Towards Classifying Propositional Probabilistic Logics

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Probabilistic logics are commonly designed by adding probabilistic operators and their semantics to a logical system. There are several axes in which one might classify probabilistic logics, depending on syntactic and semantic design decisions. By using classical propositional logic as a starting ground, the present work aims at shedding some light on which choices are relevant when classifying probabilistic logics, analysing their expressivity, applicability and computational complexity.

In this paper we consider three axes. The first one distinguishes between two cases: either a possible world is simply a truth assignment over atomic propositions (and then a probability distribution over all truth assignments gives truth values to probabilistic assessments on propositional formulas); or a possible world also contains a probability distribution over the truth assignments, allowing probabilistic formulas to be subformulas themselves [2]. The second axis is concerned with the operators and relations that are allowed in basic probabilistic formulas, which is a predicate stating that a mathematical relation (typically $\leq$, $\geq$ or $=$) between a number and an expression formed by probabilities holds. That is, this axis deals with trade-offs between expressivity and complexity in basic probabilistic formulas. The third axis is concerned with the Boolean operators that are allowed in building probabilistic formulas, which are Boolean combinations of basic probabilistic formulas.

A major decision one must make when combining probability with propositional logic is to specify whether possible worlds contain probability distributions. The natural semantics for an assertion like $P(\phi) \geq q$, which reads "the probability of ϕ being true is at least q", determines truth values based on the sum of probabilities of valuations that satisfy ϕ. By keeping a single probability distribution π over valuations (a distribution that is outside of the possible worlds, so to speak), an agent is capable of evaluate the probability of propositional formulas according to π. Intuitively, the agent considers different possible worlds, but has a unique probability distribution outside of them, so that pure logical propositions may be either true or false in which world, but probabilistic assertions are either true or false within the whole structure. In this scenario, nested probabilities are meaningless, as probabilistic assertions do not have truth value defined in a single possible world. This is the case of Nilsson's probability logic [7], probabilistic satisfiability (PSAT) [6] and coherence checking problems [1].

A more general approach embeds probability distributions "inside" possible worlds, allowing one to assess the truth value of a basic probabilistic formula, like $P(\phi) \geq q$, in a specific world. An agent can then contemplate possible worlds with different probability distributions, and compute the probability of a formula $P(\theta) \geq q$ in a possible world using the probability distribution of that world to sum the probabilities of the valuations where $P(\theta) \geq q$ is true. Hence, it is possible to evaluate the truth value of a formula such as $P(P(\theta) = q) = r$; that is, the probability assertions turn into operators inside the language, thus allowing nesting of probabilities. These higher order probabilities are useful in stochastic processes, where one can say that the probability of a transition to a state, in which the probability of a transition to other state is q, is r. With multiple agents, it is meaningful to state 'the agent 1 has probability r for the probability of agent 2 for θ being q"; formally, $P_1(P_2(\theta) = q) = r$. This is the case of the logic of Fagin and Halpern [3], for instance. This generalization naturally brings more representation power, and demands no extra computational effort under some reasonable assumptions.

A second axis to analyse is how to build basic probabilistic formulas. A simple path is to assert that some probability lies in some interval, as $P(\phi) \geq q$, but these can be obviously generalized and restricted in several ways. A more general expression would be $f(P(\phi_{i_1}), \ldots, P(\phi_{i_n})) \bowtie q$, where $f : [0,1]^n \to \mathbb{R}$ is a function, $\bowtie \in \{<, >, \leq, \geq, =\}$ is a transitive binary realtion over the reals, and $q \in \mathbb{R}$ is a number. Varying the constraints on f, $\bowtie$ and q may lead to different classes of logical systems, but just some of these choices are relevant. For instance, is well known the fact that the expressions $a = b$, $a < b$ and $a > b$ can be replaced by suitable logical combinations of $a \geq b$. Furthermore, the PSAT problem has a normal form in which the equality can simulate $\leq$ and $\geq$ [5].

The question of how to constrain f deals with how much expressiveness, and applicability, is gained in order to pay the complexity costs. Allowing linear combinations, like $a_1 P(\phi_1) + \cdots + a_n P(\phi_n) \geq q$, is usually useful, and the corresponding decision procedure for propositional logics implies no higher efforts [4]. If f may be a polynomial of degree $m > 1$, we have an increase in the computational complexity, but then it is possible to talk about conditional probabilities with different conditioning events, provided these have positive probabilities; since $P(\phi|\theta) = P(\phi \wedge \theta)/P(\theta)$, one can clear the denominators in $P(\phi_1|\theta_1) + \cdots + P(\phi_m|\theta_m) \geq q$. Other generalizations of f, like $f = \sin(P(\phi))$, does not seem to have straightforward use to compensate for the complexity.

Restricting the possible values for q only makes sense in expressions of the form $P(\phi) \bowtie q$, where there are no coefficients for $P(\phi)$. To extend classical propositional logic, it seems reasonable to *a priori* allow expressions like $P(\phi) = 1$ and $P(\phi) = 0$. We can show that if $q \in \{0, \alpha, 1\}$, where $\alpha = r/s$ is a proper irreducible fraction, then it is possible to simulate any formula of the form $P(\phi) \bowtie i/s$, for any integer $0 \leq i \leq s$, using Boolean operations and new variables. If $\alpha \in [0,1]$ is irrational, then $P(\phi) \bowtie \beta$ can be approximated with arbitrary precision. While constraining q may not decrease the expressive power, given limited precision for probability assignments, clearly it yields high complexity costs.

Once we have built the basic probabilistic formulas, our third axis control their Boolean closure. In probabilistic satisfiability and coherence check problems, it is usually asked about the consistency of a set of probability assignments – this means that only conjunctions of probabilistic formulas are permitted. Fagin et al. [4] has showed that negation and disjunction of basic probabilistic formulas do not increase the computational complexity in their logic, however the linear programming framework for probabilistic propositional logic is not applicable. We show that with the propositional language and a probability distribution out of the possible worlds, the first case of the first axis, and the closure of the basic probabilistic formulas under negation, disjunction and conjunction, it is possible to simulate a restricted case of nesting probabilities, the second case of the first axis, in which all possible worlds have the same probability distribution – pointing out some interplay between axes.

References

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