Evenly Convex Credal Sets

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Abstract

An evenly convex credal set is a set of probability measures that is evenly convex; that is, a set that is an intersection of open halfspaces. An evenly convex credal set can for instance encode preference judgments through strict and non-strict inequalities such as $\mathbb{P}(A) > 1/2$ and $\mathbb{P}(A) \leq 2/3$. This paper presents an axiomatization of evenly convex sets from preferences, where we introduce a new (and very weak) Archimedean condition.

Keywords: credal sets; sets of probability measures; preference axioms; convexity.

1. Introduction

The goal of this note is to show that relatively simple axioms on preference orderings can be used to characterize *evenly convex* sets of probability measures; that is, sets that are intersections of open halfspaces. Such sets allow assessments such as $\mathbb{P}(A) \geq 1/2$ and $1/4 < \mathbb{P}(B) \leq 3/4$; that is, strict and non-strict inequalities can be expressed on probability values.

A preference ordering is a binary relation $\succ$ on *gambles*; a gamble is a function X that yields a real number $X(\omega)$ for each *state* ω, and $X \succ Y$ is understood as "X is preferred to Y".

If a preference ordering is only a partial order, then, subject to a few additional conditions, it can be represented by a set of probability measures (Giron and Rios, 1980; Seidenfeld et al., 1990; Walley, 1991; Williams, 1975). Typically such axiomatizations of sets of probability measures focus on a single *maximal closed convex* set of probability measures. It seems that the only existing axiomatization that allows for open sets of probability measures sets has been given by Seidenfeld et al. (1995), using a more general setting where utilities are also derived, and a proof technique based on transfinite induction. Their representation result may require sets of state-dependent utilities to represent preferences; for this reason it may be a little difficult to grasp the geometric content of a preference profile. One wonders whether it is possible to capture assessments such as $\mathbb{P}(A) > 1/2$ with some intuitive construction.

Section 4 presents a concise axiomatization for *evenly convex* sets of probability measures. We use a new Archimedean condition, and emphasize the use of separating hyperplanes as much as possible, hopefully producing results that can be appreciated with moderate effort.

2. Preference orderings, sets of desirable gambles, and credal sets

In this section we present some basic concepts and results used throughout. Because some results here are in essence well-known, only very short proof sketches are mentioned for them.

Consider a finite set Ω containing n states $\{\omega_1, \ldots, \omega_n\}$. An *event* is a subset of Ω; a *gamble* is a function $X : \Omega \to \Re$. A gamble can be viewed as a n-dimensional vector. A probability measure over Ω is entirely specified by a n-dimensional vector with non-negative elements that add up to one. Given such a vector p that induces a probability measure $\mathbb{P}$, and a gamble X, the expected value of X, denoted by $\mathbb{E}_{\mathbb{P}}[X]$, is simply the inner product $X \cdot p$.

All sets we consider are subsets of $\Re^n$; throughout we assume the Euclidean topology. For a set $\mathcal{A}$, cl$\mathcal{A}$ is the closure of $\mathcal{A}$ and relint$\mathcal{A}$ is the relative interior of $\mathcal{A}$. A *cone* $\mathcal{A}$ is a set such that if $X \in \mathcal{A}$ then $\lambda X \in \mathcal{A}$ for $\lambda > 0$ (the origin may not be in $\mathcal{A}$). An *exposed ray* of a convex cone is an exposed face that is a half-line emanating from the origin (recall that an exposed face is a face that is equal to the set of points achieving the maximum of some linear function).

Most results in this paper deal with the representation of preferences:[1]

Definition 1 *A* preference ordering $\succ$ *is a strict partial order over pairs of gambles.*

Absence of preference between X and Y is indicated by $X \not\sim Y$. If $X \succ 0$, X is *desirable*; if $X \not\sim 0$, X is *neutral*.

We always assume two additional properties:

Monotonicity: If $X(\omega) > Y(\omega)$ for all $\omega \in \Omega$, then $X \succ Y$;

Cancellation: For all $\alpha \in (0, 1]$, $X \succ Y$ iff $\alpha X + (1-\alpha)Z \succ \alpha Y + (1-\alpha)Z$.

The following representation obtains:[2]

Proposition 2 *If a preference ordering $\succ$ satisfies monotonicity and cancellation, then there is a convex cone $\mathcal{D}$, not containing the origin but containing the interior of the positive octant, such that $X \succ Y$ iff $X - Y \in \mathcal{D}$.*

Cones that encode preference orderings have received attention in the literature for some time (Giron and Rios, 1980; Seidenfeld et al., 1990; Williams, 1975; Walley, 1991). In fact, the literature on *sets of desirable gambles* (Miranda and Zaffalon, 2010; Quaeghebeur, 2014; Walley, 2000) employs cones of gambles to model preferences, often assuming *admissibility*: if $X(\omega) \geq 0$ for all ω and $X(\omega) > 0$ for some ω, then $X \succ 0$. We do not assume admissibility here; indeed, admissibility cannot be satisfied in general if preferences are to be encoded by expectation with respect to probability measures (when probability values may be equal to zero). In any case, we use the term *set of desirable gambles* to refer to a convex cone $\mathcal{D}$ constructed as in Proposition 2. This proposition allows one to freely switch between preference orderings and sets of desirable gambles; we find the former to be more intuitive so we mostly employ them in the remainder of this paper.

One might think that *any* convex cone of gambles can be represented by a set of probability measures as follows: $X \in \mathcal{D}$ iff $\mathbb{E}_{\mathbb{P}}[X] > 0$ for all $\mathbb{P}$ in some set $\mathcal{K}$ of probability measures. This is not possible. Consider the set of desirable gambles depicted in Figure 1 (left). All gambles in the interior of $\mathcal{D}$ satisfy $X(\omega_1)\mathbb{P}(\omega_1) + X(\omega_2)\mathbb{P}(\omega_2) > 0$ for $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = 1/2$. No other pair of probability values (or sets of pairs of probability values) can similarly represent the interior of $\mathcal{D}$. But even this probability measure cannot represent the fact that half-border is in $\mathcal{D}$; for this half-border, $f(\omega_1)\mathbb{P}(\omega_1) + f(\omega_2)\mathbb{P}(\omega_2) = 0$. Thus some condition on boundaries is needed.

Conditions on boundaries of sets of desirable gambles inevitably focus on what "makes sense" concerning limiting behavior. For instance, Aumann (1962) has proposed the following condition:

1. A *strict partial order* is a binary relation that is irreflexive and transitive, an *equivalence* is a binary relation that is reflexive, transitive, and symmetric (a binary relation $\diamond$ is *irreflexive* when $X \diamond X$ if false for every X; it is *transitive* when $X \diamond X$ and $Y \diamond Z$ imply $X \diamond Z$; it is *symmetric* when $X \diamond Y$ implies $Y \diamond X$) (Fishburn, 1970, Section 2.3).
2. Proof sketch: Applying cancellation, $X \succ Y$ iff $X/2 - Y/2 \succ Y/2 - Y/2$ iff $X - Y \succ 0$. Now if $X \succ 0$ and $Y \succ 0$, then $0 \succ -Y$ (as $X \succ Y$ iff $-Y \succ -X$), and by transitivity we get $X + Y \succ 0$. For any $\lambda \in (0, 1)$, $X \succ 0$ iff $\lambda X \succ 0$ by cancellation. Finite induction leads to: $X \succ 0$ implies $\lambda X \succ 0$ for $\lambda > 0$, so we have the cone (monotonicity implies this cone contains every positive gamble; irreflexivity eliminates the origin).

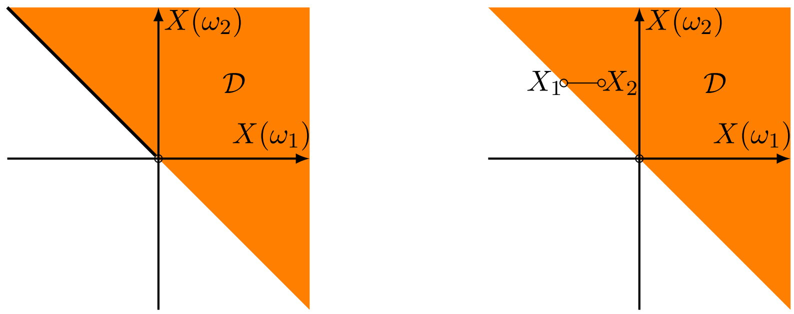

Figure 1: Left: a cone $\mathcal{D}$; one bordering ray (thick line from the origin) belongs to $\mathcal{D}$, while the other bordering ray does not belong to $\mathcal{D}$. Right: to understand the effect of Auman's continuity condition, take a similar cone $\mathcal{D}$; the gamble X_2 is inside $\mathcal{D}$ and the gamble X_1 is in the border, so the segment between X_1 and X_2 is in $\mathcal{D}$, implying that $X_1 \succ 0$ or $X_1 \not\succ 0$ by Aumann's continuity condition; the same reasoning could be repeated for $-X_1$, hence the border must be open because $X_1 \succ 0$ and $-X_1 \succ 0$ cannot happen.

Aumann's continuity: If $\alpha X + (1-\alpha)Y \succ Z$ for all $\alpha > 0$, then either $Y \succ Z$ or $Y \not\succ Z$.

If the interior of the set of desirable gambles is an open halfspace, Auman's continuity condition forces the set of desirable gambles to be open (see Figure 1 (right)). In general, if the interior of the set of desirable gambles is strictly smaller than a halfspace, Aumann's continuity condition does not imply that $\mathcal{D}$ is entirely open; it only implies that each gamble in the boundary of $\mathcal{D}$ is either desirable or neutral.

If the continuity condition is strengthened so that $\mathcal{D}$ is assumed open (Seidenfeld et al., 1990; Walley, 1991), then it is possible to find a representation of preference orderings through probabilities. Walley imposes openness by basically requiring that $X \succ 0$ implies $X - \epsilon \succ 0$ for some ϵ (Walley, 1991, Section 3.7.8, D7). Another possibility could be to require that (note that limits of sequences of gambles are always assumed pointwise):

Open continuity If $X_i \succ 0$ is false for every i, and $X = \lim_i X_i$, then $X \succ 0$ is false.

Here is the representation result under the assumption of openness:[3]

Proposition 3 *If a set of desirable gambles $\mathcal{D}$ is open, then it can be represented by a closed convex set $\mathcal{K}$ of probability measures, in the sense that $X \in \mathcal{D}$ iff $\mathbb{E}_{\mathbb{P}}[X] > 0$ for all $\mathbb{P} \in \mathcal{K}$.*

A set of probability measures is called a *credal set* (Levi, 1980). There is a significant disadvantage in assuming that a set of desirable gambles is open; namely, the representing credal set is

3. Proof sketch: Copy the proof of Theorem 11, except Part 5 (note that in Part 1 one might choose to replace Theorem 7 by some appropriate separating hyperplane theorem (Klee Jr., 1955)). Now to show that $\mathcal{K}$ is closed, show that the complement of the cone $\mathcal{C}$ in the proof of Theorem 11 is open: If $p \notin \mathcal{C}$, then there is $X \in \mathcal{D}$ such that $X \cdot p \leq 0$, and also $X - \epsilon \in \mathcal{D}$ for some $\epsilon > 0$ (as $\mathcal{D}$ is open by assumption). Consider the closed halfspace $\mathcal{H} = \{q : (X - \epsilon) \cdot q \leq 0\}$; this halfspace is disjoint from $\mathcal{C}$. Also, p is in $\mathcal{H}$ but not in its boundary (there is a ball around p inside $\mathcal{H}$ for any radius smaller than $|(X - \epsilon) \cdot p| / ||X - \epsilon||$). So the complement of $\mathcal{C}$ is open as desired.

necessarily closed. Hence one cannot say that a coin is biased simply by stating $\mathbb{P}(\mathsf{Heads}) > 1/2$. It seems that the only existing condition in the literature that can accept such assessments has been proposed by Seidenfeld et al. (1995). Their condition has two parts, but only one is necessary:

SSK-continuity If $X_i \succ Y_i$ for every i, and $\lim_i Y_i \succ Z$, then $\lim_i X_i \succ Z$, whenever limits exist.

The other part of the original condition by Seidenfeld, Schervish and Kadane can actually be derived from the previous conditions:

Proposition 4 *Suppose a preference ordering $\succ$ satisfies cancellation and SSK-continuity. If $X_i \succ Y_i$ for every i, and $W \succ \lim_i X_i$, then $W \succ \lim_i Y_i$.*

Proof The assumptions imply $X_i - Y_i \succ 0$ and then $-Y_i \succ -X_i$ for every i; similarly, $-\lim X_i = \lim -X_i \succ -W$, so by SSK-continuity, $\lim -Y_i \succ -W$ and then $W \succ \lim Y_i$. ■

In fact we might simplify SSK-continuity even more in the presence of cancellation:

Proposition 5 *Suppose $\succ$ is a preference ordering satisfying cancellation. Suppose that if $X_i \succ Y_i$ and $\lim_i Y_i \succ 0$ then $\lim_i X_i \succ 0$. Then $\succ$ satisfies SSK-continuity.*

Proof If $\{X_i\} \to X$, $\{Y_i\} \to Y$, $X_i \succ Y_i$ and $Y \succ Z$ then $\{X_i - Z\} \to X - Z$, $\{Y_i - Z\} \to Y - Z$, $X_i - Z \succ Y_i - Z$ and $Y - Z \succ 0$; if the property assumed in the statement is true, then $X - Z \succ 0$ so $X \succ Z$ as desired. ■

If a preference ordering satisfies SSK-continuity, and $\{X_i\} \to X$, $\{Y_i\} \to Y$, and $X_i \succ Y_i$, then either $X \succ Y$ or $X \not\succ Y$ (for suppose otherwise that $Y \succ X$; SSK-continuity says that if $X_i \succ Y_i$ and $Y = \lim_i Y_i \succ X$ then $\lim_i X_i \succ X$, hence $X \succ X$, a contradiction). Thus we have that SSK-continuity conveys Aumann's continuity condition. We will return to SSK-continuity when we examine whether it implies even convexity (it does not).

3. Evenly convex sets and evenly convex cones

An *evenly convex* set $\mathcal{A}$ is an intersection of open halfspaces (Fenchel, 1952). Hence an open convex set is evenly convex; also a closed convex set is evenly convex as it is an infinite intersection of halfspaces. For any set $\mathcal{A}$, its *evenly convex hull* $\mathsf{eco}\mathcal{A}$ is the intersection of all evenly convex sets containing $\mathcal{A}$; so $\mathsf{eco}\mathcal{A}$ is the intersection of all open halfspaces that contain $\mathcal{A}$. Note that $\mathsf{co}\mathcal{A} \subseteq \mathsf{eco}\mathcal{A}$, where $\mathsf{co}\mathcal{A}$ is the convex hull of $\mathcal{A}$.

There are many characterizations of evenly convex sets (Daniilidis and Martinez-Legaz, 2002; Goberna et al., 2003; Klee, 1968). In particular, we will use the following result in the proof of Theorem 9 (Daniilidis and Martinez-Legaz, 2002, Corollary 6): a convex set $\mathcal{A}$ is evenly convex iff for every $X_0 \in \mathsf{cl}\mathcal{A}\backslash\mathcal{A}$, and every $\{X_i\}_{i\geq 1} \subset \mathcal{A}$, and every $\{\lambda_i\}_{i\geq 1}$ such that $\lambda_i > 0$, we have $X_0 - \lim_i \lambda_i(X_i - X_0) \notin \mathcal{A}$ whenever the limit exists.

If $\mathcal{A}$ is evenly convex, then if $X \in \mathcal{A}$ and $Y \in \mathsf{cl}\mathcal{A}$ we have $\alpha X + (1-\alpha)Y \in \mathcal{A}$ for $\alpha \in (0,1)$ (Fenchel, 1952, Section 3.5). Consequently:

Lemma 6 *Suppose $\mathcal{A}$ is evenly convex and $0 \notin \mathcal{A}$. If X and $-X$ belong to $\mathsf{cl}\mathcal{A}$, then neither is in $\mathcal{A}$.*

Proof If $X \in \mathcal{A}$, then $-X \in \text{cl}\mathcal{A}$ implies $X/2 + (-X)/2 = 0 \in \mathcal{A}$, a contradiction; hence $X \notin \mathcal{A}$. By similar reasoning, $-X \notin \mathcal{A}$. ■

We then obtain the following separation property, that is used later:

Theorem 7 *Suppose $\mathcal{A}$ is an evenly convex cone such that $0 \notin \mathcal{A}$. If $X \notin \mathcal{A}$, then there is p such that $X \cdot p \leq 0$ and $Y \cdot p > 0$ for all $Y \in \mathcal{A}$.*

Proof Part 1) Suppose $X \in \text{cl}\mathcal{A}$, but $X \notin \mathcal{A}$. Because $\mathcal{A}$ is evenly convex, there is p and β such that $X \cdot p = \beta$ and $Y \cdot p > \beta$ for all $Y \in \mathcal{A}$ (Goberna et al., 2003, Proposition 3.1(ii)). If $\beta > 0$, then for any Y in a neighborhood of 0 we have $\epsilon Y \cdot p < \beta$ for some $\epsilon > 0$; this is a contradiction because some such Y is in $\mathcal{A}$, and for this Y we must have $\epsilon Y \cdot p > \beta$. Hence $\beta \leq 0$. We now show that actually $\beta = 0$.

For $Y \in \mathcal{A}$, $Y \cdot p > \beta = X \cdot p$, hence $(Y - X) \cdot p > 0$. Because X is in the boundary of $\mathcal{A}$, there is a gamble Y in a neighborhood of X that belongs to $\mathcal{A}$; define $q = Y - X$, and note that the segment from Y to X (excluding X) is in $\mathcal{A}$ (Fenchel, 1952, Section 3.5). That is, there is q such that $q \cdot p > 0$ and $(X + \epsilon q) \cdot p > \beta$ for $\epsilon > 0$ in a neighborhood of 0. Now for any $\lambda > 0$ we have $\lambda(X + \epsilon q) \in \mathcal{A}$. That is, $\lambda(X + \epsilon q) \cdot p > \beta$, so $X \cdot p > \beta/\lambda - \epsilon q \cdot p$. Again use $X \cdot p = \beta$, to obtain $\beta > \beta/\lambda - \epsilon q \cdot p$. Consequently, we have both $\beta \leq 0$ and $\beta > -\epsilon q \cdot p/(1 - 1/\lambda)$; take say $\lambda = 2$ to obtain the constraint $\beta > -\epsilon(2q \cdot p)$. These conditions can only be satisfied for $\epsilon > 0$ if $\beta = 0$.

Part 2) Now suppose instead that $X \notin \text{cl}\mathcal{A}$. Consider the cone $\mathcal{B} = \{\lambda X : \lambda \geq 0\}$. Using an appropriate separation result (Klee Jr., 1955, Theorem 2.5), we know that there is p such that $Y \cdot p > 0$ for $Y \in \text{cl}\mathcal{A} \backslash (\text{cl}\mathcal{A} \cap -\text{cl}\mathcal{A})$, $Y' \cdot p = 0$ for $Y' \in (\text{cl}\mathcal{A} \cap -\text{cl}\mathcal{A}) \cup (\mathcal{B} \cap -\mathcal{B})$, $Y'' \cdot p \leq 0$ for $Y'' \in \mathcal{B} \backslash (\mathcal{B} \cap -\mathcal{B})$. Clearly $\mathcal{B} \cap -\mathcal{B}$ contains just the zero gamble. Now note that $\text{cl}\mathcal{A} \cap -\text{cl}\mathcal{A}$ does not intersect $\mathcal{A}$ (if $Y \in \text{cl}\mathcal{A} \cap -\text{cl}\mathcal{A}$, then $Y \in \text{cl}\mathcal{A}$ and $-Y \in \text{cl}\mathcal{A}$, so both are not in $\mathcal{A}$ by Lemma 6). Hence there is p such that $X \cdot p \leq 0$ and $Y \cdot p > 0$ for $Y \in \mathcal{A}$. ■

4. Evenly convex sets of desirable gambles and evenly convex credal sets

In this section we consider preference orderings that can be represented by evenly convex sets of desirable gambles; such preference orderings can also be represented by evenly convex credal sets. This will allow us to consider assessments such as $1/4 \leq \mathbb{P}(\mathsf{Heads}) < 1/2$.

4.1 Evenly convex sets of desirable gambles

We introduce the following condition:

Even continuity If $X_i \succ 0$ for every i, and $Y \succ 0$ is false, then $\lim_i(\lambda_i Y - X_i) \succ 0$ is false for any sequence of $\lambda_i > 0$ such that the limit exists.

Even though the condition is somewhat long, it is quite reasonable: one cannot take an undesirable gamble Y and make it desirable, not even in the limit, by multiplying it by a positive number and subtracting from it a desirable gamble.[4]

4. One might consider a weaker condition (as suggested by a reviewer): If $X_i \succ 0$ and not $Y \succ 0$, then not $\lim_i(Y - X_i) \succ 0$. But this is implied by SSK-continuity: if $X_i \succ 0$, then if $Y \succ \lim_i X_i$ then $Y \succ 0$ by SSK-continuity, implying that if $X_i \succ 0$, then if not $Y \succ 0$ then not $Y \succ \lim_i X_i$.

To make later results more concise, we introduce the following definition:

Definition 8 *A preference ordering $\succ$ is* coherent *when it satisfies monotonicity, cancellation, and even continuity.*

We then obtain:

Theorem 9 *If a preference ordering $\succ$ is coherent, then there is an evenly convex cone $\mathcal{D}$ of gambles, not containing the origin but containing the interior of the positive octant, such that $X \succ Y$ iff $X - Y \in \mathcal{D}$.*

Proof Take the set of desirable gambles produced by Proposition 2.

For a fixed $Y \in \text{cl}\mathcal{D} \backslash \mathcal{D}$ (hence $Y \notin \mathcal{D}$) and $X_i \in \mathcal{D}$ for every i, and $\lambda_i > 0$, compute $\lambda'_i = 1+\lambda_i$ and $X'_i = \lambda_i X_i$. Clearly $\lambda'_i > 0$ and $X'_i \in \mathcal{D}$. By even continuity, $\lim_i(\lambda'_i Y - X'_i) \notin \mathcal{D}$; hence $\lim_i((1+\lambda_i)Y - \lambda_i X_i) \notin \mathcal{D}$, and then $Y - \lim_i \lambda_i(X_i - Y) \notin \mathcal{D}$. Thus $\mathcal{D}$ is evenly convex (Daniilidis and Martinez-Legaz, 2002, Corollary 6). ∎

Note that coherence implies Aumann's continuity condition:

Proposition 10 *Suppose a preference ordering $\succ$ is coherent. If $\alpha X + (1-\alpha)Y \succ Z$ for all $\alpha > 0$, then either $Y \succ Z$ or $Y \not\sim Z$.*

Proof If $X_i \succ 0$ for every i, then the fact that $\neg(0 \succ 0)$ and even continuity imply $\neg(-X \succ 0)$ for $X = \lim_i X_i$. Now, if $\alpha X + (1-\alpha)Y \succ Z$, then take $\alpha_i = 1/2^i$ and $X_i = \alpha_i(X - Z) + (1 - \alpha_i)(Y - Z)$; hence $X_i \succ 0$, implying that $\neg(Z - Y \succ 0)$, so either $Y \succ Z$ or $Y \not\sim Z$. ∎

4.2 Evenly convex credal sets

Evenly convex sets of desirable gambles can be nicely represented by evenly convex sets of probability measures, as described by the next theorem. In the next proof and later we use the nonempty cone

$$\mathcal{C} = \{p : X \cdot p > 0, \forall X \in \mathcal{D}\}.$$

Theorem 11 *If a preference ordering $\succ$ is coherent, then there is a unique maximal evenly convex credal set $\mathcal{K}$ such that $X \succ Y$ iff for all $\mathbb{P} \in \mathcal{K}$ we have $\mathbb{E}_{\mathbb{P}}[X] > \mathbb{E}_{\mathbb{P}}[Y]$.*

Proof Part 1) For any $X \notin \mathcal{D}$, there is p such that $X \cdot p \leq 0$ and $Y \cdot p > 0$ for all $Y \in \mathcal{D}$ by Theorem 7. So $\mathcal{C}$ is nonempty, and in fact it is a cone (if p' and p'' satisfy the constraints, then so does $\lambda p'$ for $\lambda > 0$ and $p' + p''$). Hence if $X \notin \mathcal{D}$ then $\exists p \in \mathcal{C} : X \cdot p \leq 0$; equivalently, if $\forall p \in \mathcal{C} : X \cdot p > 0$, then $X \in \mathcal{D}$.

Part 2) By construction, if $X \in \mathcal{D}$ then $X \cdot p > 0$ for all $p \in \mathcal{C}$; using this and Part 1, $X \in \mathcal{D} \Leftrightarrow \forall p \in \mathcal{C} : X \cdot p > 0$.

Part 3) We now show that $\mathcal{C}$ is equivalent to a set of probability measures $\mathcal{K}$. Denote by $\mathbf{1}$ a vector of ones, and $\mathbf{1}_i$ a vector whose ith element is 1 and all other elements are zero. By monotonicity, $\mathbf{1} \cdot p > 0$ for all $p \in \mathcal{C}$, so $\sum_i p_i > 0$. Also, for every $p \in \mathcal{C}$: $(\mathbf{1}_i + \epsilon) \cdot p > 0$ for every $\epsilon > 0$; hence $p_i + \epsilon \sum_j p_j > 0$ for every ϵ, implying that $p_i \geq 0$ (if $p_i < 0$ then for $\epsilon < -p_i / \sum_j p_j$ we have $p_i + \epsilon \sum_j p_j < 0$, a contradiction). Hence we can normalize each p in $\mathcal{C}$, thus obtaining a set of probability measures $\mathcal{K}$ that is a representation for $\mathcal{D}$: $X \in \mathcal{D} \Leftrightarrow \forall \mathbb{P} \in \mathcal{K} : \mathbb{E}_{\mathbb{P}}[X] > 0$.

Part 4) Take the set $\mathcal{K}$ that is equal to the intersection of $\mathcal{C}$ and the unitary simplex $\sum_i p_i = 1$: If p belongs to this intersection, it is normalized so $p \in \mathcal{K}$; and if $p \in \mathcal{K}$, then $p \in \mathcal{C}$ and also it is normalized so it belongs to the unitary simplex. Hence $\mathcal{K}$ is the intersection of two convex sets, so $\mathcal{K}$ is convex.

Part 5) The cone $\mathcal{C}$ is defined as the intersection of open halfspaces, hence by definition it is evenly convex. And $\mathcal{K}$ is the intersection of those open halfspaces and the unitary simplex (itself the intersection of open halfspaces), hence $\mathcal{K}$ is evenly convex.

Part 6) To show that $\mathcal{K}$ is the unique maximal credal set that represents $\succ$, suppose there is $\mathcal{K}'$ that represents $\succ$, and $\mathbb{P}' \in \mathcal{K}'$ but $\mathbb{P}' \notin \mathcal{K}$. If $\mathbb{P}' \notin \mathcal{K}$, then by the definition of $\mathcal{K}$ we must have some $X \in \mathcal{D}$ such that $\mathbb{E}_{\mathbb{P}'}[X] \leq 0$. However, because $\mathcal{K}'$ represents $\succ$, for any $X \in \mathcal{D}$ we must have $\mathbb{E}_{\mathbb{P}}[X] > 0$ for all $\mathbb{P} \in \mathcal{K}'$; that is, $\mathbb{E}_{\mathbb{P}'}[X] > 0$. Hence we get a contradiction, implying that no representing credal set can contain probability measures outside of $\mathcal{K}$. ■

In fact *many* sets of probability measures may encode the same ordering. For instance, if a representing $\mathcal{K}$ is a closed set, then the set of its extreme points ext$\mathcal{K}$ is an equivalent representation for $\succ$; that is, $X \succ Y \Leftrightarrow \forall \mathbb{P} \in \mathsf{ext}\mathcal{K} : \mathbb{E}_{\mathbb{P}}[X] > \mathbb{E}_{\mathbb{P}}[Y]$.

Theorem 12 *Suppose $\succ$ is a coherent preference ordering, and the credal set $\mathcal{K}$ has been built as in the proof of Theorem 11. A credal set $\mathcal{K}'$ represents $\succ$ iff $\mathsf{eco}\mathcal{K}' = \mathcal{K}$.*

Proof We need only to consider preferences with respect to the zero gamble.

Take a credal set $\mathcal{K}'$ such that $\mathsf{eco}\mathcal{K}' = \mathcal{K}$. Clearly if $X \succ 0$ then $\forall \mathbb{P} \in \mathcal{K} : \mathbb{E}_{\mathbb{P}}[X] > 0$ then $\forall \mathbb{P} \in \mathcal{K}' : \mathbb{E}_{\mathbb{P}}[f] > 0$ as $\mathcal{K}' \subseteq \mathsf{eco}\mathcal{K}'$. Now suppose $\forall \mathbb{P} \in \mathcal{K}' : \mathbb{E}_{\mathbb{P}}[X] > 0$. Consider that $\mathsf{eco}\mathcal{K}'$ is the set of all p such that $Y \cdot p > 0$ for all Y such that for all $q \in \mathcal{K}'$ we have $Y \cdot q > 0$. As X satisfies the last set of inequalities, then $X \cdot p > 0$ for all $p \in \mathsf{eco}\mathcal{K}'$, hence $\mathbb{E}_{\mathbb{P}}[X] > 0$ for all $\mathbb{P} \in \mathcal{K}$, and then $X \succ 0$. Hence $\mathcal{K}'$ represents $\succ$.

Now suppose $\mathcal{K}'$ represents $\succ$. Then its elements must satisfy the constraints $X \cdot p > 0$ for all $X \in \mathcal{D}$. Suppose $\mathcal{K}'$ also satisfies a nontrivial constraint $Y \cdot p > \alpha$ for some Y and α; that is, there is p' that satisfies all other constraints but such that $Y \cdot p' \leq \alpha$. Because every p is a probability measure, $(Y - \alpha) \cdot p > 0$ is an equivalent constraint. Hence $(Y - \alpha) \cdot p > 0$ for all $p \in \mathcal{K}'$; because $\mathcal{K}'$ represents $\succ$, $Y - \alpha$ is a desirable gamble. However there is $p' \in \mathcal{K}$ such that $(Y - \alpha) \cdot p' \leq 0$, implying $Y - \alpha \notin \mathcal{D}$, a contradiction. So there is no additional nontrivial strict linear inequality that distinguishes $\mathcal{K}'$ and $\mathcal{K}$, and consequently they share the same evenly convex hull. ■

This theorem shows that if two evenly convex sets are different, then they represent distinct preference orderings. Figure 2 shows several different credal sets that have the same evenly convex hull, and hence represent the same coherent preference ordering.

4.3 A bit of duality

Additional insight can be obtained by investigating the duality between cl$\mathcal{D}$ and cl$\mathcal{C}$. As $\mathcal{C}$ is nonempty, $\mathsf{cl}\mathcal{C} = \{p : \forall X \in \mathcal{D} : X \cdot p \geq 0\}$; hence cl$\mathcal{C}$ is by definition the dual cone[5] of $\mathcal{D}$,

5. Given a convex set $\mathcal{A}$, its polar set is $\mathcal{A}^\circ = \{p : \forall X \in \mathcal{A} : X \cdot p \leq 1\}$ (Brondsted, 83); if $\mathcal{A}$ is a convex cone its polar set is equal to its polar cone, defined as $\{p : \forall X \in \mathcal{A} : X \cdot p \leq 0\}$ (because any inequality with right hand

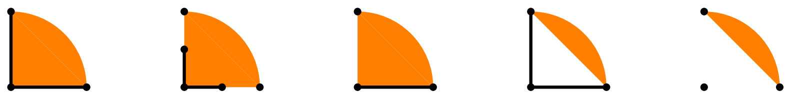

Figure 2: Five credal sets with the same evenly convex hull (the first credal set is evenly convex). Filled dots, thick lines and darker (orange) regions are in the credal sets.

denoted by $\mathcal{D}^\star$ (Boyd and Vandenberghe, 2004, Section 2.6). Then $(\mathrm{cl}\mathcal{C})^\star$ is just the closure of $\mathcal{D}$, as $\mathrm{cl}\mathcal{D} = \mathcal{D}^{\star\star}$ (Brondsted, 83, Theorem 6.2). Also, if a cone $\mathcal{F} \subset \mathcal{D}$ (say a proper face of $\mathcal{D}$), then $\mathcal{D}^\star \subset \mathcal{F}^\star$, and if we have several cones $\{\mathcal{D}_i\}_i$, then $(\cup_i \mathcal{D}_i)^\star = \cap \mathcal{D}_i^\star$ (Lay, 1982, Theorem 23.3).

It is also possible to establish a connection between the faces of $\mathrm{cl}\mathcal{D}$ and $\mathrm{cl}\mathcal{C}$. The following definition is necessary: for any face $\mathcal{F}$ of a closed convex cone $\mathcal{A}$, define its dual face $\mathcal{F}^\triangle = \mathcal{A}^\star \cap \mathcal{F}^\perp$ (Stoer and Witzgall, 1970, Section 2.13), where the superscript $\perp$ denotes orthogonal complement (that is, $\mathcal{B}^\perp = \{p : \forall X \in \mathcal{B} : X \cdot p = 0\}$). If for two faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{A}$ we have that $\mathcal{F}_1$ is a face of $\mathcal{F}_2$, then $\mathcal{F}_2^\triangle$ is face of $\mathcal{F}_1^\triangle$ (Tam, 1985, Proposition 2.4). In fact, if all faces of $\mathcal{A}$ are exposed, then the mapping between faces of $\mathcal{A}$ and its dual is one-to-one and onto, in such a way that $\mathcal{F}_1$ is a face of $\mathcal{F}_2$ iff $\mathcal{F}_2^\triangle$ is face of $\mathcal{F}_1^\triangle$ (Tam, 1985, Corollary 2.6). In particular if $\mathrm{cl}\mathcal{D}$ is generated by a finite number of gambles, then all its faces are exposed and the mapping is indeed one-to-one and onto the faces of $\mathrm{cl}\mathcal{C}$ (Stoer and Witzgall, 1970, Theorem 2.13.2). Of course, this applies similarly to faces of $\mathrm{cl}\mathcal{C}$ and its dual.

We can further refine these connections between $\mathcal{D}$ and $\mathcal{C}$. For instance, if a face of $\mathrm{cl}\mathcal{D}$ does intersect $\mathcal{D}$, its dual face does not intersect $\mathcal{C}$:

Theorem 13 *If $\mathcal{F}$ is a face of $\mathrm{cl}\mathcal{D}$, and $\mathcal{F} \cap \mathcal{D} \neq \emptyset$, then $\mathcal{F}^\triangle \cap \mathcal{C} = \emptyset$.*

Proof Suppose $\mathcal{F} \cap \mathcal{D} \neq \emptyset$. Pick $X \in \mathcal{F} \cap \mathcal{D}$. For any $p \in \mathcal{F}^\perp$ we must have $X \cdot p = 0$, so p cannot be in $\mathcal{C}$; hence $\mathcal{F}^\perp \cap \mathcal{C} = \emptyset$ and consequently $\mathcal{F}^\triangle \cap \mathcal{C} = \mathcal{D}^\star \cap \mathcal{F}^\perp \cap \mathcal{C} = \emptyset$. ∎

The converse can be shown for finitely generated faces:[6]

Theorem 14 *If $\mathcal{F}$ is a finitely generated face of $\mathrm{cl}\mathcal{D}$, and $\mathcal{F} \cap \mathcal{D} = \emptyset$, then $\mathcal{F}^\triangle \cap \mathcal{C} \neq \emptyset$.*

Proof We have that $\mathcal{F}$ is the conic hull of a finite set of gambles $\{X_1, \dots, X_n\}$. Suppose that no element of $\mathcal{F}^\perp = \{p : \forall X \in \mathcal{F} : X \cdot p = 0\}$ belongs to $\mathcal{C}$. Then for each $p \in \mathcal{C}$ there is at least a $X \in \mathcal{F}$ such that $X \cdot p > 0$. Write X as $\sum_i \alpha_i X_i$ (where all $\alpha_i \geq 0$) to obtain that $\sum_i \alpha_i X_i \cdot p > 0$; if we have $X_i \cdot p \geq 0$ for all X_i, then it must be that $X_i \cdot p > 0$ for at least one X_i. (To conclude that $X_i \cdot p \geq 0$ for all X_i, reason as follows. As any $Y \in \mathcal{F}$ is in the boundary of $\mathcal{D}$, for all such Y we have, for all $p \in \mathcal{C}$ and all $\epsilon > 0$, that $(Y + \epsilon) \cdot p > 0$. So for all $Y \in \mathcal{F}$ and all $p \in \mathcal{C}$ we must have $Y \cdot p \geq 0$ to satisfy $Y \cdot p > -\epsilon \sum_i p_i$ for all $\epsilon > 0$.) Consequently the convex combination $Z = \sum_{i=1}^n X_i/n$ must satisfy $Z \cdot p > 0$ for all $p \in \mathcal{C}$, and then $Z \in \mathcal{D}$. But Z must belong to $\mathcal{F}$, so Z cannot be in $\mathcal{D}$ by assumption. Hence there must be an element of $\mathcal{F}^\perp$ in $\mathcal{C}$; this proves the theorem as $\mathcal{F}^\perp \cap \mathcal{C} = \mathcal{F}^\perp \cap \mathcal{C} \cap \mathrm{cl}\mathcal{C} = \mathcal{F}^\perp \cap \mathcal{C} \cap \mathcal{D}^\star = \mathcal{F}^\triangle \cap \mathcal{C}$. ∎

side larger than zero is redundant). The dual cone is simply the mirror image of the polar cone: $\mathcal{A}^\star = -\mathcal{A}^\circ$. Also, $\mathcal{A}^{\circ\circ} = \{X : \forall p \in \mathcal{A}^\circ : X \cdot p \leq 0\} = \{X : \forall -p \in -\mathcal{A}^\circ : X \cdot p \leq 0\} = \{X : \forall p \in \mathcal{A}^\star : X \cdot p \geq 0\} = \mathcal{A}^{\star\star}$.

6. Whether or not Theorem 14 holds for general faces is an open question.

4.4 Back to SSK-continuity

Note that SSK-continuity is satisfied by coherent preference orderings:

Proposition 15 *If $\succ$ is a coherent preference ordering, then SSK-continuity holds.*

Proof Take $\{X_i\} \to X$ and $\{Y_i\} \to Y$ such that $X_i \succ Y_i$. Take the representing credal set $\mathcal{K}$; any probability measure $\mathbb{P} \in \mathcal{K}$ satisfies $\mathbb{E}_{\mathbb{P}}[X_i - Y_i] > 0$, so $\lim_i \mathbb{E}_{\mathbb{P}}[X_i - Y_i] \geq 0$; then $\mathbb{E}_{\mathbb{P}}[\lim_i X_i] \geq \mathbb{E}_{\mathbb{P}}[\lim_i Y_i]$ as the state space is finite, hence $\mathbb{E}_{\mathbb{P}}[X] \geq \mathbb{E}_{\mathbb{P}}[Y]$. If additionally $Y \succ Z$, then $\mathbb{E}_{\mathbb{P}}[Y] > \mathbb{E}_{\mathbb{P}}[Z]$ for every $\mathbb{P} \in \mathcal{K}$, so $\mathbb{E}_{\mathbb{P}}[X] > \mathbb{E}_{\mathbb{P}}[Z]$ for every $\mathbb{P} \in \mathcal{K}$, and then $X \succ Z$ as desired. ■

The natural question is whether SSK-continuity implies even continuity. It does not; but to appreciate the matter, it is interesting to note that SSK-continuity implies even continuity in an important case. Start by considering a consequence of SSK-continuity that is quite reasonable as a property of preferences:

Proposition 16 *Suppose $\succ$ is a preference ordering satisfying monotonicity and SSK-continuity. If $\alpha W + (1-\alpha)X \succ Y \succ 0$ for $\alpha \in (0, 1]$, then $X \succ 0$.*

Proof Take $\alpha_i = 1/2^i$, $X_i = \alpha_i W + (1-\alpha_i)X$ and $Y_i = Y$. As $X_i \succ Y_i$, $\{X_i\} \to X$, $\{Y_i\} \to Y$, and $Y \succ 0$, SSK-continuity implies $X \succ 0$ as desired. ■

This result leads to:

Proposition 17 *Suppose $\succ$ is a preference ordering satisfying monotonicity, cancellation, and SSK-continuity, with representing set of desirable gambles $\mathcal{D}$. If $X \in \mathcal{D}$ and $Y \in \mathsf{cl}\mathcal{D}$, then $\alpha X + (1 - \alpha)Y \in \mathcal{D}$ for $\alpha \in (0, 1)$.*

Proof Take $X \in \mathcal{D}$, $Y \in \mathsf{cl}\mathcal{D}$, $\alpha \in (0, 1)$, and $Z = \alpha X + (1-\alpha)Y$. For some $\delta > 0$ we have $Y + \delta \in \mathsf{relint}\mathcal{D}$ by monotonicity; hence $\beta(Y + \delta) + (1-\beta)Y \in \mathcal{D}$ for $\beta \in (0, 1]$ (Rockafellar, 1970, Theorem 6.1). Note that $Y = \gamma Z - \alpha\gamma X$ where $\gamma = (1-\alpha)^{-1}$; thus $\beta(\gamma Z - \alpha\gamma X + \delta) + (1-\beta)(\gamma Z - \alpha\gamma X) \succ 0$. Hence $\beta(\gamma Z + \delta) + (1-\beta)(\gamma Z) \succ \alpha\gamma X$ for $\beta \in (0, 1]$. By assumption $X \succ 0$, so $\alpha\gamma X \succ 0$; by Proposition 16, we obtain $\gamma Z \succ 0$, hence $Z \in \mathcal{D}$ as desired. ■

As noted by (Fenchel, 1952, Section 3.5), a cone $\mathcal{A}$ whose closure is the intersection of finitely many closed halfspaces is evenly convex iff it satisfies: if $X \in \mathcal{A}$ and $Y \in \mathsf{cl}\mathcal{A}$, then the segment between X and Y is in $\mathcal{A}$. Hence:

Theorem 18 *Suppose $\succ$ is a preference ordering satisfying monotonicity, cancellation, and SSK-continuity, with representing set of desirable gambles $\mathcal{D}$. If the closure of $\mathcal{D}$ is the intersection of finitely many closed halfspaces, then $\mathcal{D}$ is evenly convex.*

That is, SSK-continuity produces even convexity of the set of desirable gambles, and therefore of the representing credal set, when only finitely many assessments affect preferences. However, in general SSK-continuity does not enforce even convexity of sets of desirable gambles.

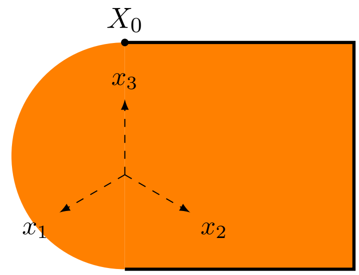

Figure 3: The set $\mathcal{B}$ in Example 1, viewed from point $(1, 1, 1)$.

To understand how this is possible, take a coherent preference ordering $\succ'$ and its representing set of desirable gambles $\mathcal{D}'$. Suppose $\mathcal{D}'$ contains a *non-exposed* but extreme ray R_0 that goes through gamble X_0 (that is, $R_0 = \{\lambda X_0 : \lambda > 0\}$). Define $\mathcal{D}'' = \mathcal{D}' \backslash R_0$; this is still a convex set (hence a convex cone) containing the positive octant. We can then define a preference ordering $\succ''$ as $X \succ'' Y$ iff $X - Y \in \mathcal{D}''$. Note that $\mathcal{D}''$ is not an evenly convex set, but $\succ''$ built as described satisfies SSK-continuity as we argue in the remainder of this section. However, before we plunge into the arguments, consider a concrete example:

Example 1 *Suppose* $\Omega = \{\omega_1, \omega_2, \omega_3\}$*; a gamble is a triple of numbers* (x_1, x_2, x_3)*, meaning* $(X(\omega_1), X(\omega_2), X(\omega_3))$*. Consider* $\mathcal{B}$ *as the union of the* open *circle with center* $(1/4, 1/4, 1/2)$ *and radius* $\sqrt{3/2}$ *drawn on the simplex consisting of* $x_1 + x_2 + x_3 = 1$*, and the* closed *polygon with four vertices* $(3/4, 3/4, -1/2)$*,* $(-1/4, -1/4, 3/2)$*,* $(-2, 3/2, 3/2)$*,* $(-1, 5/2, -1/2)$*, and take* $X_0 = (-1/4, -1/4, 3/2)$*, a non-exposed extreme point of* $\mathcal{B}$*. Figure 3 depicts the set* $\mathcal{B}$*. Take the cone* $\mathcal{D}''$ *as the set of all rays emanating from the origin and going through points of* $\mathcal{B}$ *except* X_0*. This cone* $\mathcal{D}''$ *produces a preference ordering that satisfies SSK-continuity.* □

We now show that the preference ordering $\succ''$ induced by $\mathcal{D}''$ satisfies SSK-continuity.

As the cone $\mathcal{D}'$ is evenly convex, we can build its representing credal set $\mathcal{K}'$. By construction $X \succ'' 0$ implies that for all $\mathbb{P} \in \mathcal{K}'$ we have $\mathbb{E}_{\mathbb{P}}[X] > 0$; also by construction $X \succ'' 0$ implies $X \notin R_0$. Also, if for all $\mathbb{P} \in \mathcal{K}'$ we have $\mathbb{E}_{\mathbb{P}}[X] > 0$ *and* $X \notin R_0$, then $X \succ'' 0$. That is, we have the representation: $X \succ'' 0 \Leftrightarrow (X \notin R_0) \wedge (\forall \mathbb{P} \in \mathcal{K}' : \mathbb{E}_{\mathbb{P}}[X] > 0)$.

By Proposition 5, we need to show that $\{X_i\} \to X$, $\{Y_i\} \to Y$, $X_i \succ'' Y_i$, $Y \succ'' 0$ imply $X \succ'' 0$. If $Y \in R_0$, then $Y \succ'' 0$ is false and there is nothing to prove; hence assume that $Y \notin R_0$. We distinguish two cases: $X \notin R_0$ and $X \in R_0$.

Take $X \notin R_0$. To prove that $X \succ'' 0$, note that $\mathbb{E}_{\mathbb{P}}[X_i - Y_i] > 0$ for every $\mathbb{P} \in \mathcal{K}'$, so $\lim_i \mathbb{E}_{\mathbb{P}}[X_i - Y_i] \geq 0$ and therefore $\mathbb{E}_{\mathbb{P}}[X] \geq \mathbb{E}_{\mathbb{P}}[Y]$ for $\mathbb{P} \in \mathcal{K}'$. Thus $\mathbb{E}_{\mathbb{P}}[X] \geq \mathbb{E}_{\mathbb{P}}[Y] > \mathbb{E}_{\mathbb{P}}[0] = 0$ and then $\mathbb{E}_{\mathbb{P}}[X] > 0$ for every $\mathbb{P} \in \mathcal{K}'$, implying $X \succ'' 0$ as desired.

Now take $X \in R_0$; note that X is in an extreme ray of $\text{cl}\mathcal{D}'$. In the next paragraph we show that if $\{X_i\} \to X$, $\{Y_i\} \to Y$, $X_i \succ'' Y_i$, then $Y \succ'' 0$ must be false. Hence it is irrelevant to consider $X \in R_0$ as the premise of SSK-continuity is never satisfied in this case, and the proof is finished.

To conclude we show that, if $\mathcal{A}$ is a convex cone, $\{X_i\} \to X$, $\{Y_i\} \to Y$, $X_i - Y_i \in \mathcal{A}$, and X belongs to an extreme ray of $\text{cl}\mathcal{A}$ but $X \notin \mathcal{A}$, then $Y \notin \mathcal{A}$. We have that $Y \in \text{cl}\mathcal{A}$ and, as $X_i - Y_i \in \mathcal{A}$ for every i, $X - Y \in \text{cl}\mathcal{A}$ (the closure is the set of limiting points). So we have both Y and $X - Y$ in $\text{cl}A$. If $Y \neq \lambda X$, then $X/2$ is the convex combination $Y/2 + (X - Y)/2$ of two

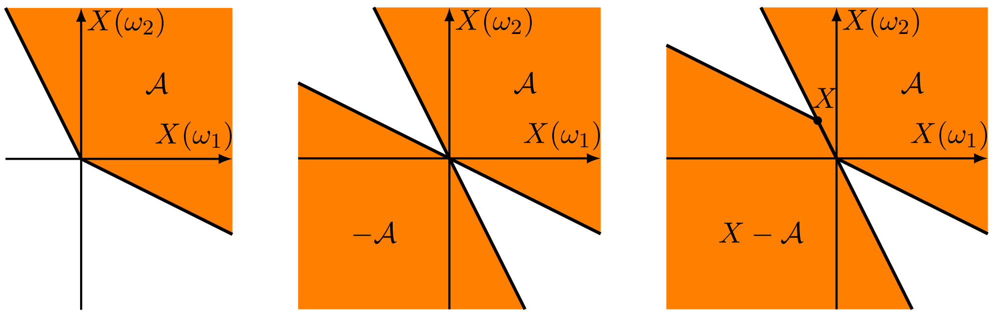

Figure 4: A closed convex cone $\mathcal{A}$ (left), the cones $\mathcal{A}$ and $-\mathcal{A}$ (middle), and the cones $\mathcal{A}$ and $X-\mathcal{A}$ for X in an extreme ray of $\mathcal{A}$ (right).

points not in the ray containing X, a contradiction with the assumption that X is in an extreme ray of cl$\mathcal{A}$. So $Y = \lambda X$ for some λ, and then $Y \notin \mathcal{A}$. (This result is illustrated by Figure 4: Y must belong to the closure of $\mathcal{A}$ and to the closure of $X - \mathcal{A}$, so it belongs to the line from the origin through X.)

5. Conclusion

We have presented a few axioms on preference orderings that, together, imply a representation through evenly convex credal sets. This representation lets one handle assessments of strict inequality for probabilities, and go beyond what can be done with closed convex credal sets. The main idea is to adopt a novel Archimedean condition (even continuity) that implies even convexity. A similar representation can be obtained using SSK-continuity in many, but not all, cases.

Future work should look at natural and similar extensions, as well as to conditioning and independence. It should also be possible to use our proposed Archimedean condition to obtain *general* sets of probabilities, mimicking results by Seidenfeld et al. (2010).

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