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# Towards classifying propositional probabilistic logics

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This paper examines two aspects of propositional probabilistic logics: the nesting of probabilistic operators, and the expressivity of probabilistic assessments. We show that nesting can be eliminated when the semantics is based on a single probability measure over valuations; we then introduce a classification for probabilistic assessments, and present novel results on their expressivity. Logics in the literature are categorized using our results on nesting and on probabilistic expressivity.


## 1. Introduction

As even a cursory look at the literature reveals, there are many ways to mix probabilities and propositional logic. Indeed, propositional probabilistic logics appear in several fields, from philosophy to economics and artificial intelligence, with real applications from material discovery [14] to model checking [2]. However, there are relatively few proposals for the classification of propositional probabilistic logics based on their expressivity [10,18,36,39].

In this paper we examine two aspects of propositional probabilistic logics that can be used to classify them. First, we examine the nesting of probabilistic operators, which we identify not only as a major syntactic decision, but also as a decision that drives the semantics. Second, we investigate the expressivity of probabilistic assessments, a topic that has received scant attention in the literature but that has interesting consequences.

We restrict our study to propositional probabilistic logics that adopt classical propositional logic as a starting point; we also assume that all models have a finite number of possible worlds, so that issues of countable additivity are irrelevant. Finally, we only deal with "unconditional" probabilities, leaving conditional assessments to future work.[1]

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[1] We consider some logics that can make statements constraining the value of $P(\phi \wedge \theta)/P(\theta)$, given $P(\theta) > 0$, in Section 4.3; but this is not the same as constraining the value of $P(\phi|\theta)$ in the general case (in which $P(\theta)$ may be 0).

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After a brief review of relevant concepts in Section 2, we focus on the nesting of probabilistic operators and its semantic consequences in Section 3. That is, we examine the meaning of expressions such as $P(P(\phi) = 0.5) = 0.7$, understood as the "probability of the probability of a propositional formula $\phi$ being 0.5 is 0.7". We contribute with a novel analysis of the relationship between nesting and semantics. In Section 4 we discuss ways to specify probability assessments; we call it the *probabilistic expressivity* of a logic. For instance, a particular logic may allow one to state $P(\phi) = 1/2$, while another logic may allow $P(\phi_1)^2 + 2P(\phi_2) \leqslant q$ where $\phi_1$ and $\phi_2$ are propositional formulas and $q$ is restricted to be a rational number. The extent to which one can express probabilistic appraisals strongly affects expressivity and complexity of the whole logic in question.

Some decisions one can take when mixing probabilities and propositions may not have much effect on expressivity and complexity; other decisions can have dramatic effect. Based on our results, we sketch a scheme for classification of propositional probabilistic logics in Section 5.

## 2. Preliminaries

In this section we offer a brief review of propositional logic and probabilistic satisfiability, so as to fix notation and terminology.

First, we define a logic as a *satisfaction system*, as it is done in [3]:

**Definition 1.** Let $L$ be a non-empty set (a language) and $M$ be a non-empty class (of models). A *logic* is a tuple $(L, M, \models)$ in which $\models \subseteq M \times L$ is a relation.

### *2.1. The propositional language and its semantics*

The language of propositional logic consists of a set of formulas formed by propositions combined with logical connectives, possibly with punctuation elements. We assume a countably infinite set of symbols $X = \{x_1, x_2, x_3, \ldots\}$ corresponding to atomic propositions. We have the unary connective $\neg$ (negation) and the binary connectives $\vee$ (disjunction), $\wedge$ (conjunction) and $\rightarrow$ (implication), plus parentheses (dropped whenever possible). Every proposition is a formula; moreover, if $\phi$ is a formula, then $\neg\phi$ is a formula; if $\phi_1$ and $\phi_2$ are formulas, then $(\phi_1 \vee \phi_2)$, $(\phi_1 \wedge \phi_2)$ and $(\phi_1 \rightarrow \phi_2)$ are formulas. The set of all formulas built using only these guidelines is the language of propositional logic, denoted by $L_{PL}$. Additionally, $\phi_1 \leftrightarrow \phi_2$ (bi-implication) denotes $(\phi_1 \rightarrow \phi_2) \wedge (\phi_2 \rightarrow \phi_1)$, and $\top$ denotes $x_i \vee \neg x_i$ for some $x_i$.

Each atomic proposition can assume a truth value, either true or false, represented respectively by 1 and 0. A *truth assignment*, or *valuation*, is a function $v : X \rightarrow \{0, 1\}$ that takes atomic propositions to truth values. Valuations may have their domain extended to all of $L_{PL}$, as follows. Let $\phi_1$ and $\phi_2$ be formulas from the propositional language; then: $v(\phi_1 \wedge \phi_2) = 1$ if and only if $v(\phi_1) = 1$ and $v(\phi_2) = 1$; $v(\phi_1 \vee \phi_2) = 1$ if and only if $v(\phi_1) = 1$ or $v(\phi_2) = 1$; $v(\neg\phi_1) = 1$ if and only if $v(\phi) = 0$; $v(\phi_1 \rightarrow \phi_2) = 1$ if and only if $v(\phi_1) = 0$ or $v(\phi_2) = 1$. Let $V$ be the set of all valuations over $X$. The classical propositional logic $L_{PL}$ is the tuple $(L_{PL}, V, \models)$ where $\models \subseteq V \times L_{PL}$ is a relation such that $v \models \phi$ iff $v(\phi) = 1$ for every $(v, \phi) \in V \times L_{PL}$. A propositional formula $\phi$ is *satisfiable* when it is possible to find a valuation $v$ such that $v \models \phi$.

We will identify a logical formula $\phi$ with a set of *possible worlds*, to be suitably defined later. Such a set is denoted by $[\![\phi]\!]$. In this paper we always have, for the logics and semantics we define, that $[\![\neg\phi]\!] = \overline{[\![\phi]\!]}$, $[\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \cup [\![\phi_2]\!]$, and $[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!]$.

### *2.2. Probability theory and probabilistic satisfiability*

A probability measure attaches real numbers to events that are subsets of a set $\Omega$, the possibility space. All probability spaces in this paper are finite, hence we can take that any subset of $\Omega$ is an event. A probability measure $P$ is such that $P(\Omega) = 1$, $P(A) \geqslant 0$ for any event $A$, and $P(A \cup B) = P(A) + P(B)$ for any disjoint events $A$ and $B$. A function from elements of $\Omega$ to the interval $[0,1]$, that adds up to one, is a *probability mass* over $\Omega$; a probability mass induces a probability measure.

Probability values may be interpreted as relative frequencies, propensities, objective or subjective degrees of belief, betting rates [32,18,25]; we are not concerned with the interpretation of probability values in this paper.

By $P(\phi)$, where $\phi$ is a formula, one typically means the probability of the set $[\![\phi]\!]$ of possible worlds in which $\phi$ is true. This is indeed the sort of semantics we adopt throughout. Given a probability mass $\pi$ over the set of possible worlds, $P_\pi(\cdot)$ denotes the induced probability measure on formulas by $P_\pi(\phi) = \pi([\![\phi]\!])$. For classical propositional logic, we can equate possible worlds with valuations, and we obtain:

$$P_\pi(\phi) = \pi\big([\![\phi]\!]\big) = \sum \big\{\pi(v_i) \big| v_i(\phi) = 1\big\}. \tag{1}$$

Note that alternative approaches to semantics of probabilistic assessments are possible, as can be found for instance in the work of Haenni et al. [19].

We now formalize the notion of a probabilistic assessment. Call *elementary probabilistic formula* an expression such as $P(\phi) \geqslant q$, where $\phi$ is a propositional formula and $q$ is a rational number. Suppose one has a set of elementary probabilistic formulas; we say that these assessments are *consistent* if there is a probability measure $P_\pi(\cdot)$ on $L_{PL}$ such that $P_\pi(\phi_i) \geqslant q_i$ for each formula $P(\phi_i) \geqslant q_i$.

The problem of verifying consistency of a set of elementary probabilistic formulas is the *probabilistic satisfiability* (PSAT) problem [17]. If there are $n$ atomic propositions in the formulas considered, one must find a probability mass $\pi(.)$ over the $2^n$ possible worlds that induces the measure $P_\pi(.)$. We say an instance of PSAT problem is satisfiable if there is such probability mass $\pi$, and it is unsatisfiable otherwise.

PSAT has been rediscovered several times, and an analytic version was actually proposed by Boole [4]. Hailperin [20], Bruno and Gilio [5], and Nilsson [33] suggested a linear programming approach. Consider the problem of deciding the consistency of a set of elementary probabilistic formulas $\{P(\phi_i) \geqslant q_i | 1 \leqslant i \leqslant m\}$, where $\phi$ are propositions involving $n$ atoms. Let $A$ be an $(m \times 2^n)$-matrix whose elements are $a_{ij} = v_j(\phi_i)$, let $q$ be the $(m \times 1)$-vector $[q_1, \ldots, q_m]'$. The problem is satisfiable iff there is a $(2^n \times 1)$-vector $\pi$ that is solution to $A\pi = q$, $\pi \geqslant 0$ and $\sum \pi = 1$. Intuitively, in the matrix $A$ we have a column for each valuation and a row for each formula. The vector $\pi$ is a probability mass over the valuations. If there is a feasible solution for $\pi$, then there is a probability mass $\pi$ that satisfies the problem; otherwise, the problem is unsatisfiable.

PSAT is an NP-complete problem [17]; if there is a solution, there is a solution with only $m+1$ valuations (columns) receiving positive probability. Kavvadias and Papadimitriou [29] and Jaumard et al. [28] have shown how column generation methods can handle large problems, and several approaches have since appeared [30,13,24,8]. Note that this linear programming approach can be applied to other probabilistic logics, as discussed by Andersen and Hooker [1] and Jaumard et al. [27].

## 3. To nest, or not to nest

The simplest class of propositional probabilistic logics, discussed in Section 3.1, is one where nesting is not allowed, and where the semantics is given by a probability measure over the set of valuations. Section 3.2 considers another class of propositional probabilistic logics, where nesting is allowed but the semantics is still based on a probability measure over valuations. The main result of this section is that this second class collapses into the first one. This fact was noted by Ognjanovic and Raškovic [35] for first-order probability logics, even though no formal result on this was given.

A third class of propositional probabilistic logics, discussed in Section 3.3, allows nesting *and* takes each possible world to be associated with a valuation and with a probability measure over valuations. Here nesting truly increases expressivity. Finally, a fourth class is contemplated in Section 3.4, where nesting is not allowed and the semantics takes each possible world to be associated with a valuation and a probability

**Table 1**
List of most important definitions and symbols.

| Concept/symbol | Explanation |
|---|---|
| $L_{PL}$ | the usual propositional language over $X = \{x_1, x_2, \dots\}$ |
| elementary probabilistic formula | a formula like $P(\phi) \geqslant q$, with $q \in Q$ and $\phi \in L_{PL}$ |
| basic probabilistic formula | any probabilistic assessment like $f(P(\phi_1), \dots, P(\phi_m)) \bowtie q$ |
| $L_{BPF}$ | the set of all basic probabilistic formulas with $\phi_i \in L_{PL}$ |
| $L_{PPL}$ | a language formed by the Boolean closure of $L_{BPF}$ |
| $\mathcal{L}_{PPL}$ | a logic over $L_{PPL}$ with standard probabilistic semantics |
| $L_{NPL}$ | a language: $L_{PPL}$ plus nesting of probabilistic operators |
| $\mathcal{L}_{PN}$ | a logic over $L_{NPL}$ with models for higher-order probability |
| $Fr(n)$ | the set $\{0, 1/n, 2/n, \dots, (n-1)/n, 1\}$ |
| $\mathcal{L}_{\bowtie S}$ | sublogics of $\mathcal{L}_{PPL}$ with assignments restricted to $P(\phi) \bowtie q$, $q \in S$ |
| $\mathcal{L}'_{\bowtie S}$ | sublogics of $\mathcal{L}_{NP}$ with assignments restricted to $P(\phi) \bowtie q$, $q \in S$ |

measure over valuations. It is hard to imagine useful applications for any such logic; indeed, we have not been able to find any representative of this fourth class in the literature.

There are two related classification schemes in the literature. In the classification scheme of Ognjanovic et al. [36], the first category above corresponds to logic $LPP_2$, while their logic $LPP_1$ allows nesting and adopts possible worlds endowed with probability measures. Ikodinovic et al. [26] have refined the classification of $LPP_2$ based on the probabilistic expressivity, what we discuss in Section 4.3.

A different classification is proposed by Williamson [39], who distinguishes "internal" and "external" logics. Such classification scheme is also explored by Demey et al. [10]; we comment on it in Section 3.5 and indicate why we prefer our scheme based on nesting and its semantics.

In the following sections, we introduce and use several definitions and symbols; the most important are listed in Table 1 with a short explanation.

### *3.1. No nesting*

Consider a logic that does not allow nesting, and where the semantics is based on a probability measure over the set of valuations (similar to PSAT problems). A possible world is simply a valuation. In such logics, each propositional formula is true or false in each possible world, but each probabilistic assessment is true or false in a pair consisting of the set of possible worlds and a probability measure over them.

PSAT problems are defined within a rather simple language that consists of conjunctions of elementary probabilistic formulas. There are two dimensions to generalize here; first, we might consider more elaborate probabilistic assessments; second, we might consider Boolean operators among probabilistic assessments.

Call *basic probabilistic formula* an expression such as $f(P(\phi_1), \dots, P(\phi_n)) \bowtie q$, where $f : [0,1]^n \to \mathbb{R}$ is a function, $q \in \mathbb{Q}$ is a rational number, $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$, and $\phi_1, \dots, \phi_n$ are formulas in the language of interest. In this section each $\phi_i$ is a propositional formula. We discuss in more detail the specification of $f$, $\bowtie$ and $q$ in Section 4, in which constraints on $f$, $\bowtie$ and $q$ (and the relationship among them) are investigated. For instance, both $P(x_1) \geqslant 0.2$ and $P(x_1)P(x_2) - 2P(x_3) < 0$ are basic probabilistic formulas, but the first one is an elementary probabilistic formula as well. The set of basic probabilistic formulas with $\phi_i \in L_{PL}$, for all $i$, is denoted by $L_{BPF}$.

A model for basic probabilistic formulas is defined below:

**Definition 2.** A *PPL-model* is a structure $\mathcal{M} = (W, \pi)$ where $W$ is a finite set of possible worlds (identified with valuations) and $\pi$ is a probability distribution over $W$.

Let $M_{PPL}$ be the set of all such models.[2] If a formula $\theta$ is equal to $f(P(\phi_1), \dots, P(\phi_n)) \bowtie q$, then we say that a structure $\mathcal{M}$ models $\theta$, written $\mathcal{M} \models \theta$, if $f(P_\pi(\phi_1), \dots, P_\pi(\phi_n)) \bowtie q$. Here $f$, $\bowtie$ and $q$ appear

[2] Note that we restrict a model to have a finite set of worlds $W$ by relying on the fact that, within the semantics adopted, if a formula has a model, then it has a model with finite $W$ [36]. This result holds for the more general logic in Section 3.3.

both as elements of $\theta$ and as elements of a mathematical expression; the function $f$ applied to numbers $P_\pi(\phi_1), \dots, P_\pi(\phi_n)$ returns a number that must be related to $q$ through $\bowtie$.

We can now consider conjunctions of basic probabilistic formulas, but also we can consider disjunctions and negations of them. That is, we can define the language $L_\wedge$ as follows: if $\phi \in L_{BPF}$, then $\phi \in L_\wedge$; and if $\phi_1, \phi_2 \in L_\wedge$, then $(\phi_1 \wedge \phi_2) \in L_\wedge$. The definition of $\mathcal{M} \models \theta_1 \wedge \theta_2$, for $\theta_2, \theta_2 \in L_\wedge$, is simply: $\mathcal{M} \models \phi_1$ and $\mathcal{M} \models \phi_2$. Similarly, we can define the language $L_{\wedge,\vee}$ by adding disjunction; in such a language one could easily express assessments such as $P(\phi) \in [0, 0.2] \cup [0.3, 0.4]$.[3] However, we have not found in the literature any logic that allows for conjunction and disjunction of basic probabilistic formulas, and does not allow for negation of basic probabilistic formulas. Hence we define the language $L_{PPL}$ by adding $\neg$ and $\vee$ to $L_\wedge$:

- if $\phi \in L_{BPF}$, then $\phi \in L_{PPL}$;
- if $\phi_1, \phi_2 \in L_{PPL}$, then $(\phi_1 \wedge \phi_2), (\phi_1 \vee \phi_2) \in L_{PPL}$;
- if $\phi \in L_{PPL}$, then $\neg\phi \in L_{PPL}$.

And then we add to the semantics of $L^\wedge$: $\mathcal{M} \models \phi_1 \vee \phi_2$ iff $\mathcal{M} \models \phi_1$ or $\mathcal{M} \models \phi_2$; $\mathcal{M} \models \neg\phi$ iff $\mathcal{M} \not\models \phi$. The logic $\mathcal{L}_{PPL}$ is then the tuple $(L_{PPL}, M_{PPL}, \models)$.

As an example of a logic that instantiates $\mathcal{L}_{PPL}$, note that De Bona et al. [9] propose a disjunctive linear programming approach for the satisfiability problem of a logic that allows for conjunction, disjunction and negation of elementary probabilistic formulas. As another example, Fagin et al. [12] describe a language where each basic probabilistic formula can only use linear combinations of probabilities, and where all Boolean operators can be used to combine these restricted basic probabilistic formulas. They prove that the satisfiability problem for that language is NP-complete.

We will return to propositional probabilistic logics without nesting in Section 3.4, after we examine some properties of nesting.

### 3.2. Nesting, but not quite

Now suppose we allow probabilistic operators to be within the scope of other probabilistic operators. For instance, consider the following syntax for a language $L_{NPL}$ that extends $L_{PPL}$ with nesting:

- Every atomic proposition is in $L_{NPL}$;
- If $\phi \in L_{NPL}$, then $\neg\phi \in L_{NPL}$;
- If $\phi_1, \phi_2 \in L_{NPL}$, then $(\phi_1 \vee \phi_2)$, $(\phi_1 \wedge \phi_2)$, $(\phi_1 \to \phi_2)$ are in $L_{NPL}$.
- If $\phi_1, \phi_2, \dots, \phi_n \in L_{NPL}$, then $f(P(\phi_1), \dots, P(\phi_n)) \bowtie q \in L_{NPL}$.

Suppose also that we wish to keep each valuation as a possible world, and to define satisfiability in terms of a probability measure over valuations. We cannot resort to a semantics based on the expression $P_\pi(\phi) = \sum\{\pi(w) | w \models \phi\}$, because $\phi$ may itself be a probabilistic formula that does not have truth value at any particular valuation. We need some adjustments to the semantics.

One could change slightly the semantics, as follows. Say $(\mathcal{M}, w) \models f(P(\phi_1), \dots, P(\phi_n)) \bowtie q$ iff $\mathcal{M} \models f(P(\phi_1), \dots, P(\phi_n)) \bowtie q$. Note that we use $(\mathcal{M}, w) \models \phi$ instead of $w \models \phi$ as the probability distribution $\pi \in \mathcal{M}$ may be used to compute the truth value of $\phi$. This implies that a basic probabilistic formula has the same truth value in all possible worlds (having probability 1 or 0). This sort of semantics captures the assumption that “if one knows a probability, this probability is known with probability one” (also known as the *lifting principle*), discussed for instance by Uchii [38].

[3] Note that the same assessment can be produced using a basic probabilistic formula such as $(P(\phi) - 0)(P(\phi) - 0.2)(P(\phi) - 0.3)(P(\phi) - 0.4) \leqslant 0$. This illustrates the importance of taking into account the expressivity of assessments, as we discuss in Section 4.

As formulas are evaluated in a single possible world, there is no problem with formulas outside the scope of a basic probabilistic subformulas, like $x_1$ in $x_1 \wedge P(x_2) = 0.5$, because worlds are propositional valuations. As probabilistic formulas are syntactically introduced at the same level of pure propositional formulas, the Boolean operators (disjunction, conjunction, negation) are all inherited from classical propositional logic.

Semantically, the formulas of $L_{NPL}$ are interpreted in a structure $\mathcal{M} = (W, \pi)$, in which $W$ is a finite set of possible worlds (valuations) and $\pi$ is a probability measure over them. We say a pair $(\mathcal{M}, w_j)$ models a formula $\phi \in L_{NPL}$, denoting by $(\mathcal{M}, w_j) \models \theta$, if (for $\phi_1, \phi_2, \ldots, \phi_n \in L_{NPL}$):

- $\theta$ is an atomic proposition $x_i$ and $w_j(x_i) = 1$;
- $\theta = (\phi_1 \wedge \phi_2)$, $(\mathcal{M}, w_j) \models \phi_1$ and $(\mathcal{M}, w_j) \models \phi_2$;
- $\theta = \neg\phi_1$ and $(\mathcal{M}, w_j) \not\models \phi$;
- $\theta = f(P(\phi_1), \ldots, P(\phi_n)) \bowtie q$ and $f(P_\pi(\phi_1), \ldots, P_\pi(\phi_n)) \bowtie q$.

Here again $f, \bowtie \in \{<, >, \leqslant, \geqslant, =\}$ and $q \in \mathbb{Q}$ are used in two different contexts, as part of a logical formula, and with their usual mathematical meaning. Let $M_{PPL'}$ be the set of all pairs $(\mathcal{M}, w)$. The logic $\mathcal{L}_{NNQ}$ is the tuple $(L_{NPL}, M_{PPL'}, \models)$.

As an example, De Bona et al. [9] discuss a logic that is a restriction of $\mathcal{L}_{NNQ}$; in that logic, assessments are restricted to elementary probabilistic formulas. Another example is the propositional fragment of Halpern's [21] first-order logic with probabilities on possible worlds, in which nesting of probabilistic operators is allowed, but models have only a single probability measure.

We now show that, under this semantics, $L_{NPL}$ has a normal form whose syntax dispenses with nesting; that is, any formula in $L_{NPL}$ can be polynomially rewritten with nesting elimination. The conclusion we wish to suggest is that a logic that adopts nesting, but whose semantics is based on a probability measure over valuations, does not add any real expressivity to the logics described in Section 3.1.

Formally, we say a formula $\phi \in L_{NPL}$ is in normal form if $\phi \in L_{PPL}$. The following theorem[4] shows that every formula $\phi \in L_{NPL}$ can be transformed (in polynomial time) into a formula $\theta \in L_{PPL}$, preserving satisfiability. As fresh atoms are added in this procedure, $\phi$ and $\theta$ are not logically equivalent, but from a model for one we can recover a model for the other.

**Theorem 1.** *For every $\phi \in L_{NPL}$, there exists $\theta \in L_{PPL}$ (computed in polynomial time) such that $\phi$ is satisfiable if, and only if, $\theta$ is. Furthermore, $(\mathcal{M}, w) \models \theta$ implies $(\mathcal{M}, w') \models \phi$ for some $w'$.*

These results indicates that one does not gain expressivity by allowing for syntactic higher-order probabilities in a semantic framework that is based on a single probability distribution. In the next section we examine a different semantics that lends useful meaning to nesting.

### *3.3. Proper nesting*

Given the results in the previous section, we might look for semantics that associate with each possible world a probability measure over the possible worlds, as follows [11].

Consider the syntax of $L_{NPL}$ as described in Section 3.2. Take the formulas to be interpreted in the following a structure:

**Definition 3.** A *PN-model* is a structure $\mathcal{M} = (W, \Pi, v)$, where $W$ is a finite set of possible worlds $w_i$, $\Pi = \{\pi_1, \ldots, \pi_{|W|}\}$ is a set of probability distributions over the possible worlds, where $\pi_i$ is associated with $w_i$, and $v$ is a function that assigns a valuation $v(w)$ to each world $w$.

[4] The proofs of all theorems have been placed in a separate appendix at the end of the paper.

Note that different worlds $w$ and $w'$ may be associated with the same valuation, while they may be associated with different probability distributions – that is the reason why valuations are not identified with possible worlds anymore. To save notation, when $v$ is clear from context, we say $w(x_i) = 1$ when $v(w)(x_i) = 1$.

We then say a pair $(\mathcal{M}, w_j)$ models a formula $\phi \in L_{NPL}$, denoting it by $(\mathcal{M}, w_j) \models \phi$, if, for $\phi_1, \phi_2 \in L_{NPL}$:

- $\phi$ is an atomic proposition $x_i$ and $w_j(x_i) = 1$;
- $\phi = \phi_1 \wedge \phi_2$, $(\mathcal{M}, w_j) \models \phi_1$ and $(\mathcal{M}, w_j) \models \phi_2$;
- $\phi = \neg\phi_1$ and $(\mathcal{M}, w_j) \not\models \phi$;
- $\phi = f(P(\phi_1), \ldots, P(\phi_n)) \bowtie q$ and $f(P_{\pi_j}(\phi_1), \ldots, P_{\pi_j}(\phi_n)) \bowtie q$.

Here $P_{\pi_j}(\phi)$ denotes $\sum\{\pi_j(w_i) | (\mathcal{M}, w_i) \models \phi\}$. Denote by $M_{PN}$ the set of all pairs $(\mathcal{M}, w)$ to define the logic $\mathcal{L}_{PN} = (L_{NPL}, M_{PN}, \models)$.

When every particular world is measurable, this semantics builds a modified Kripke structure in which the accessibility relation arcs from each particular possible world are labeled with non-negative numbers that add up to one. Note that if all possible worlds are always associated with identical measures, then the semantics of Section 3.2 is recovered. When $\mathcal{M} = (W, \Pi, v)$ is such that all probability measures are equal to $\pi$, a structure $\mathcal{M} = (W, \pi)$ with the semantic rules from Section 3.2 satisfies the same formulas at the same valuations.

An even more general semantics would be produced if each possible world might be associated with more than a probability measure, perhaps each one of them corresponding to distinct agents. For instance, one might pursue such a scheme to encode actions in stochastic processes [11]. Here we focus on the single-agent scenario, and restrict our semantics to a probability measure per possible world.

### *3.4. No nesting, again*

We have so far considered three classes of propositional probabilistic logics. The first one does not allows nesting, while the third allows nesting and provides a rich semantics for it. Our contribution has been to indicate that the second class, where one uses the syntax of the third class but the semantics basically belongs to the first class, is not an interesting class with respect to expressivity.

One might consider a fourth class, where nesting is not allowed, but one adopts the semantics of the third class; namely, each possible world is associated with a probability measure over possible worlds. As far as we could find, there has been no proposal in the literature for such a logic. In fact, this particular combination of syntax and semantics does not seem sensible. Suppose we have a formula $\phi \in L_{PPL}$, without nesting. Consider a structure $\mathcal{M} = (W, \Pi, v)$. To assess the truth value of $\phi$ in a pair $(\mathcal{M}, w_i)$, every basic probabilistic formula $f(P(\phi_1), \ldots, P(\phi_n)) \bowtie q$ that is a subformula of $\phi$ must have its truth value computed using the probability distribution $\pi_i$, associated with the world $w_i$. Therefore, the truth value of $\phi$ in $(\mathcal{M}, w_i)$ is independent of probability measures in $\Pi$ other than $\pi_i$. That is, if there is a model for $\phi$, there is a model in which all probability distributions in $\Pi$ are the same.

### *3.5. Internal and external probabilistic logics*

We have seen that propositional probabilistic logics can be divided into two groups depending on whether they allow nesting or not, and that each group can be most naturally associated with one type of semantics. A different classification has been proposed by Williamson [39]; he divides propositional probabilistic logics into *external* and *internal* ones. In an external logic, “probabilities are attached to sentences of a logical language,” while in an internal logic, “sentences incorporate statements about probabilities.” The PSAT problem represents a situation where the probabilistic language is external to propositional logic.

In an internal probabilistic logic, the language incorporates probabilities, and formulas possibly contain probabilistic assessments on subformulas.

Such distinction between external and internal is interesting, but artificial; any external probabilistic logic can be transformed into an internal logic that belongs to the first class discussed previously, and indeed this was done in Section 3.1 as we moved from PSAT to $L_\wedge$. Additionally, every logic that allows nesting must include probabilistic operators in the language, and thus qualifies as internal for Williamson [39]; hence, the logics discussed in Sections 3.2–3.3 are all internal, despite their semantic difference. Altogether, the classification proposed here seems to capture interesting features that escape the external/internal dichotomy.

## 4. The probabilistic expressivity

In this section we analyze the construction of basic probabilistic formulas with respect to identifying the format restrictions relevant to the expressivity/complexity trade-off. From here on we consider just two classes of propositional probabilistic logics: with no nesting (Section 3.1) and with proper nesting (Section 3.3).

A basic probabilistic formula has the form $f(P(\phi_1), \dots, P(\phi_n)) \bowtie q$. Varying the constraints on $f$, $\bowtie$ and $q$ may lead to different classes of logical systems, but just some of these choices are relevant. With no constraints on $f$, restricting $q$ and $\bowtie$ has no impact on complexity. So, we shall study the effectiveness of limiting the domain of $q$ and $\bowtie$ according to a class of functions for $f$. Based on the common approaches found in the literature to assign probabilities to logical formulas, we focus on four classes of functions $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \mathcal{F}_4$:

- $\mathcal{F}_1$ contains only the identity function;
- $\mathcal{F}_2$ is the class of linear combinations with rational coefficients, that is $f(P(\phi_1), \dots, P(\phi_n)) = \sum_{i=1}^{n} a_i P(\phi_i)$, with $a_i \in \mathbb{Q}$;
- $\mathcal{F}_3$ is the class of all polynomials over the reals with rational coefficients, for all arities $n \in \mathbb{N}$;
- $\mathcal{F}_4$ is the class of all computable functions.

We say a basic probabilistic formula $f(P(\phi_1), \dots, P(\phi_n)) \bowtie q$ is in $\mathcal{F}_i$ when $f \in \mathcal{F}_i$; and we say a logic is in $\mathcal{F}_i$ when all its basic probabilistic formulas are in $\mathcal{F}_i$.

The concept of *probability assertion* defined by Scott and Krauss [37], when limited to the propositional case, can be viewed as a Boolean combination of basic probabilistic formulas in $\mathcal{F}_3$ with minor differences, such as integer coefficients. However, such definition does not capture some probabilistic operators found in the literature, as we shall see in Section 4.3, reason why we introduce the more general class $\mathcal{F}_4$.

### *4.1. The relation $\bowtie$*

For each class $\mathcal{F}_i$, we can investigate whether the choice of $\bowtie$ is important or not. It is a well known fact that expressions like $a = b$, $a < b$ and $a > b$ can be replaced by suitable logical combinations of $a \geqslant b$ (or $a \leqslant b$). With no restrictions on $q$, $f(P(\phi_1), \dots, P(\phi_n)) \geqslant q$ is equivalent to $-f(P(\phi_1), \dots, P(\phi_n)) \leqslant -q$, and if $f \in \mathcal{F}_i$, for $i \in \{2, 3, 4\}$, then $f'(P(\phi_1), \dots, P(\phi_n)) = -f(P(\phi_1), \dots, P(\phi_n)) \in \mathcal{F}_i$. When $f \in \mathcal{F}_1$, $P(\phi) \geqslant q$ can be replaced by $P(\neg\phi) \leqslant 1-q$. Second, if the syntax allows for conjunction of basic probabilistic formulas, then $f(P(\phi_1), \dots, P(\phi_n)) = q$ can be replaced by $(f(P(\phi_1), \dots, P(\phi_n)) \geqslant q) \wedge (f(P(\phi_1), \dots, P(\phi_n)) \leqslant q)$, for any $f$. Furthermore, when a basic probabilistic formula can be negated, $f(P(\phi_1), \dots, P(\phi_n)) < q$ can be replaced by $\neg(f(P(\phi_1), \dots, P(\phi_n)) \geqslant q)$. We trivially conclude that any $\bowtie \in \{<, >, \leqslant, \geqslant\}$ can simulate all $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$, provided negation and conjunction of basic probabilistic formulas and free choice of $q$. Less trivial is the fact that $=$ can simulate any $\bowtie \in \{<, >, \leqslant, \geqslant\}$ in probability assignments with $f \in \mathcal{F}_1$. Finger and De Bona [13] presented a normal form for the probabilistic satisfiability problem

(PSAT), from the logic system of Nilsson [33], in which assignments of the form $P(\phi) \geqslant q$ are replaced by $P(x) = q$ and $P(x \to \phi) = 1$, where $x$ is a fresh atom. By a similar argument, if $f \in \mathcal{F}_2$, then $f(P(\phi_1), \dots, P(\phi_n)) = \sum_{i=1}^{n} a_i P(\phi_i) \geqslant q$ can be simulated by a conjunction of basic probabilistic formulas with $\bowtie \in \{=\}$ and $f \in \mathcal{F}_2$. Therefore, at least while $f \in \mathcal{F}_2$, one can fix a $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$ without losing expressivity, if conjunction and negation can be applied to basic probabilistic formulas.

Even when we cannot negate basic probabilistic formulas, in some cases assignments like $P(\phi) > q$ can be replaced by $P(\phi) \geqslant q + \epsilon$ preserving satisfiability, if we properly choose $\epsilon > 0$. As PSAT can be formulated as a linear programming problem involving a $\{0, 1\}$-matrix [29], the right value for $\epsilon$ in order to preserve satisfiability can be obtained by upper bounding this matrix determinant using a Hadamard's result from [16]. This holds for any probabilistic logics (with or without nesting) whose basic probabilistic formulas are in $\mathcal{F}_2$.

### *4.2. The number q*

Now we turn our attention to the number $q$ at the right-hand side of a basic probabilistic formula. When the function $f \in \mathcal{F}_{3,4}$, the independent term of $f$, a rational number, can overcome any constraint on the values $q$ can take. If $f$ is restricted to $\mathcal{F}_2$, the coefficients in the left-hand side of the basic probabilistic formula are arbitrary rational numbers, and an independent term $a_0$ can be inserted with $a_0 P(\top)$. However, if $f \in \mathcal{F}_1$, then the expressivity becomes sensitive to the domain of $q$. We start the investigation by constraining $q$ to a single value in basic probabilistic formulas in $\mathcal{F}_1$, $P(\phi) \bowtie q$.

In the simplest and most common case, $f \in \mathcal{F}_1$ ($f$ is the identity function), and probability assignments are of the form $P(\phi) \bowtie q$. This can be denoted as a unary operator $P_{\bowtie q}$ applied to a proposition $\phi$. For a finite number of possible worlds, a probability 1 is equivalent to necessity, and $P_{\bowtie q}$ can be viewed as a generalization of the modal operators $\Box$ and $\Diamond$, when interpreted as necessity and possibility, since $\Box\phi = P_{=1}\phi$ and $\Diamond\phi = \neg P_{=1}\neg\phi = P_{>0}\phi$. In order to keep these operators, a single value for $q$, either 0, or 1, is needed. From any of the operators $P_{\leqslant 0}, P_{=0}, P_{\geqslant 1}, P_{=1}$, the others are trivially recovered, as $P(\phi) \geqslant 1$ is equivalent to $P(\neg\phi) \leqslant 0$.

If we fix $P_{\bowtie q}$ with $q \notin \{0, 1\}$, the language may have no power to represent the absolute truth or falsity, then becoming subclassical. For instance, suppose we have only the operator $P_{\geqslant 0.5}$. It is not clear how one could posit that a proposition $\phi$ is the case. For a logic with no nesting, all purely propositional formulas would be in the scope of $P_{\geqslant 0.5}$ and nothing can be known for sure. In the case with proper nesting, a propositional formula $\phi$ distinguishes a set of possible worlds $[\![\phi]\!]$ in which it is true in a model $\mathcal{M}$, but it may be the case that $(\mathcal{M}, w) \models \phi \wedge P(\phi) = 0.5$ for all $w \in [\![\phi]\!]$. Again, $P_{\geqslant 0.5}$ is useless to ascribe which proposition is certainly true. The polynomial reduction from SAT to PSAT is such that $\phi_1, \dots, \phi_m$ is (Boolean) satisfiable iff $P_{=1}\phi_1, \dots, P_{=1}\phi_m$ is (probabilistically) satisfiable, indicating that an operator $P_{=1}$ recovers the representation power from classical propositional logic.

Suppose we are allowed to assign probability 1 to a formula, and our basic probabilistic formula has the form $P(\phi) \geqslant q$ (denoted by $P_{\geqslant q}\phi$). If we allow for a value for $q$ different from 1 or 0, then other probability assignments can be made by adding new atoms, using conjunction of basic probabilistic formulas. For example, if we have only the operator $P_{\geqslant 0.8}$, an assignment $P_{\geqslant 0.6}\phi$ could be simulated by $P_{\geqslant 0.8}y_1 \wedge P_{\geqslant 0.8}y_2 \wedge P_{\geqslant 1}(y_1 \wedge y_2 \leftrightarrow \phi)$, in which $y_i$ are new atoms. From Kolmogorov's axioms, we know that $P(y_1) + P(y_2) = P(y_1 \vee y_2) + P(y_1 \wedge y_2)$ and $P(y_1 \vee y_2) \leqslant 1$, so we can conclude that $P(y_1 \wedge y_2) \geqslant 0.6$. As $y_1 \wedge y_2 \leftrightarrow \phi$ is true in every world with positive probability, we must have $P(\phi) = P(y_1 \wedge y_2) \geqslant 0.6$ in any model for $P_{\geqslant 0.8}y_1 \wedge P_{\geqslant 0.8}y_2 \wedge P_{\geqslant 1}(y_1 \wedge y_2 \leftrightarrow \phi)$. We generalize this result for propositional probabilistic logics without nesting, to later extend it to logics with nesting. First, some definitions.

**Definition 4.** Let $S$ be a subset of $[0, 1]$ and let $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$ be a relation. We say that $L_{\bowtie S}$ is the class of languages $L \subset L_{PPL}$ in which the basic probabilistic formulas have the form $P(\phi) \bowtie q$, for some

$q \in S$ and $\phi \in L_{PL}$, and whose syntax allows (at least) for the conjunction of them. Then $\mathcal{L}_{\bowtie S}$ is the class of logics $(L, M_{PPL}, \models)$ with $L \in L_{\bowtie S}$ and $\models$ from Section 3.1.

The language of logics in $\mathcal{L}_{=\{0.5,1\}}$, for instance, contains basic probabilistic formulas like $P(\phi_1) = 1$ and $P(\phi_2) = 0.5$, that can be combined with conjunction (and optionally negation and disjunction). For an $n \in \mathbb{N}_{>0}$ (a positive natural number), we denote by $Fr(n)$ the set of proper fractions $\{\frac{i}{n} | 0 \leqslant i \leqslant n\}$. The language of a logic in the class $\mathcal{L}_{=Fr(7)}$ would contain basic probabilistic formulas like $P(\phi_1) = 2/7$ or $P(\phi_2) = 6/7$, for $\phi_i \in L_{PL}$.

**Theorem 2.** *Let $m/n$ be an irreducible proper fraction. For every logic $\mathcal{L}_A = (L_A, M_{PPL}, \models) \in \mathcal{L}_{\geqslant Fr(n)}$ there is a logic $\mathcal{L}_B = (L_B, M_{PPL}, \models) \in \mathcal{L}_{=\{m/n,1\}}$, such that for every formula $\theta \in L_A$ there is a formula $\theta' \in L_B$ such that $\theta'$ is satisfiable iff $\theta$ is; furthermore, $(W, \pi) \models \theta'$ implies $(W, \pi) \models \theta$. The syntax of $L_B$ differs from that of $L_A$ only in the formation of basic probabilistic formulas.*

We can generalize the theorem above to simulate any logics in $L_{\bowtie Fr(n)}$, for $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$, as we have argued for the indifference of the $\bowtie$ choice, at least when basic probabilistic formulas can be negated.

**Corollary 1.** *Let $m/n$ be an irreducible proper fraction. For every logic $\mathcal{L}_A = (L_A, M_{PPL}, \models) \in \mathcal{L}_{\bowtie Fr(n)}$, for $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$, there is a logic $\mathcal{L}_B = (L_B, M_{PPL}, \models) \in \mathcal{L}_{=\{m/n,1\}}$, such that for every formula $\theta \in L_A$ there is a formula $\theta' \in L_B$ such that $\theta'$ is satisfiable iff $\theta$ is; furthermore, $(W, \pi) \models \theta'$ implies $(W, \pi) \models \theta$.*

These results indicate that a single value for $q$, besides 1, can be used to assign arbitrary probabilities with limited precision to basic probabilistic formulas in $\mathcal{F}_1$. Of course, the precision is closely related to the denominator in $q$, and a logic with a single probabilistic operator like $P_{\geqslant 1/2}$ (that can simulate $P_{\leqslant 1/2}$ and $P_{=1/2}$) loses considerable representation power. In practice, the number of decimals used to assess a probability is usually bounded naturally by the uncertainty inherent to human reasoning, for subjective probabilities, and physical experiments and measurements, for objective ones. A single probabilistic operator like $P_{=501/1000}$ could be used to represent any probability assignment with 3 decimals of precision, for example.

One drawback of using only one value for $q$, except from 1, in basic probabilistic formulas in $\mathcal{F}_1$ is related to computational complexity. By the method used in the proof of Theorem 2, if one wants to assess the probability $P(\phi) \geqslant 15/32$ using only an operator $P_{=1/32}$, around $3 \times 15$ basic probabilistic formulas are needed, together with $2 \times 15$ fresh atoms. For a general $P(\phi) \geqslant i/n$ to be simulated by a single operator $P_{=1/n}$, $O(i)$ assignments and fresh atoms are necessary. This asymptotic upper bound can be improved to $O(\log i)$, if we use $P_{=k}$ to obtain $P_{=2k}$ for the induction in the part (II) of the proof of Theorem 2. Generally, for approximating an arbitrary assignment $P(\phi) \bowtie q$ with precision $1/n$, using a logic in $\mathcal{L}_{=\{1/n,1\}}$, a total of $O(\log n)$ basic probabilistic formulas, and fresh atoms, has to be used. If it is not an intractable explosion for computing, it certainly is for human readability – which suggests the use of arbitrary $P_{\bowtie q}$ as abbreviations.

What would happen if $q$ could be irrational? An irrational value, with infinite precision, may find application for instance in the subjective probability a rational agent assigns to the event of a random point in a square to be inside the inscribed circle. However, this is quite debatable for at least two reasons. On the one hand, the best bounds known to any irrational value (like $\pi$) are rational; on the other hand, the precision of physical experiments and measurements is naturally limited.

The impact of allowing $q$ to be any real number in $[0, 1]$ is not in our scope, due to issues related to (un)decidability and uncountable languages. We just investigate the expressivity of a logic in $\mathcal{L}_{=\{q,1\}}$, for an irrational $q$ denoted in a formula by a special symbol, keeping the language countable.

If the language syntax of a logic $\mathcal{L} \in \mathcal{L}_{=\{q,1\}}$ has only conjunction of basic probabilistic formulas, it is straightforward to see that $\mathcal{L}$ is decidable. Using the linear programming techniques shown in [23], we can substitute a parameter $q'$ for the occurrences of $q$ in a given formula and find the (rational) upper and lower bounds for the values of $q'$ that keep the consistency of the set of probabilistic assignments. The formula is satisfiable iff $q$ lies in between these bounds. This procedure works for the set of feasible points for $q'$ is a single closed interval – what is not necessarily the case when negation and disjunction can be applied to basic probabilistic formulas. To cope with the general case, one can use a disjunctive normal form and check the satisfiability of each conjunct separately using the former procedure. Negated assignments like $\neg P(\phi) = q$ can be replaced by $P(\phi) \geqslant q + \epsilon \vee P(\phi) \leqslant q - \epsilon$, for a suitably chosen $\epsilon > 0$, as commented in Section 4.1.

A result from [31] allows us to use the same techniques from the first parts of the proof of Theorem 2 to show how it is possible to approximate any basic probabilistic formula in $\mathcal{F}_1$ with arbitrary precision using a single irrational value. The following fact is a particular case from that result, sufficient for our needs.

**Fact 1.** *(See [31].) If $x$ is irrational, the sequence of fractional parts of $x, 2x, 3x, \ldots$ is dense in $[0,1]$.*

For logics in $\mathcal{L}_{\geqslant Fr(n)}$, we have shown how to construct formulas preserving satisfiability in logics in $\mathcal{L}_{=\{m/n,1\}}$. Now, with a logic in $\mathcal{L}_{=\{q,1\}}$, for irrational $q$, we can only approximate arbitrary basic probabilistic formulas with $q \in [0,1] \cap \mathcal{Q}$. This notion is formalized below.

**Definition 5.** Let $\mathcal{L}_A = (L_A, M_{PPL}, \models)$ (or $\mathcal{L}_A = (L_A, M_{PN}, \models)$) be a propositional probabilistic logic with $\models$ from Section 3.1 (or Section 3.3) such that all basic probabilistic formulas in $L_A$ are in $\mathcal{F}_1$. We say a logic $\mathcal{L}_B$ simulates $\mathcal{L}_A$ with arbitrary precision if for every $\epsilon > 0$ and every formula $\theta$ in $L_A$, there is a formula $\theta_\epsilon$ in $L_A$ and a formula $\theta'_\epsilon$ in $L_B$ such that:

- $\theta_\epsilon$ is formed from $\theta$ replacing each $P(\phi) \bowtie q$ by $P(\phi) \bowtie q'$, such that $|q - q'| < \epsilon$;
- $\theta_\epsilon$ is satisfiable iff $\theta'_\epsilon$ is;
- every model of $\theta'_\epsilon$ is a model of $\theta_\epsilon$.

We can now prove the next result:

**Theorem 3.** *Let $q \in [0,1]$ be an irrational number. For every logic $\mathcal{L}_A = (L_A, M_{PPL}, \models) \in \mathcal{L}_{\geqslant[0,1]\cap\mathbb{Q}}$ there is a logic $\mathcal{L}_B = (L_B, M_{PPL}, \models) \in L_{=\{q,1\}}$, such that $\mathcal{L}_B$ simulates $\mathcal{L}_A$ with arbitrary precision. The syntax of $L_B$ only differs from $L_A$ in the formation of basic probabilistic formulas.*

Analogously to Theorem 2, Lemma 3 has the following corollary:

**Corollary 2.** *Let $q \in [0,1]$ be an irrational number. For every logic $\mathcal{L}_A = (L_A, M_{PPL}, \models) \in \mathcal{L}_{\bowtie[0,1]}$, for $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$, there is a logic $\mathcal{L}_B = (L_B, M_{PPL}, \models) \in L_{=\{q,1\}}$, such that $\mathcal{L}_B$ simulates $\mathcal{L}_A$ with arbitrary precision. The syntax of $L_B$ only differs from $L_A$ in the formation of basic probabilistic formulas.*

It seems that a unique value (like $1/\pi$) is enough to assign any probability to a formula with the desired precision. Nevertheless, the complexity conveys the burden. To illustrate that, suppose we are left with a single operator $P_{=q}$, in which $q \in (0.49999, 0.5)$ is an irrational number. To approximate an operator like $P_{=0.1}$ with a coarse precision of $\epsilon = 0.05$, the proof of Theorem 3 requires the use of roughly $10^4$ basic probabilistic formulas and fresh atoms. We are not in a position to bound this quantity in function of $q$ and $\epsilon$, but this case exemplifies how the size of formulas can grow.

The results above were proved for logics with no nesting, where there is no higher-order probability and a model has just a single probability distribution over possible worlds. Both theorems, and their corollaries, can be extended to propositional probabilistic logics with proper nesting, changing only parts (III) and (IV) of the proof of Theorem 2.

**Definition 6.** Let $S$ be a subset of $[0,1]$ and let $\bowtie \in \{<,>,\leqslant,\geqslant,=\}$ be a relation. We say that $L'_{\bowtie S}$ is the class of languages $L \subset L_{NPL}$ in which the basic probabilistic formulas have the form $P(\phi) \bowtie q$, for some $q \in S$ and $\phi \in L_{PL}$. Then $\mathcal{L}'_{\bowtie S}$ is the class of logics $(L, M_{PN}, \models)$ with $L \in L'_{\bowtie S}$ and $\models$ from Section 3.3.

We can extend Theorem 2 to propositional probabilistic logics with proper nesting.

**Theorem 4.** *Let $m/n$ be an irreducible proper fraction. For every logic $\mathcal{L}'_A = (L_A, M_{PN}, \models) \in \mathcal{L}'_{\bowtie Fr(n)}$, for $\bowtie \in \{<,>,\leqslant,\geqslant,=\}$ there is a logic $\mathcal{L}'_B = (L_B, M_{PN}, \models) \in \mathcal{L}'_{=\{m/n,1\}}$, such that for every formula $\theta \in L_A$ there is a formula $\theta' \in L_B$ such that $\theta'$ is satisfiable iff $\theta$ is; furthermore, $(\mathcal{M}, w_i) \models \theta'$ implies $(\mathcal{M}, w_i) \models \theta$.*

Combining Theorem 3 with Theorem 4, one could also prove the following result.

**Theorem 5.** *Let $q \in [0,1]$ be an irrational number. For every logic $\mathcal{L}'_A = (L_A, M_{PN}, \models) \in \mathcal{L}'_{\geqslant[0,1]\cap\mathbb{Q}}$ there is a logic $\mathcal{L}'_B = (L_B, M_{PN}, \models) \in \mathcal{L}'_{=\{q,1\}}$, such that $L_B$ simulates $L_A$ with arbitrary precision.*

After all this discussion, we can conclude that restrictions on the value of $q$ for basic probabilistic formulas in $\mathcal{F}_1$ can constrain the expressivity of the logical system by limiting the precision of the probability assignment. This limitation becomes severe when $q$ is a fraction with a small denominator. An operator $P_{\geqslant 0.5}$ by itself cannot simulate anything different from $P(\phi) \bowtie 0.5$, at least using the methods we presented. The fractional $\xi$ systems from [6] use an operator "probable" to state $P(\phi) > \xi$ for a fixed rational $0.5 \leqslant \xi < 1$. It is important to note that this kind of limitation in the expressivity is not accompanied by a gain in the efficiency of the decision procedures inherent to the logic. If a probability 1, or absolute truth, can be expressed, then the classical propositional logic is there embedded, whose satisfiability problem is known to be NP-complete, same class of PSAT.

### 4.3. The function $f$

Constraints on the function $f$ can have the most impact over the construction of basic probabilistic formulas. As we have seen, only in $\mathcal{F}_1$ the choice of $q$ is important. Although assignments in $\mathcal{F}_1$ are by far the most common and intuitive, $\mathcal{F}_2$-like constructions appear in the literature as well. The non-linearity of $\mathcal{F}_3$ makes things difficult to compute, but we shall see applications in conditional probabilities. The class $\mathcal{F}_4$ is just the whole class of computable functions, and almost all probabilistic logics ever described fall into the former classes.

Typically, probabilistic logics allow only for basic probabilistic formulas in $\mathcal{F}_1$, with some undesirable limitations. For instance, it may be the case that there is no numerical estimate for a probability, but only the relation between probabilities is known, like $P(\phi_1) \geqslant P(\phi_2)$. This is the scenario of *qualitative* probabilities, in which one can just state that a proposition is more probable than another – instantiated in [15]. The corresponding basic probabilistic formula, $P(\phi_1) - P(\phi_2) \geqslant 0$, is an example of the applicability of a function $f \in \mathcal{F}_2$.

Recall that a basic probabilistic formula in $\mathcal{F}_2$ has the form $\sum_{i=1}^{n} a_i P(\phi_i)$, with $a_i \in \mathbb{Q}$. Thus, it is possible to assert that "event $\phi_1$ is at least twice as probable as $\phi_2$", formally $P(\phi_1) - 2P(\phi_2) \geqslant 0$. As Hansen and Jaumard [23] have shown, the linear programming approach to PSAT, and to the corresponding entailment problem, can be straightforwardly applied to decide the consistency of a set of basic probabilistic formulas in $\mathcal{F}_2$, when the subformulas $\phi$ are all purely propositional (without nesting). Moreover, the propositional probabilistic logic with no nesting of Fagin et al. [12] permits linear combination of probabilities in its basic probabilistic formulas, possibly combined with conjunction, disjunction and negation; and it is shown that the satisfiability problem is still NP-complete, as SAT and PSAT, although the linear programming formulation is not applicable in this case.

For probabilistic logics with proper nesting, it is not known how the computational complexity is affected when $f$ moves from $\mathcal{F}_1$ to $\mathcal{F}_2$. Fagin and Halpern [11] presented a probabilistic logic with proper nesting and linear combination of probabilities and proved its satisfiability problem to be PSPACE-complete. Ognjanovic et al. [36] use this fact to show that the satisfiability problem of $LPP_1$, a probabilistic logic with proper nesting and basic probabilistic formulas in $\mathcal{F}_1$, is in PSPACE, but only an NP lower bound is shown. Using reasonable assumptions to constrain the probability distributions for each possible world, Fagin and Halpern [11] showed their logic stays in NP.

The similar complexity of basic probabilistic formulas for $\mathcal{F}_2$ and $\mathcal{F}_1$ suggests that logical systems with formulas in both classes could be classified together. However, there remain two reasons for treating $\mathcal{F}_1$ and $\mathcal{F}_2$ separately: on the one hand, logics with basic probabilistic formulas in $\mathcal{F}_2$ are strictly more expressive than those in $\mathcal{F}_1$; on the other hand, all limitations imposed on the value of $q$ just affect the representation power of the latter.

As we move from $\mathcal{F}_2$ to $\mathcal{F}_3$, we can use any polynomial over probabilities to construct a basic probabilistic formula. Non-linear terms in probability assessments allow expressing independence and linear combination of conditional probabilities. A way to express the independence of $\phi_1$ and $\phi_2$ is to assert $P(\phi_1 \wedge \phi_2) - P(\phi_1)P(\phi_2) = 0$, which is a basic probabilistic formula in $\mathcal{F}_3$. Furthermore, since $P(\phi_1|\phi_2) = P(\phi_1 \wedge \phi_2)/P(\phi_2)$, given $P(\phi_2) > 0$, a linear combination of conditional probabilities yields a polynomial when one clears the denominators. When all conditioning events in a linear combination are the same, the expression $\sum_{i=1}^{m} a_i P(\phi_i|\theta_i) \geqslant q$ remains in $\mathcal{F}_2$ when denominators are cleared; in general, however, for $m$ different conditioning events $\theta_i$, such a basic probabilistic formula has an $f$ that is a polynomial of degree $m$.

For a satisfiability problem to be in NP, there ought to be a model whose size is polynomial in relation to the formula size. Basic probabilistic formulas like $P(\phi)^2 = 0.5$ may require models with probabilities that are large in "length", as noted by Fagin et al. [12]. The authors have shown a procedure in PSPACE to solve the satisfiability problem of their logic (without nesting) with basic probabilistic formulas in $\mathcal{F}_3$. The logic $L_n^{QU,\times}$ in [22] is a logic in $\mathcal{F}_3$ with proper nesting whose satisfiability problem is claimed to be still PSPACE-complete. Such differences in expressivity and computational complexity makes the boundary between $\mathcal{F}_2$ and $\mathcal{F}_3$ a good guide to classify probabilistic logics.

When $f \notin \mathcal{F}_3$, it is difficult to find an application that pays the associated complexity cost. A possible use of basic probabilistic formulas with $f \notin \mathcal{F}_3$ would be in information theory. As the entropy of an event $\phi$ is computed as $-P(\phi)\log P(\phi) - P(\neg\phi)\log P(\neg\phi)$, the expression $-P(\phi_1)\log P(\phi_1) - P(\neg\phi_1)\log P(\neg\phi_1) \geqslant -m(P(\phi_2)\log P(\phi_2) + P(\neg\phi_2)\log P(\neg\phi_2))$ means that $\phi_1$ conveys at least $m$ times more information than $\phi_2$. Computing functions like the logarithm requires a computational cost that depends on the precision wanted, since we are generally approximating irrational numbers. This incapacity of giving upper bounds for the time to compute functions $f \notin \mathcal{F}_3$ is a reason to classify all of them in $\mathcal{F}_4$. It is not clear that this kind of basic probabilistic formula is useful in real applications, and this is another reason we do not distinguish among functions $f \notin \mathcal{F}_3$.

Ognjanovic and Raskovic [34] proposed logics without nesting and with a probabilistic operator $Q_F$, in which $F$ is a set from a family $O$ of recursive subsets of $[0,1] \cap \mathbb{Q}$. The authors show that these operators cannot simulate, nor be simulated by, finite combination of operators $P_{\geqslant q}$, for rational $q$. Ikodinovic et al. [26] introduced a hierarchy for such probability logics based on the family $O$, while we classify them all together. As $F$ is recursive, there is a function $I_F$ that returns 1 iff a given number is in $F$. This is represented by the basic probabilistic formula $I_F(P(\phi)) = 1$ in $\mathcal{F}_4$.

Out of the four main classes of propositional probabilistic logics $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$, the first three have a definite computational complexity class for the satisfiability problem of propositional probabilistic logics with no nesting: respectively NP-complete, NP-complete and PSPACE; the fourth class has no bound. For propositional probabilistic logics with proper nesting, the three first classes are PSPACE, PSPACE-complete and PSPACE-complete. The power to represent knowledge also increases. Note that, in $\mathcal{F}_1$ an entire spec-

**Table 2**
Proposed logics in the literature, without nesting, and with a semantics where satisfiability is based on a single probability measure over valuations.

| | Conjunction only | Conjunction, negation and disjunction |
|---|---|---|
| $\mathcal{F}_1$ | Georgakopoulos et al. [17], Nilsson [33] | De Bona et al. [9], Ognjanovic et al. [36] |
| $\mathcal{F}_2$ | Hansen and Jaumard [23] | Fagin et al. [12] |
| $\mathcal{F}_3$ | Cozman et al. [7] | Fagin et al. [12] |
| $\mathcal{F}_4$ | | Ognjanovic and Raskovic [34] |

**Table 3**
Proposed logics in the literature, with nesting, and with a semantics where each possible world is associated with a probability measure.

| | |
|---|---|
| $\mathcal{F}_1$ with fixed $q$ | Burgess [6] |
| $\mathcal{F}_1$ | Ognjanovic et al. [36] |
| $\mathcal{F}_2$ | Fagin and Halpern [11], Gärdenfors [15] |
| $\mathcal{F}_3$ | Halpern [22] |
| $\mathcal{F}_4$ | |

trum of subclasses may be derived from the restriction of $q$ in $P(\phi) = q$. For $q = 1/2$, $q = 1/3$, $q = 1/4$, etc., we have logical systems with different expressivity but the same computational complexity for the related satisfiability problem, in logics with and without nesting.

## 5. Conclusion

We have analyzed features of propositional probabilistic logics, with the goal of better understanding and classifying them. We have focused on two issues; namely, nesting of probabilistic operators, and expressivity of probabilistic assessments.

It seems that there are two classes of propositional probabilistic logics that must be differentiated; one does not allow nesting, and adopts a semantics where satisfiability is decided with a single probability measure over valuations; the other class allows nesting and adopts a semantics that associates a probability measure with each possible world. Within the first group, one can find various proposals in the literature, with various forms of probabilistic formulas, each variously adopting only conjunctions of probabilistic formulas, or full Boolean combinations of probabilistic formulas.

Several classes can be considered when we examine the expressivity of probabilistic assessments. The way probability is added to the classical propositional logic is central to determine the representation power of the resulting logic, and also the computational complexity of inference.

Tables 2 and 3 indicate relevant references in the literature.

## Appendix A. Proofs of theorems and corollaries

**Theorem 1.** *For every $\phi \in L_{NPL}$, there exists $\theta \in L_{PPL}$ (computed in polynomial time) such that $\phi$ is satisfiable if, and only if, $\theta$ is. Furthermore, $(\mathcal{M}, w) \models \theta$ implies $(\mathcal{M}, w') \models \phi$ for some $w'$.*

**Proof.** The proof of this theorem can be split into Lemmas 1 and 2. Putting them together, one has a procedure to transform a formula into the normal form, preserving satisfiability and the relation between the models, that is the desired proof for the theorem. Note that in such lemmas, $\mathcal{M} = (W, \pi)$, and $(\mathcal{M}, w) \models \phi$ follows the semantics from Section 3.1. □

**Lemma 1.** *For every $\phi \in L_{NPL}$, there exists a $\theta \in L_{NPL}$ in which all purely propositional subformulas are subformulas of a basic probabilistic formulas, such that $\phi$ is satisfiable if, and only if, $\theta$ is. Furthermore, $\theta$ is computed in linear time and $(\mathcal{M}, w_i) \models \theta$ implies that there is a $w_j$ within the same structure $\mathcal{M}$ such that $(\mathcal{M}, w_j) \models \phi$.*

**Proof.** We assume that $\phi$ has $n$ atomic propositions $x_1, \ldots, x_n$. Let $I$ be the set of indexes of all atomic propositions $x_i$ occurring in $\phi$ outside the scope of a probability assignment. To build $\theta$, for all $i \in I$, substitute $P(y_i) \geqslant 1$, where $y_i$ is a fresh atomic proposition, for all occurrences of $x_i$ out of the scope of a basic probabilistic formula; this is done in linear time in the size of $\phi$. We now to prove that $\theta$ is satisfiable iff $\phi$ is.

($\leftarrow$) Suppose $\phi$ is satisfied by a pair $(\mathcal{M}, w_{j*})$, with $\mathcal{M} = (W, \pi)$. Create a structure $\mathcal{M}' = (W', \pi')$, where for each $w_j \in W$ there exists a $w'_j \in W'$ such that $w_j(x_i) = 1$ iff $w'_j(x_i) = 1$ for all $1 \leqslant i \leqslant n$. Make $w'_j(y_i) = 1$ iff $w'_{j*}(x_i) = 1$ and $\pi(w_j) = \pi'(w'_j)$, for all $i \in I$ and $1 \leqslant j \leqslant |W|$. Clearly, $(\mathcal{M}', w'_{j*}) \models P(y_i) \geqslant 1$ iff $(\mathcal{M}, w_{j*}) \models x_i$, for all $i \in I$, so $(\mathcal{M}', w'_{j*}) \models \theta$.

($\rightarrow$) Suppose now that $(\mathcal{M}, w_{j*}) \models \theta$, with $\mathcal{M} = (W, \pi)$. There is a world $w_{k*} \in W$ such that $(\mathcal{M}, w_{k*}) \models x_i$ iff $(\mathcal{M}, w_{j*}) \models P(y_i) \geqslant 1$. By the construction of $\theta$ from $\phi$, $(\mathcal{M}, w_{k*}) \models \phi$. □

**Lemma 2.** *For every $\phi \in L_{NPL}$, there exists a $\theta \in L_{PPL}$ in which all basic probabilistic formulas are not subformulas of another basic probabilistic formulas, such that $\phi$ is satisfiable if, and only if, $\theta$ is. Furthermore, $\theta$ is computed in polynomial time and $(\mathcal{M}, w) \models \theta$ implies $(\mathcal{M}, w) \models \phi$.*

**Proof.** To prove by induction, we show how to decrease the number of nested probabilities, keeping the satisfiability and the connection between the models. Given a formula $\phi$ with nested probabilities, construct $\phi'$ by substituting a new atomic proposition $y$ for a basic probabilistic formula $f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q$ that is subformula of another basic probabilistic formula. Define $\phi'' = \phi' \wedge ((P(y) \geqslant 1) \vee (P(y) \leqslant 0)) \wedge (\neg(P(y) \geqslant 1) \vee (f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q)) \wedge ((P(y) \geqslant 1) \vee \neg(f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q))$. Clearly, this can be done in polynomial time. Now we need to prove that $\phi''$ is satisfiable iff $\phi$ is.

($\leftarrow$) Suppose $(\mathcal{M}, w_{j*}) \models \phi$, with $\mathcal{M} = (W, \pi)$. We can change $\mathcal{M}$ to satisfy $\phi''$. For each $w_j \in W$, make $w_j \models y$ iff $(\mathcal{M}, w_{j*}) \models f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q$ to form $W'$. Create a structure $\mathcal{M}' = (W', \pi)$. If $(\mathcal{M}, w_{j*}) \models f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q$, then $(\mathcal{M}', w_j) \models y$ for all $w_j \in W'$ and $(\mathcal{M}', w_{j*}) \models P(y) \geqslant 1$. Else, $(\mathcal{M}', w_j) \models \neg y$ for all $w_j \in W'$, and $(\mathcal{M}', w_{j*}) \models P(y) \leqslant 0$. Anyway, $(\mathcal{M}', w_{j*}) \models ((P(y) \geqslant 1) \vee (P(y) \leqslant 0)) \wedge (\neg(P(y) \geqslant 1) \vee (f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q)) \wedge ((P(y) \geqslant 1) \vee \neg(f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q))$. Hence, $(\mathcal{M}', w_{j*}) \models \phi''$.

($\rightarrow$) Note that the last two clauses in $\phi''$ state that $P(y) \geqslant 1 \leftrightarrow f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q$. Therefore, if $(\mathcal{M}, w_{i*}) \models \phi''$, $(\mathcal{M}, w_{i*}) \models P(y) \geqslant 1$ iff $(\mathcal{M}, w_{i*}) \models f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q$. But $(\mathcal{M}, w_{i*}) \models P(y) \geqslant 1$ iff $(\mathcal{M}, w_i) \models P(y) \geqslant 1$ for every $w_i \in W$; and $(\mathcal{M}, w_{i*}) \models f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q$ iff $(\mathcal{M}, w_i) \models f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q$ for every $w_i \in W$. Hence, for every $w_i \in W$, $(\mathcal{M}, w_i) \models P(y) \geqslant 1$ iff $(\mathcal{M}, w_i) \models f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q$. Then, due to the clause $((P(y) \geqslant 1) \vee P(y) \leqslant 0)$, $(\mathcal{M}, w_i) \models y$ iff $(\mathcal{M}, w_i) \models f(P(\psi_1), \ldots, P(\psi_n)) \bowtie q$ for every $w_i$ such that $\pi(w_i) > 0$. Finally, as $(\mathcal{M}, w_{i*}) \models \phi''$, $(\mathcal{M}, w_{i*}) \models \phi$.

By iterating the process of building $\phi''$, a formula $\theta$ without nested probabilities is reached. As the number of nested probabilities has a linear the upper bound in the size of $\phi$, and each iteration takes no more than linear polynomial in the size of $\phi$, the whole process of building $\theta$ is polynomial in time. □

**Theorem 2.** *Let $m/n$ be an irreducible proper fraction. For every logic $\mathcal{L}_A = (L_A, M_{PPL}, \models) \in \mathcal{L}_{\geqslant Fr(n)}$ there is a logic $\mathcal{L}_B = (L_B, M_{PPL}, \models) \in \mathcal{L}_{=\{m/n,1\}}$, such that for every formula $\theta \in L_A$ there is a formula $\theta' \in L_B$ such that $\theta'$ is satisfiable iff $\theta$ is; furthermore, $(W, \pi) \models \theta'$ implies $(W, \pi) \models \theta$. The syntax of $L_B$ differs from that of $L_A$ only in the formation of basic probabilistic formulas.*

**Proof.** We start the proof by choosing a logic $\mathcal{L}_B = (L_B, M_{PPL}, \models) \in L_{=\{m/n,1\}}$ such that differs $L_B$ from $L_A$ only in the syntax of the basic probabilistic formula (preserving the presence/absence of negation and disjunction). We will construct $\theta'$ from $\theta$ by eliminating all basic probabilistic formulas $P(\phi) \geqslant i/n$ with $i \notin \{m, 1\}$, while preserving satisfiability and a relation between the models. In the proof, we say a formula

$\theta'$ in $L_B$ simulates a formula $\theta$ in $L_A$ if two conditions hold: $\theta'$ is satisfiable iff $\theta$ is, and every model of $\theta'$ is a model of $\theta$.

We split the proof into four parts: (I) we simulate an assignment $P(\phi) = 1/n$ in $L_B$; (II) we simulate any assignment $P(\phi) = i/n$ in $L_B$; (III) we show how to substitute any $P(\phi) \geqslant i/n$ with $i \notin \{m, 1\}$ in $\theta$ using the former results; (IV) it is proven that $\theta'$ is satisfiable iff $\theta$ is, and every model of $\theta'$ is a model of $\theta$.

(I) Let $\theta$ be the formula $P(\phi) = 1/n$ in $L_A$. As $m/n$ is proper and irreducible, the greatest common divisor of $m$ and $n$ is 1, hence there exist $x \in \mathbb{N}_{>0}$ such that $xm \equiv 1(\text{mod } n)$ ($xm$ leaves rest 1 when divided by $n$). We start $\theta'$ in $L_B$ with $x$ atomic propositions $y_1, y_2, \dots, y_x$ (not occurring in $\phi$), assigning probabilities to them, $\bigwedge_{j=1}^{x} P(y_j) = m/n$. At this point, we have $P_\pi(y_1) + P_\pi(y_2) + \cdots + P_\pi(y_x) = xm/n$, for every $(W, \pi)$ satisfying $\theta'$. Let $Y$ be the set $\{y_1, y_2, \dots, y_x\}$ and let $C^k Y$ denote the formula $\bigvee\{y_{i_1} \wedge y_{i_2} \wedge \cdots \wedge y_{ik} | 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant x\}$, that is, $C^k Y$ is the disjunction of all possible conjuncts with $k$ distinct formulas from the set $Y = \{y_1, y_2, \dots, y_x\}$. A variation of the inclusion/exclusion principle states that $P(y_1) + P(y_2) + \cdots + P(y_x) = P(C^1 Y) + P(C^2 Y) + \cdots + P(C^x Y)$. So, every model of $\theta'$ has $P_\pi(C^1 Y) + P_\pi(C^2 Y) + \cdots + P_\pi(C^x Y) = mx/n$. Let $j^*$ be the smallest integer greater than $mx/n$, and assign the probability $P(C^j) = 0$ (abbreviation of $P(\neg C^j) = 1$), for $j^* + 1 \leqslant j \leqslant x$. Now $\theta' = \bigwedge_{j=1}^{x} P(y_j) = m/n \bigwedge_{j=j^*+1}^{x} P(\neg C^j Y) = 1$. It is the case that $P_\pi(C^1 Y) + P_\pi(C^2 Y) + \cdots + P_\pi(C^{j^*} Y) = mx/n$ in all models of $\theta'$. Note that $mx = (j^* - 1)n + 1$. So $P_\pi(C^1 Y) + P_\pi(C^2 Y) + \cdots + P_\pi(C^{j^*} Y) = (j^* - 1) + 1/n$ yields $P\pi(C^{j^*} Y) \geqslant 1/n$. If we impose $P_\pi(C^j Y) = 1$ for $1 \leqslant j \leqslant j^* - 1$, with $\theta' = \bigwedge_{j=1}^{x} P(y_j) = m/n \bigwedge_{j=j^*+1}^{x} P(\neg C^j Y) = 1 \bigwedge_{j=1}^{j^*-1} P(C^j Y) = 1$, then $P\pi(C^{j^*} Y) = 1/n$. To achieve $P\pi(\phi) \geqslant 1/n$ in every model of $\theta'$ we add to it $P(C^{j^*} Y \to C^{j^*} Y) = 1$, obtaining $\theta' = P(C^{j^*} Y \to \phi) = 1 \bigwedge_{j=1}^{x} P(y_j) = m/n \bigwedge_{j=j^*+1}^{x} P(\neg C^j Y) = 1 \bigwedge_{j=1}^{j^*-1} P(C^j Y) = 1$. As desired, for every model $(W, \pi)$ of $\theta'$ is also a model of $\theta = P(\phi) \geqslant 1/n$. In parts (II) and (III), we denote such $\theta'$ in $L_B$ as $P_{1/n}\phi$, and each occurrence of it will have fresh auxiliary atoms $y_i$. We now show that $\bigwedge_{j=1}^{x} P(y_j) = m/n \bigwedge_{j=j^*+1}^{x} P(\neg C^j Y) = 1 \bigwedge_{j=1}^{j^*-1} P(C^j Y) = 1$ has a model $(W_y, \pi_y)$ such that it can be extended to satisfy any $P_{1/n} z_1$, in which $z_1$ is a fresh atom. Consider a set of $n$ possible worlds $W_y = \{w_1, \dots, w_n\}$ and a probability distribution over them $\pi_y(w_i) = 1/n$, for $1 \leqslant i \leqslant n$. Let $\langle x \rangle$ denote the fractional part of a number $x$. Define $w_i(y_j) = 1$ iff $\langle (j-1)m/n \rangle < i/n \leqslant \langle jm/n \rangle$ or it is the case that $\langle jm/n \rangle < i/n \leqslant \langle jm/n \rangle$. Note that, $P_{\pi_y}(y_j) = \sum\{\pi_y(w_i) | w_i(y_j)\} = m/n$ for every $y_j$. As $mx = (j^* - 1)n + 1$, each world $w_i$, for $2 \leqslant i \leqslant n$, satisfies exactly $j^* - 1$ atoms $y_j$, and $w_1$ satisfies $j^*$ atoms, so that no world satisfy $j^* + 1$ atoms. Hence, $(W_y, \pi_y) \models \bigwedge_{j=j^*+1}^{x} P(\neg C^j Y) = 1 \bigwedge_{j=1}^{j^*-1} P(C^j Y) = 1$ and $(W_y, \pi_y) \models \bigwedge_{j=1}^{x} P(y_j) = m/n \bigwedge_{j=j^*+1}^{x} P(\neg C^j Y) = 1 \bigwedge_{j=1}^{j^*-1} P(C^j Y) = 1$. For a fresh atom $z_1$, we extend $W_y$ in such a manner that $w_i(z_1) = 1$ iff $i = 1$, assuring $P_{\pi_y}(z_1) = 1/n$ and satisfying $P_{i/n} z_1$.

(II) Let $V$ denote the set such that $i \in V$ iff we can simulate $P(\phi) = i/n$ in $L_B$. Initially we only know that $1, m, n \in V$, and in the this part of the proof we show, by induction, that $i \in V$, for all $i \in \{1, \dots, n\}$. The basis of the induction is done in part (I), and $1 \in V$. Suppose now that $k \in V$ for a $1 \leqslant k \leqslant n - 1$, so there is a formula in $L_B$ that simulates $P(\phi) = k/n$, which we denote by $P_{k/n}\phi$. We want to simulate $P(\phi) = (k+1)/n$. Start $\theta'$ assigning probabilities to fresh atomic propositions $z_k$ and $z'_k$: $P_{1/n} z'_k \wedge P_{k/n} z_k$. Since $1 + k \leqslant n$, we can consistently add $P(z_k \wedge z'_k) = 0$, obtaining $\theta' = P_{1/n} z'_k \wedge P_{k/n} z_k \wedge P(z_k \wedge z'_k) = 0$. As $P_\pi(z_k) + P_\pi(z'_k) = P_\pi(z_k \vee z'_k) + P_\pi(z_k \wedge z'_k) = (1+k)/n$, for every $\pi$, all models $(W, \pi)$ of $\theta'$ are such that $P_\pi(z_k \vee z'_k) = (1+k)/n$. So, with $P(\phi \leftrightarrow z_k \vee z'_k) = 1$ in $\theta'$, one simulates $P(\phi) = (k+1)/n$. Finally, we have $\theta' = P_{1/n} z'_k \wedge P_{k/n} z_k \wedge P(z_k \wedge z'_k) \leqslant 0 \wedge P(\phi \leftrightarrow z_k \vee z'_k) = 1$, and $k + 1 \in V$. By induction on $k$, $i \in V$ for all $i \in \{1, \dots, n\}$. We denote by $P_{i/n}\phi$ the formula in $L_B$ simulating $P(\phi) = i/n$ in $L_A$. Now, lets show a model for $P_{i/n}\phi$, providing that $\phi$ is satisfiable. Initially, we need a model $(W_z, \pi_z)$ such that $\pi_z(z'_j) = 1/n$, $\pi_z(z_j) = j/n$, for $1 \leqslant j \leqslant i - 1$, and $z_1, z'_1, z'_2, \dots, z'_{i-1}$ are all two by two disjoint atoms. Consider a set of $i + 1$ possible worlds $W_z = \{w_0, w_1, \dots, w_i\}$ such that $w_j(z_k) = 1$ iff $j < k$, $w_j(z'_k) = 1$ iff $k = j$, and $w_i(z_k) = w_i(z'_k) = 0$, for all $0 \leqslant j \leqslant i-1$ and $1 \leqslant k \leqslant i-1$. Let $\pi_z$ be a probability distribution over $W_z$ such that $\pi_z(w_j) = 1/n$, if $j < i$, and $\pi_z(w_i) = (n-i)/n$. Note that $(W_z, \pi_z) \models P(z_k \vee z'k \leftrightarrow z_{k+1}) = 1$ for every $1 \leqslant k \leqslant i - 2$ and $(W_z, \pi_z) \models P(z_k \wedge z'_k) = 0$ for every $1 \leqslant k \leqslant i - 1$. Furthermore, $(W_z, \pi_z) \models P(z_k) = k/n$ and $(W_z, \pi_z) \models P(z'_k) = 1/n$ for every $1 \leqslant k \leqslant i - 1$. Now, to model $P_{1/n} z'_1$, it is required the combination of the models $(W_z, \pi_z)$ and $(W_y, \pi_y)$ to form $(W_{z,y}, \pi_{z,y})$. Start forming $W_{z,y}$ by collapsing the worlds $w_1 \in W_z$ and $w_1 \in W_y$ in a new possible world $w_{1,1}$ such that $w_{1,1}(\phi) = 1$ iff $w_1(\phi) = 1$, for $w_1 \in W_z$ or $w_1 \in W_y$. For each world $w_j \in W_z \setminus \{w_1\}$ and $w_k \in W_y \setminus \{w_1\}$, create a new world $w_{j,k}$ in $W_{z,y}$ such that $w_{j,k}(\phi) = 1$ iff $w_j(\phi) = 1$ or $w_k(\phi) = 1$. Make $\pi_{z,y}(w_{1,1}) = 1/n$ and distribute the remaining probabilities proportionally, $\pi_{z,y}(w_{j,k}) = (1 - \pi_{z,y}(w_{1,1}))\frac{\pi_z(w_j)}{1-\pi_z(w_1)}\frac{\pi_y(w_k)}{1-\pi_y(w_{1,1})}$. It is the case that $\pi_{z,y}(y_i) = \pi_y(y_i)$, $\pi_{z,y}(z_i) = \pi_z(z_i)$ and $\pi_{z,y}(z'_i) = \pi_z(z'_i)$. As $P_{\pi_{z,y}}(z'_1) = \pi_{z,y}(w_{1,1}) = 1/n$, $(W_{z,y}, \pi_{z,y}) \models P_{1/n} z'_1$. To satisfy another formula $P_{1/n} z$ ($z$ is an atom), we add fresh atoms $y$ and repeat the process of combining models, merging $W_{z,y}$ with a new disjoint $W_y$. Observe that now there is a set of worlds $\{w_{j,k} \in W_{z,y} | w_{j,k}(z) = 1\}$ that will be collapsed with $w_1 \in W_y$ to build worlds $w_{j,k,1}$, each with probability $\pi_{z,y}(w_{j,k})$, such that $w_{j,k,1}(x) = 1$ iff $w_{j,k}(x) = 1$ or $w_1(x) = 1$ ($w_1 \in W_y$), for any atom $x$. By iterating this procedure, one reaches a model for $P_{i/n} z_i$, in which $z_i$ is an atom, defining $w(z_i) = 1$ iff $w(z_{i-1} \vee z'_{i-1}) = 1$. We denote by $(W_{z_i,y}, \pi_{z_i,y})$ such a model of $P_{i/n} z_i$.

(III) Let $\theta$ be an arbitrary formula in $L_A$, possibly with disjunction and negation. The basic idea to form $\theta'$ is to substitute $P_{i/n} z_i \wedge P(z_i \to \phi) = 1$ for every $P(\phi) \geqslant i/n$ with $i \notin \{m, 1\}$ in $\theta$, but when there is negation in the second level of the syntax, there are some subtleties. Suppose $\theta = \neg P(\phi) \geqslant i/n$. Since $\theta'$ and $\theta$ are not logically equivalent, due to the fresh atoms, $\theta' = \neg(P_{i/n} z_i \wedge P(z_i \leftarrow \phi) = 1)$ may be satisfied by a model $(W, \pi)$ such that $P_\pi(\phi) \geqslant i/n$, not satisfying $\theta$. Note that this only happens when there is negation or disjunction in the syntax, otherwise a model must satisfy all basic probabilistic formulas. If there is conjunction or negation in the syntax of $L_A$ (and then in $L_B$), do the following: for every basic probabilistic formula $P(\phi) \geqslant i/n$ in $\theta$, replace it by $P(z_i \to \phi) = 1$, and then make $\theta'$ be the conjunction of $\theta$ with $P_{i/n} z_i \wedge (P(z_i \to \phi) = 1 \vee P(\phi \to z_i) = 1)$ for every basic probabilistic formula replaced. Note that, if $L_B$ has negation and conjunction over basic probabilistic formulas in its syntax, disjunction can be simulated, so that $P_{i/n} z_i \wedge (P(z_i \to \phi) = 1 \vee P(\phi \to z_i) = 1)$ is a well-formed formula.

(IV) When $P(z_i \to \phi) = 1$ and $\theta'$ are satisfied in a model $(W, \pi)$, $P_{i/n} z_i$ forces that $P_\pi(\phi) \geqslant i/n$, as desired. But with negation in $L_B$, it is possible that $(W, \pi) \models \theta'$ and $(W, \pi) \not\models P(z_i \to \phi) = 1$. Then the clause $(P(z_i \to \phi) = 1 \vee P(\phi \to z_i) = 1)$ implies $(W, \pi) \models P(\phi \to z_i) = 1$, and we have $P_\pi(\phi) < i/n$, for $P_\pi(z_i \wedge \neg\phi) > 0$; thus $(W, \pi) \models \theta$. Hence, if $\theta'$ is satisfiable, $\theta$ also is. Now, from a model $(W, \pi)$ of $\theta$ we have to construct a model $(W', \pi')$ of $\theta'$. Firstly, we need to change $W$, forming $W'$, assigning suitable truth values to the fresh atoms. For each world in $w_i \in W$, there is a set $w'_i \subset W'$ such that $w_i(x_k) = w'(x_k)$ for every $w' \in w'_i$ and atomic proposition $x_k$ occurring in $\theta$. Let $[\![\phi]\!]$ be the set $\{w \in W | w(\phi) = 1\}$. Suppose that there were $h$ basic probabilistic formulas to be replaced in $\theta$. Each formula $P(\phi) = i/n$ in $\theta$ has a corresponding $P_{i/n} z_i$ in $\theta'$ built from the set of fresh variables $Z_g \cup Y_g$, in which $Y_g$ contains the fresh atoms $y$ used to construct $P_{1/n}$ in part (I) and $Z_g$ contains the fresh atoms $z$ used in parts (II) and (III). Each formula $P_{i/n} z_i$ is satisfiable by a model $(W_{z_i,y}, \pi_{z_i,y})$. Let's replace the basic probabilistic formulas one by one, starting from $\theta_0 = \theta$ and constructing $\theta_{i+1}$ from $\theta_i$, while showing a model of the former based on a model of the later. From $\theta$, we substitute $P(z_i \to \phi) = 1$ for a formula $P(\phi) \geqslant i/n$ (with $i \notin \{m, 1\}$) and make $\theta_1 = \theta \wedge P_{i/n} z_i \wedge (P(z_i \to \phi) = 1 \vee P(\phi \to z_i) = 1)$. If there is only conjunction in $L_A$ and $L_B$, then $\theta_1$ is built from $\theta$ simply substituting $P(z_i \to \phi) = 1 \wedge P_{i/n} z_i$ for $P(\phi) \geqslant i/n$. To construct a model $(W_1, \pi_1)$ to $\theta_1$ from a model $(W, \pi)$ of $\theta_0 = \theta$ there exist two cases to be analyzed: (a) when $P_\pi(\phi) \geqslant i/n$; and (b) when $P_\pi(\phi) < i/n$. We point out that (b) does not make sense if there is only conjunction of basic probabilistic formulas in $L_B$. Consider the model $(W_{z_i,y}, \pi')$ of $P_{i/n} z_i$ and the sets of worlds $[\![z_i]\!], [\![\neg z_i]\!] \subset W_{z_i,y}$. Take $W_1 = W_{z_i,y} \times W$, such that $(w, w')(x) = 1$ iff $w(x) = 1$ or $w'(x) = 1$, for every atom $x$. In case (a), for every $w \in [\![\phi]\!]$ and $w' \in [\![z_i]\!]$, define $\pi_1((w, w')) = \pi(w)\pi'(w')/P_\pi(\phi)$. For every $w \in [\![\phi]\!]$ and $w' \in [\![\neg z_i]\!]$, define $\pi_1((w, w')) = \pi(w)\pi'(w')(P_\pi(\phi) - P_{\pi'}(z_i))/(P_{\pi'}(z_i)P_\pi(\phi))$. For every $w \in [\![\neg\phi]\!]$ and $w' \in [\![z_i]\!]$, define $\pi_1((w, w')) = 0$. For every $w \in [\![\neg\phi]\!]$ and $w' \in [\![\neg z_i]\!]$, define $\pi_1((w, w')) = \pi(w)\pi'(w')/P_{\pi'}(\neg z_i)$. Note that $(W_1, \pi_1) \models \theta \wedge P(z_i \to \phi) = 1 \wedge P_{i/n}(z_i)$, hence $(W_1, \pi_1) \models \theta_1$. In case (b), for every $w \in [\![\phi]\!]$ and $w' \in [\![z_i]\!]$, define $\pi_1((w, w')) = \pi(w)\pi'(w')/P_{\pi'}(z_i)$. For every $w \in [\![\phi]\!]$ and $w' \in [\![\neg z_i]\!]$, define $\pi_1((w, w')) = 0$. For every $w \in [\![\neg\phi]\!]$ and $w' \in [\![z_i]\!]$, define $\pi_1((w, w')) = \pi(w)\pi'(w')(P_{\pi'}(z_i) - P_\pi(\phi))/(P_{\pi'}(z_i)P_\pi(\neg\phi))$. For every $w \in [\![\neg\phi]\!]$ and $w' \in [\![\neg z_i]\!]$, define $\pi_1((w, w')) = \pi(w)\pi'(w')/P_\pi(\neg\phi)$. Now note that $(W_1, \pi_1) \models$

$\theta \wedge P(\phi \to z_i) = 1 \wedge P_{i/n}(z_i)$ and $(W_1, \pi_1) \not\models P(z_i \to \phi) = 1$, hence $(W_1, \pi_1) \models \theta_1$. If this routine is repeated, transforming $\theta_i$ into $\theta_{i+1}$ and $(W_i, \pi_i)$ into $(W_{i+i}, \pi_{i+1})$, until all basic probabilistic formulas in $\theta$ are replaced, one has a model $(W_h, \pi_h)$ of $\theta_h = \theta'$ that is also a model of $\theta$. Finally, we conclude that $\theta'$ is satisfiable iff $\theta$ is. □

**Corollary 1.** *Let $m/n$ be an irreducible proper fraction. For every logic $\mathcal{L}_A = (L_A, M_{PPL}, \models) \in \mathcal{L}_{\bowtie Fr(n)}$, for $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$, there is a logic $\mathcal{L}_B = (L_B, M_{PPL}, \models) \in \mathcal{L}_{=\{m/n,1\}}$, such that for every formula $\theta \in L_A$ there is a formula $\theta' \in L_B$ such that $\theta'$ is satisfiable iff $\theta$ is; furthermore, $(W, \pi) \models \theta'$ implies $(W, \pi) \models \theta$.*

**Proof.** Using Theorem 2, we simulate any $P(\phi) \geqslant i/n$, thus any $P(\phi) \leqslant i/n$, via $P(\neg\phi) \geqslant 1 - i/n$, and hence $P(\phi) = i/n$. With negation over the basic probabilistic formulas (possibly absent in a logic without nesting), one recovers $<$ and $>$. □

**Theorem 3.** *Let $q \in [0, 1]$ be an irrational number. For every logic $\mathcal{L}_A = (L_A, M_{PPL}, \models) \in \mathcal{L}_{\geqslant[0,1]}$ there is a logic $\mathcal{L}_B = (L_B, M_{PPL}, \models) \in L_{=\{q,1\}}$, such that $\mathcal{L}_B$ simulates $\mathcal{L}_A$ with arbitrary precision. The syntax of $L_B$ only differs from $L_A$ in the formation of basic probabilistic formulas.*

**Proof.** It suffices to show how to simulate a basic probabilistic formula, and the rest of the proof is equivalent to parts (III) and (IV) of Theorem 2. Suppose we want a precision $\epsilon > 0$ such that we can simulate $P(\phi) \geqslant r'$, with $|r' - r| < \epsilon$. By Fact 1, for any $\epsilon > 0$, there is an $n \in \mathbb{N}_{>0}$ such that, if $r'$ is the fractional part of $nq$, then $|r' - r| < \epsilon$. We use $n$ fresh atoms from the set $Y = \{y_1, \ldots, y_n\}$. Recall from the proof of Theorem 2 the definition of $C^kY$ and the equality $P(y_1) + P(y_2) + \cdots + P(y_n) = P(C^1Y) + P(C^2Y) + \cdots + P(C^nY)$. The formula $\bigwedge_{i=1}^n P(y_i) = q$ imply $P_\pi(C^1Y) + P_\pi(C^2Y) + \cdots + P_\pi(C^nY) = nq$ for every model $(W, \pi)$ satisfying it. Let $j^*$ be the least integer greater than $nq$, and let $r'$ denote the fractional part of $nq$, such that $nq = j^* - 1 + r'$. Assigning probability zero to $C^jY$ for all $j^* + 1 \leqslant j \leqslant n$ yields $P_\pi(C^1Y) + P_\pi(C^2Y) + \cdots + P_\pi(C^{j^*}Y) = nq$ in any model $(W, \pi)$. Assigning $P(C^jY) = 1$, for $1 \leqslant j \leqslant j^* - 1$, we have $P(C^{j*}Y) = r'$. Finally, $P(C^{j*}Y \to \phi) = 1$ entails $P_\pi(\phi) \geqslant r'$ in every model, with $|r' - r| < \epsilon$, as desired. To construct $\theta'_\epsilon$, if there is only conjunction in $L_A$ (and in $L_B$), then replace each $P(\phi) \geqslant r'$ in $\theta_\epsilon$ by $\bigwedge_{i=1}^n P(y_i) = q \bigwedge_{j=j^*+1}^n P(C^jY) = 0 \bigwedge_{j=1}^{j^*-1} P(C^jY) = 1 \wedge P(C^{j*}Y \to \phi) = 1$. If disjunction or negation are allowed, substitute $P(C^{j*}Y \to \phi) = 1$ for each $P(\phi) \geqslant r'$ in $\theta_\epsilon$, and make $\theta'_\epsilon = \theta_\epsilon \wedge \bigwedge_{i=1}^n P(y_i) = q \bigwedge_{j=j^*+1}^n P(C^jY) = 0 \bigwedge_{j=1}^{j^*-1} P(C^jY) = 1 \wedge (P(C^{j*}Y \to \phi) = 1 \vee P(\phi \to C^{j*}Y) = 1)$. The rest of the proof is completely analogous to the proof of Theorem 2. □

**Corollary 2.** *Let $q \in [0, 1]$ be an irrational number. For every logic $\mathcal{L}_A = (L_A, M_{PPL}, \models) \in \mathcal{L}_{\bowtie[0,1]}$, for $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$, there is a logic $\mathcal{L}_B = (L_B, M_{PPL}, \models) \in L_{=\{q,1\}}$, such that $\mathcal{L}_B$ simulates $\mathcal{L}_A$ with arbitrary precision. The syntax of $L_B$ only differs from $L_A$ in the formation of basic probabilistic formulas.*

**Proof.** Completely similar to the proof of Corollary 1. □

**Theorem 4.** *Let $m/n$ be an irreducible proper fraction. For every logic $\mathcal{L}'_A = (L_A, M_{PN}, \models) \in \mathcal{L}'_{\bowtie Fr(n)}$, for $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$ there is a logic $\mathcal{L}'_B = (L_B, M_{PN}, \models) \in \mathcal{L}'_{=\{m/n,1\}}$, such that for every formula $\theta \in L_A$ there is a formula $\theta' \in L_B$ such that $\theta'$ is satisfiable iff $\theta$ is; furthermore, $(\mathcal{M}, w_i) \models \theta'$ implies $(\mathcal{M}, w_i) \models \theta$.*

**Proof.** We will just sketch the proof, illustrating the main ideas but not exhaustively covering the details. Firstly, note that $\mathcal{L}'_B \in \mathcal{L}'_{=\{m/n,1\}}$ inherits negation and disjunction from propositional logic, since it is a propositional probabilistic logic with proper nesting (from Section 3.3). Therefore, by simulating $P_{\geqslant i/n}$ for any $i$, any $P_{\bowtie i/n}$ for $\bowtie \in \{<, >, \leqslant, \geqslant, =\}$ is easily recovered using conjunction and negation. To construct $\theta'$, we use parts (I) and (II) from Theorem 2, just substituting $(\mathcal{M}, w_i)$ for $(W, \pi)$ and $\pi_i$ for $\pi$, to obtain the formulas $P_{i/n}z_i$ in $L'_B$. The significant change is in part (III), where we use the formulas $P_{i/n}z_i$ to transform $\theta$ into $\theta'$. Consider initially $\psi = \theta$. The algorithm is now iterative:

1. For each basic probabilistic formula $P(\phi) \geqslant i/n$ in $\psi$ that is *not* a subformula of another basic probabilistic formula in $\psi$, replace $P(\phi) \geqslant i/n$ by $P(z_i \to \phi) = 1$;
2. For each $P(\phi) \geqslant i/n$ replaced, substitute $\psi \wedge P_{i/n}z_i \wedge (P(z_i \to \phi) = 1 \vee P(\phi \to z_i) = 1)$ for $\psi$; (until here, the procedure is the same as that from the proof of Theorem 2)
3. For each $P(\phi) \geqslant i/n$ replaced, make $\psi = \phi$ and go to step 1; if no basic probabilistic formula was transformed, we have $\theta'$.

The key difference from the proof of Theorem 2 is that the formula $P_{i/n}z_i \wedge (P(z_i \to \phi) = 1 \vee P(\phi \to z_i) = 1)$ must be in the same probabilistic scope as the basic probabilistic formula being replaced. Such formula only forces the equivalence between $P(\phi) \geqslant i/n$ and $P(z_i \to \phi) = 1$ when their truth value is computed with the same probability distribution $\pi_i$ of world $w_i$. Consider the basic probabilistic formula $P(\phi) \geqslant i/n$ in $\theta$ which is subformula of the maximum number of basic probabilistic formulas. Now let $\psi$ be the maximal formula containing $P(\phi) \geqslant i/n$ as subformula but with no nested probability on $\phi$. That is, $\psi$ is the either $\theta$ or such that $P(\psi) \geqslant j/n$ is a subformula of $\theta$. The procedure above replaces $P(\phi) \geqslant i/n$ by $P(z_i \to \phi) = 1$ in $\psi$ and substitutes $\psi \wedge P_{i/n}z_i \wedge (P(z_i \to \phi) = 1 \vee P(\phi \to z_i) = 1)$ for $\psi$. It is straightforward to see that a model of the transformed $\psi$ is still a model of $\psi$, by the same reasons of Theorem 2. By iterating this process, one can note that a model of $\theta'$ is a model of $\theta$. To show how to construct a model for $\theta'$ from a model of $\theta$ takes considerably more effort, but the way is similar to that from part (IV) of the proof of Theorem 2, just iterating for each step of the algorithm above. Suppose $(\mathcal{M}, w) \models \theta$, with $\mathcal{M} = (W, \Pi, v)$. It is possible to construct a model $\mathcal{M}' = (W', \Pi', v')$ by replicating the whole set $W$ for each $w \in W$, such that $|W'| = |W| + |W|^2$. Start with $W' = W$. For each ordered pair $(w_i, w_j) \in W$, construct a $w_{i,j} \in W'$ such that $v'(w_{i,j}) = v(w_j)$, $\pi'_i(w_{i,j}) = \pi_i(w_j)$, $\pi'_{i,j}(w_k) = \pi_j(w_k)$ and give zero probability mass to other cases. We have a two-level model for $\theta$ in which each world $w_{i,j}$ in the second level is such that only one world $(w_i)$ possibly assigns a positive probability to it. We repeat this process of adding a level in $W'$ according to the highest number of basic probabilistic formulas with a common subformula. In the end, if $\theta$ has nested probabilistic operators corresponding to $l$-order probabilities, we will have a model $\mathcal{M}' = (W', \Pi', v')$ with $|W'| = |W|^1 + |W|^2 + \dots + |W|^{l+1}$. This “tree-like” model has the property that, except from the first level, each world has only one “parent” possibly assigning positive probability to it. Now, starting from the basic probabilistic formulas that are subformula of the highest number of another basic probabilistic formulas (level $l + 1$, with $|W|^{l+1}$ worlds), one can repeat the process of part (IV) of the proof of Theorem 2 for each world and each level of $\mathcal{M}'$ to build a model for $\theta'$, although it is not shown here. □

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