Graphoid Properties of Epistemic Irrelevance and Independence

Fabio G. Cozman and Peter Walley *Escola Politécnica, University of São Paulo*

The objective of this paper is to study properties of irrelevance and independence for imprecise probability models. We consider properties that are based on the graphoid axioms, as these axioms have been proposed as an abstract model for irrelevance [2] and for independence [4].

We use two equivalent models for imprecise probabilities: set of probability measures and coherent lower previsions (a coherent lower prevision is always the lower expectation with respect to a set of probability measures) [5]. There are several concepts of independence in the theory of imprecise probabilities [1, 3, 5], and none of these concepts is universally accepted. In this paper we adopt the concepts of epistemic irrelevance and epistemic independence that were proposed by Walley [5, Chapter 9]:

Epistemic irrelevance Y is epistemically irrelevant to X given Z, denoted by the expression $(Y \text{ IR } X \mid Z)$, if we have $\underline{E}[f(X)|Y=y, Z=z] = \underline{E}[f(X)|Z=z]$ for all possible values of Y and Z and all functions $f(X)$.

Epistemic independence X and Y are epistemically independent given Z, denoted by the expression $(X \text{ IN } Y \mid Z)$, if $(X \text{ IR } Y \mid Z)$ and $(Y \text{ IR } X \mid Z)$.

We assume that all random variables have only finitely many possible values. The following properties of epistemic irrelevance, based on the graphoid properties, are demonstrated:

Theorem 1 *Epistemic irrelevance satisfies, for all sets of variables W, X, Y, Z:*

Direct redundancy: $(X \text{ IR } Y \mid X)$.

Direct decomposition: $(X \text{ IR } (W,Y) \mid Z) \Rightarrow (X \text{ IR } Y \mid Z)$.

Direct weak union: *if* $\underline{P}(Y=y|Z=z) > 0$ *for all possible* y, z, *then* $(X \text{ IR } (W,Y) \mid Z) \Rightarrow (X \text{ IR } W \mid (Y,Z))$.

Reverse redundancy: $(Y \text{ IR } X \mid X)$.

Reverse decomposition: *if* $\underline{P}(Y=y|Z=z) > 0$ *for all possible* y, z, *then* $((W,Y) \text{ IR } X \mid Z) \Rightarrow (Y \text{ IR } X \mid Z)$.

Reverse weak union: *if* $\underline{P}(Y=y|Z=z) > 0$ *for all possible* y, z, *then* $((W,Y) \text{ IR } X \mid Z) \Rightarrow (W \text{ IR } X \mid (Y,Z))$.

Reverse contraction: $(Y \text{ IR } X \mid Z), (W \text{ IR } X \mid (Y,Z)) \Rightarrow ((W,Y) \text{ IR } X \mid Z)$.

Using Theorem 1, we obtain:

Theorem 2 *Epistemic independence satisfies*

Symmetry: $(X \text{ IN } Y \mid Z) \Rightarrow (Y \text{ IN } X \mid Z)$.

Redundancy: $(X \text{ IN } Y \mid X)$.

Decomposition: *if* $\underline{P}(Y = y|Z = z) > 0$ *for all possible* y, z*, then*
$(X \text{ IN } (W, Y) \mid Z) \Rightarrow (X \text{ IN } Y \mid Z)$.

Weak union: *if* $\underline{P}(Y = y|Z = z) > 0$ *for all possible* y, z*, then*
$(X \text{ IN } (W, Y) \mid Z) \Rightarrow (X \text{ IN } W \mid (Y, Z))$.

Simple examples can be constructed to show that epistemic irrelevance can violate any statement of a graphoid property that is not in Theorem 1.

The distinction between irrelevance and independence is not commonly made in formal studies; for example, both Dawid [2] and Pearl [4] justify the graphoid axioms using the terms "irrelevance" and "independence" almost interchangeably. But irrelevance and independence do not have identical meaning in ordinary language. Our results and examples show that most versions of the graphoid properties can fail even in simple cases, and that this failure is consistent with our intuitive understanding of irrelevance. This suggests that structures that are more general than graphoids, and especially ones that weaken the symmetry, contraction and intersection axioms, deserve further attention.

References

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[Escola Politécnica, Universidade de São Paulo
Av. Prof. Mello Moraes, 2231, Cidade Universitária,
São Paulo, SP, Brazil]

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