implies that in the limit $Re_\tau \to \infty$, $\frac{d}{dy}\overline{U} = E_+ \frac{u_\tau}{\kappa_* y}$ in the equilibrium region $\delta_\nu \ll y \ll \delta$ where we may expect production to balance dissipation.

Concluding remarks I

1. According to classical similarity scalings, as $Re_\tau \to \infty$, $E \sim u_\tau^2$ independently of y in the equilibrium range $\delta_\nu \ll y \ll \delta$. If this is true, then the log-law will be recovered from $\frac{d}{dy}\overline{U} = E_+ \frac{u_\tau}{\kappa_* y}$ but with a Karman constant that is proportional to κ_* which is inversely proportional to C_s, the number of stagnation points (number of “eddies”?) within a volume δ_ν^3 at the upper edge of the buffer layer. Why would anyone expect this number to be universal (same for pipes and channels, lab ones and DNS ones)? If it is not, then the Karman constant might not be universal either.

Concluding remarks II

2. However, various DNS and experiments seem to suggest that E does not scale as u_τ^2 in the equilibrium region as $Re_\tau \to \infty$. If this is the case, then there is, strictly speaking, no log-law and mean flow data fitted by a log-law may yield non-universal Karman “constants” as a result of $\kappa_* = B_1^2/C_s$ but also as a result of the weak dependence of E on y, and its consequence on $\frac{d}{dy}\overline{U} = E_+ \frac{u_\tau}{\kappa_* y}$.

If E does not scale with u_τ^2, then $\epsilon \sim E u_\tau / y$ (NOT $\epsilon \sim u_\tau^3/y$).

NEW INTERMEDIATE ASYMPTOTICS

Alternative intermediate asympotitics

Instead of assuming independence of $\frac{d}{dy}\overline{U}$ on ν and δ where $\delta_\nu \ll y \ll \delta$ (which neglects small effect of inactive motions),

apply such asympotic assumptions to τ, which implies $\tau \sim y/u_\tau$.

Then use $P \sim \epsilon$ to obtain $\frac{d}{dy}\overline{U}$.

Then, $l_s \sim \sqrt{\nu\tau}$ leads to $\kappa_* = B_1^2/C_s$.

And two final points of caution

(i) B_1 may have its own weak (logarithmic?) dependencies on Re_τ and y_+ if small-scale intermittency effects are taken into account (see Mazellier & V PoF **20**, 014102). These dependencies will cause weak dependencies of κ_* on Re_τ and y_+.

(ii) For $\frac{d}{dy}\overline{U} = E_+ \frac{u_\tau}{\kappa_* y}$ to hold, the Reynolds number must be so large that $- < uv > \approx u_\tau^2$ and $P \approx \epsilon$. If some small cross-stream diffusion of turbulent kinetic energy remains and, for example, $P \approx 0.9\epsilon$ as often seems to be the case, then the value of the measured $1/\kappa_*$ will be 90 % of C_s/B_1^2.

More details can be found in:
V. Dallas, J.C. Vassilicos & G.F. Hewitt 2009 Stagnation von Karman coefficient. Phys. Rev. E **80**, 046306.

FOR MORE

SEE http://www3.imperial.ac.uk/tmfc