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Evenly convex credal sets ☆

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ABSTRACT

An evenly convex credal set is a set of probability measures that is evenly convex; that is, a set that is an arbitrary intersection of open halfspaces. An evenly convex credal set can for instance encode preference judgments through strict and non-strict inequalities such as $\mathbb{P}(A) > 1/2$ and $\mathbb{P}(A) \leq 2/3$. This paper presents an axiomatization of evenly convex sets from preferences, where we introduce a new (and very weak) Archimedean condition. We examine the duality between preference orderings and credal sets; we also consider assessments of almost preference and natural extensions. We then discuss regular conditioning, a concept that is closely related to evenly convex sets.

1. Introduction

The goal of this paper is to show, first, that relatively simple axioms on preference orderings can be used to characterize *evenly convex* sets of probability measures; that is, sets that are arbitrary intersections of open halfspaces. Evenly convex sets allow assessments such as $\mathbb{P}(A) \geq 1/2$ and $1/4 < \mathbb{P}(B) \leq 3/4$: strict and non-strict inequalities can be expressed on probability values. Central to our results is a new (very weak) Archimedean condition. We then examine the definition of conditioning under such axioms, as well as concepts of almost preference and natural extension.

A preference ordering is a binary relation $\succ$ on *gambles*; a gamble is a function X that yields a real number $X(\omega)$ for each *state* ω, and $X \succ Y$ is understood as "X is preferred to Y". If a preference ordering satisfies a few conditions, to be discussed later, then the ordering can be represented by a single probability measure (that is, by an additive set-function that assigns a nonnegative number to each event, such that $\mathbb{P}(\Omega) = 1$). It is not always reasonable to assume that a precise probability value can be attached to every possible event: one might be willing to attach probability values only to a few events, or perhaps one might associate probability intervals with events, or even impose weaker constraints on probability values.

If a preference ordering is only a partial order, then, subject to a few additional conditions, it can be represented by a set of probability measures [16,28,33,35,36]. Typically such axiomatizations of sets of probability measures focus on *maximal closed convex* sets of probability measures. It seems that the only existing axiomatization that allows for open sets of probability measures sets has been given by Seidenfeld, Schervish, and Kadane [29], using a general setting where utilities are also derived, and a proof technique based on transfinite induction. Their representation result may require sets of state-dependent utilities to represent preferences; for this reason it may be a little difficult to grasp the geometric content of a preference profile. One wonders whether it is possible to capture assessments such as $\mathbb{P}(A) > 1/2$ with some intuitive construction.

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Section 3 presents our axiomatization for evenly convex sets of probability measures. We use the new Archimedean condition, and emphasize the use of separating hyperplanes as much as possible, hopefully producing results that can be appreciated with moderate effort. We study the connection of our Archimedean condition with other conditions in the literature, and we examine the duality between preference orderings and sets of probability measures. Section 4 looks at assessments of "almost preference", and studies the natural extension of sets of assessments. Finally, Section 5 discusses *regular conditioning*, a popular form of conditioning that is intimately related to evenly convex sets.

2. Preference orderings, sets of desirable gambles, and credal sets

In this section we present some basic concepts and results used throughout. Because some results here are in essence well-known, only very short proofs are given for them.

Consider a finite set Ω containing n states $\{\omega_1, \ldots, \omega_n\}$. An *event* is a subset of Ω; a *gamble* is a function $X : \Omega \to \Re$. A gamble can be viewed as a n-dimensional vector. A probability measure over Ω is entirely specified by a n-dimensional vector with non-negative elements that add up to one. Given such a vector p that induces a probability measure $\mathbb{P}$, and a gamble X, the expected value of X, denoted by $\mathbb{E}_{\mathbb{P}}[X]$, is simply the inner product $X \cdot p$.

All sets we consider are subsets of $\Re^n$; throughout we assume the Euclidean topology. For a set $\mathcal{A}$, $\mathsf{cl}\mathcal{A}$ is the closure of $\mathcal{A}$ and $\mathsf{relint}\mathcal{A}$ is the relative interior of $\mathcal{A}$. A *cone* $\mathcal{A}$ is a set such that if $X \in \mathcal{A}$ then $\lambda X \in \mathcal{A}$ for $\lambda > 0$ (the origin may not be in $\mathcal{A}$). If $\mathcal{B}$ is a convex set, the *smallest convex cone* containing $\mathcal{B}$ is $\{\lambda X : \lambda > 0, X \in \mathcal{B}\}$ [27, Corollary 2.6.3]. An *exposed ray* of a convex cone is an exposed face that is a half-line emanating from the origin (recall that an exposed face is a face that is equal to the set of points achieving the maximum of some linear function).

Most results in this paper deal with the representation of preferences[1]:

Definition 1. A *preference ordering* $\succ$ is a strict partial order over pairs of gambles.

Absence of preference between X and Y is indicated by $X \not\sim Y$. If $X \succ 0$, X is *desirable*; if $X \not\sim 0$, X is *neutral*.

We always assume two additional properties:

Monotonicity: If $X(\omega) > Y(\omega)$ for all $\omega \in \Omega$, then $X \succ Y$;
Cancellation: For all $\alpha \in (0, 1]$, $X \succ Y$ iff $\alpha X + (1 - \alpha)Z \succ \alpha Y + (1 - \alpha)Z$.

The following representation obtains:

Proposition 2. *If a preference ordering $\succ$ satisfies monotonicity and cancellation, then there is a convex cone $\mathcal{D}$, not containing the origin but containing the interior of the positive orthant, such that $X \succ Y$ iff $X - Y \in \mathcal{D}$.*

The proof is short and instructive:

Proof. First, $X \succ Y$ iff $(X+Z)/2 \succ (Y+Z)/2$ iff $(X+Z)/4+0/2 \succ (Y+Z)/4+0/2$ iff $X+Z \succ Y+Z$ (applying cancellation). Hence $X \succ Y$ iff $X - Y \succ 0$. Also $Y \succ 0$ iff $0 \succ -Y$; if $X \succ 0$ and $Y \succ 0$ we have $X \succ 0 \succ -Y$ and by transitivity we obtain $X \succ -Y$, thus $X + Y \succ 0$. If $X \succ 0$, then $\lambda X \succ 0$ for any $\lambda \in (0, 1]$ by cancellation, and by finite induction we get $\lambda X \succ 0$ for any $\lambda > 0$. Hence $\succ$ can be represented by a cone that contains every positive gamble (by monotonicity) and does not contain the zero gamble (because $X \succ X$ is not allowed). □

As shown in this proof, we can capture a preference ordering by focusing on preferences with respect to the zero gamble, or, equivalently, by focusing on a convex cone of gambles. Cones that encode preference orderings have received attention for some time [16,28,33,35,36]. The literature on *sets of desirable gambles* [22,25,34] employs cones of gambles to model preferences, often assuming the following property of *admissibility*: if $X(\omega) \geq Y(\omega)$ for all ω and $X(\omega) > Y(\omega)$ for some ω, then $X \succ Y$. We do not assume admissibility here.[2]

In this paper we use the term *set of desirable gambles* to refer to a convex cone $\mathcal{D}$ that represents a preference ordering as in Proposition 2. This proposition allows one to freely switch between preference orderings and sets of desirable gambles.

One might think that *any* convex cone of gambles could be represented by a set $\mathcal{K}$ of probability measures as follows: $X \in \mathcal{D}$ iff $\mathbb{E}_{\mathbb{P}}[X] > 0$ for all $\mathbb{P} \in \mathcal{K}$. This is not possible. Consider the set of desirable gambles depicted in Fig. 1 (left).[3] All

[1] A *strict partial order* is a binary relation that is irreflexive and transitive; an *equivalence* is a binary relation that is reflexive, transitive, and symmetric (a binary relation $\diamond$ is *irreflexive* when $X \diamond X$ is false for every X; it is *transitive* when $X \diamond Y$ and $Y \diamond Z$ imply $X \diamond Z$; it is *symmetric* when $X \diamond Y$ implies $Y \diamond X$) [15, Section 2.3].

[2] Admissibility cannot be satisfied in general if preferences are to be encoded by expectation with respect to probability measures that may assign probability zero to events. That is, suppose we want to have that $X \succ Y$ iff $\mathbb{E}_{\mathbb{P}}[X] > \mathbb{E}_{\mathbb{P}}[Y]$; we may face X and Y such that $X(\omega) = Y(\omega)$ for all ω except that $X(\omega') > Y(\omega')$ for ω' with $\mathbb{P}(\omega') = 0$ (in this case $X \succ Y$ due to admissibility but $\mathbb{E}_{\mathbb{P}}[X] = \mathbb{E}_{\mathbb{P}}[Y]$).

[3] In all figures, the interior of sets of desirable gambles appear in pink, and their boundaries appear in red. Sets of probability measures appear in orange or purple; other sets appear in blue.

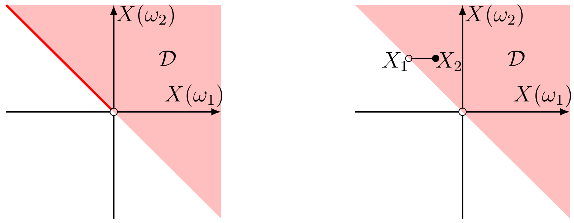

Fig. 1. Left: a cone $\mathcal{D}$; one bordering ray (thick line from the origin) belongs to $\mathcal{D}$, while the other bordering ray does not belong to $\mathcal{D}$. Right: to understand the effect of Auman's continuity condition, take a similar cone $\mathcal{D}$; the gamble X_2 is inside $\mathcal{D}$ and the gamble X_1 is in the border, so the segment between X_1 and X_2 is in $\mathcal{D}$, implying that $X_1 \succ 0$ or $X_1 \not\sim 0$ by Aumann's continuity condition; the same reasoning could be repeated for $-X_1$, hence the border must be open because $X_1 \succ 0$ and $-X_1 \succ 0$ cannot happen. (Colors in the figures appear in the web version of this article.)

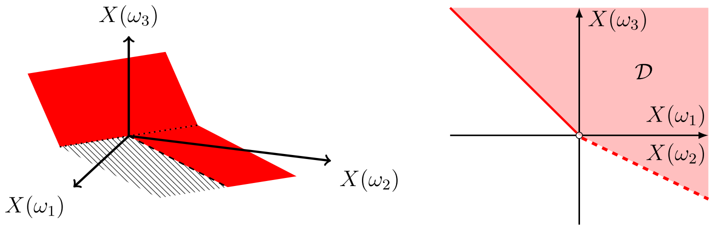

Fig. 2. A cone $\mathcal{D}$ that satisfies Aumann's condition. Left: the cone depicted in three dimensions; $\mathcal{D}$ contains the positive orthant and is bounded by two planes shown in the picture (the intersection of the two planes appears as a dotted line that does not belong to $\mathcal{D}$); the intersection of the planes with $\mathcal{D}$ appears in red, and part of the boundary of $\mathcal{D}$ is not in $\mathcal{D}$ (that part is hatched). Right: looking at $\mathcal{D}$ from point $(1,-1,0)$ to $(0,0,0)$; the dashed line segment corresponds to the boundary of $\mathcal{D}$ that is only partially in $\mathcal{D}$.

gambles in the interior of $\mathcal{D}$ satisfy $X(\omega_1)\mathbb{P}(\omega_1) + X(\omega_2)\mathbb{P}(\omega_2) > 0$ for $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = 1/2$. No other pair of probability values (or sets of pairs of probability values) can similarly represent the interior of $\mathcal{D}$. But even this probability measure cannot represent the fact that half the border is in $\mathcal{D}$; for X in this half-border, $X(\omega_1)(1/2) + X(\omega_2)(1/2) = 0$.

As a digression, note that a set of desirable gambles satisfying admissibility can always be represented by a convex set of *lexicographic* probability measures[4] [2,8,10,24,25,28]. Even though lexicographic probabilities have their appeal, we do not deal with them further in this paper.

To obtain a representation based on (non-lexicographic) probabilities, we must consider conditions on boundaries of sets of desirable gambles. Such conditions inevitably describe limiting behavior. For instance, Aumann [1] has proposed the following condition:

Aumann's continuity: If $\alpha X + (1-\alpha)Y \succ Z$ for all $\alpha > 0$, then either $Y \succ Z$ or $Y \not\sim Z$.

If the interior of the set of desirable gambles is an open halfspace, Auman's continuity condition forces the set of desirable gambles to be open (see Fig. 1 (right)). In fact, when absence of preference is an equivalence, then Aumann's condition implies that $\mathcal{D}$ is an open halfspace, and its representing set $\mathcal{K}$ is a singleton — this result, alluded to in the second paragraph of Section 1, is well-known and discussed further in Appendix A.

In general, if the interior of the set of desirable gambles $\mathcal{D}$ is strictly smaller than a halfspace, Aumann's continuity condition does not imply that $\mathcal{D}$ is entirely open. All that Aumann's continuity condition requires of $\mathcal{D}$ is that, if both X and $-X$ belong to the boundary of $\mathcal{D}$, then $X \not\sim 0$.[5] Fig. 2 shows a cone $\mathcal{D}$ that respects Aumann's continuity condition; note that a face of the closure of $\mathcal{D}$ is only partially in $\mathcal{D}$.

If the continuity condition is strengthened so that $\mathcal{D}$ is assumed open [28,33], then it is possible to find a representation of preference orderings through probabilities. Walley imposes openness by basically requiring that $X \succ 0$ implies $X - \epsilon \succ 0$ for some $\epsilon > 0$ [33, Section 3.7.8, D7]. Another possible condition is (note that limits of sequences of gambles are always assumed pointwise):

Open continuity If $X_i \succ 0$ is false for every i, and $X = \lim_i X_i$, then $X \succ 0$ is false.

[4] Recall that a lexicographic probability measure is an ordered sequence of probability measures [3]; given a gamble X, the expected value $\mathbb{E}[X]$ is then a vector of expected values (one per measure in the lexicography).

[5] To see this, reason as follows. Aumann's condition can be understood as requiring that each gamble on the boundary of $\mathcal{D}$ is either desirable or neutral. So, take X on the boundary of $\mathcal{D}$. Then there are two possibilities: either $-X$ is also on the boundary of $\mathcal{D}$, or not. If $-X$ is also on the boundary, then $X \not\sim 0$ by the assumption on $\mathcal{D}$. If $-X$ is not on the boundary of $\mathcal{D}$, then $-X$ is not in $\mathcal{D}$ (otherwise we would have, for some $\epsilon > 0$, both $X + \epsilon/2$ and $-X - \epsilon$ in $\mathcal{D}$, implying $(X + \epsilon/2)/2 + (-X - \epsilon)/2 = -\epsilon/4 \succ 0$, a violation of monotonicity), and consequently $0 \succ X$ is false because otherwise $-X \succ 0$, a contradiction. Hence either $X \succ 0$ or $X \not\sim 0$ under Aumann's condition.

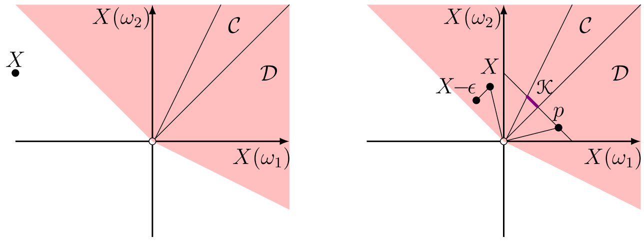

Fig. 3. Left: Part 1 of the proof of Proposition 3; the set of desirable gambles $\mathcal{D}$ and the convex cone $\mathcal{C}$, and a gamble $X \notin \mathcal{D}$. Right: Part 5 of the proof of Proposition 3; the set $\mathcal{K}$ (intersection of $\mathcal{C}$ and the unitary simplex), the point p outside of $\mathcal{C}$, a corresponding X (note that $X \cdot p = 0$), and $X - \epsilon$.

Here is the representation result under the assumption of openness:

Proposition 3. *If a set of desirable gambles $\mathcal{D}$ is open, then it can be represented by a closed convex set $\mathcal{K}$ of probability measures, in the sense that $X \in \mathcal{D}$ iff $\mathbb{E}_{\mathbb{P}}[X] > 0$ for all $\mathbb{P} \in \mathcal{K}$.*

Proof. Part 1) Build the set $\mathcal{C} = \{p : X \cdot p > 0, \forall X \in \mathcal{D}\}$. Take any $X \notin \mathcal{D}$ (there must be some such X, otherwise the zero gamble would be in $\mathcal{D}$). The relative interior of $\mathcal{D}$ and of the singleton $\{X\}$ are disjoint. Consequently, there exists a vector p such that $X \cdot p \leq 0$ and $W \cdot p \geq 0$ for all $W \in \mathcal{D}$, with $W \cdot p > 0$ for some $W \in \mathcal{D}$ [27, Theorems 11.3, 11.7]. We cannot have $W \cdot p = 0$ for $W \in \mathcal{D}$; if we had, then there would be $\lambda > 0$ such that $W - \lambda p \in \mathcal{D}$ (because $\mathcal{D}$ is open and full dimensional due to monotonicity), and then $(W - \lambda p) \cdot p = -\lambda ||p|| < 0$, yielding a contradiction. Fig. 3 illustrates the main ideas in this argument. Hence, for $X \notin \mathcal{D}$ there is $p \in \mathcal{C}$ such that $X \cdot p \leq 0$ (so $\mathcal{C}$ is nonempty); equivalently, if $\forall p \in \mathcal{C} : X \cdot p > 0$, then $X \in \mathcal{D}$.

Parts 2, 3, 4) Just copy here the Parts 2, 3, 4 in the proof of later Theorem 9. By doing so, obtain that $X \in \mathcal{D} \Leftrightarrow \forall p \in \mathcal{C} : X \cdot p > 0$, and moreover that there is a convex set of probability measures $\mathcal{K}$ that is the intersection of $\mathcal{C}$ and the unitary simplex, such that $X \in \mathcal{D} \Leftrightarrow \forall \mathbb{P} \in \mathcal{K} : \mathbb{E}_{\mathbb{P}}[X] > 0$.

Part 5) To show that $\mathcal{K}$ is closed, show that the complement of the cone $\mathcal{C}$ is open: If $p \notin \mathcal{C}$, then there is $X \in \mathcal{D}$ such that $X \cdot p \leq 0$, and also $X - \epsilon \in \mathcal{D}$ for some $\epsilon > 0$ (as $\mathcal{D}$ is open by assumption). Consider the closed halfspace $\mathcal{H} = \{q : (X - \epsilon) \cdot q \leq 0\}$; this halfspace is disjoint from $\mathcal{C}$. Also, p is in $\mathcal{H}$ but not on its boundary (there is a ball around p inside $\mathcal{H}$ for any radius smaller than $|(X - \epsilon) \cdot p| / ||X - \epsilon||$). So for any point not in $\mathcal{C}$ there is a ball, around the point, also not in $\mathcal{C}$; thus the complement of $\mathcal{C}$ is open as desired. $\square$

A set of probability measures is called a *credal set* [21]. Note that, given a set of desirable gambles and the credal set $\mathcal{K}$ built in the proof of Proposition 3, *any* credal set with the same convex hull as $\mathcal{K}$ is also a representation for $\mathcal{D}$. However, $\mathcal{K}$ is the unique maximal credal set that represents $\mathcal{D}$ (indeed, if two sets of probability measures represent $\mathcal{D}$, their union also does so).

There is a significant disadvantage in assuming that a set of desirable gambles is open; namely, the representing credal set is necessarily closed. Hence one cannot say that a coin is biased simply by stating $\mathbb{P}(\text{Heads}) > 1/2$. It seems that the only existing condition in the literature that can accept such assessments has been proposed by Seidenfeld, Schervish, and Kadane [29]. Their condition has two parts, but only one is necessary here[6]:

SSK-continuity If $X_i \succ Y_i$ for every i, and $\lim_i Y_i \succ Z$, then $\lim_i X_i \succ Z$, whenever limits exist.

The other part of the original condition by Seidenfeld, Schervish, and Kadane can actually be derived from the previous conditions:

Proposition 4. *Suppose a preference ordering $\succ$ satisfies cancellation and SSK-continuity. If $X_i \succ Y_i$ for every i, and $W \succ \lim_i X_i$, then $W \succ \lim_i Y_i$.*

Proof. The assumptions imply $X_i - Y_i \succ 0$ and then $-Y_i \succ -X_i$ for every i; similarly, $-\lim X_i = \lim -X_i \succ -W$, so by SSK-continuity, $\lim -Y_i \succ -W$ and then $W \succ \lim Y_i$. $\square$

Later we use the following simplified version of SSK-continuity:

Proposition 5. *Suppose $\succ$ is a preference ordering satisfying cancellation. Suppose that if $X_i \succ Y_i$ and $\lim_i Y_i \succ 0$ then $\lim_i X_i \succ 0$. Then $\succ$ satisfies SSK-continuity.*

[6] Note that their setting is more general, as they derive probabilities and utilities from preferences, and in that setting their complete condition may be necessary.

Fig. 4. Six credal sets with the same evenly convex hull. Filled dots, thick lines and filled regions are in the credal sets (filled regions are open). The first set is evenly open; in fact it is the evenly convex hull of all sets. The first three sets are convex; their convex hulls are distinct. The fourth set is not convex and its convex hull is equal to the second set. The fifth set is not convex, and the sixth set is not even connected; both have the same convex hull that is different from the convex hulls of all previous sets.

Proof. If $\{X_i\} \to X$, $\{Y_i\} \to Y$, $X_i \succ Y_i$ and $Y \succ Z$ then $\{X_i - Z\} \to X - Z$, $\{Y_i - Z\} \to Y - Z$, $X_i - Z \succ Y_i - Z$ and $Y - Z \succ 0$; if the property assumed in the statement is true, then $X - Z \succ 0$ so $X \succ Z$ as desired. □

If a preference ordering satisfies SSK-continuity, and $\{X_i\} \to X$, $\{Y_i\} \to Y$, and $X_i \succ Y_i$, then either $X \succ Y$ or $X \not\sim Y$ (for suppose otherwise that $Y \succ X$; SSK-continuity says that if $X_i \succ Y_i$ and $Y = \lim_i Y_i \succ X$ then $\lim_i X_i \succ X$, hence $X \succ X$, a contradiction). Thus we have that SSK-continuity conveys Aumann's continuity condition.

We will return to SSK-continuity when we examine whether it implies even convexity (it does not).

3. Evenly convex sets of desirable gambles and evenly convex credal sets

In this section we consider preference orderings that can be represented by evenly convex sets of desirable gambles; such preference orderings can also be represented by evenly convex credal sets. This will allow us to consider assessments such as $1/4 \leq \mathbb{P}(\text{Heads}) < 1/2$.

An *evenly convex* set $\mathcal{A}$ is an arbitrary intersection of open halfspaces [14]. Hence an open convex set is evenly convex; also a closed convex set is evenly convex as it is an infinite intersection of halfspaces. For any set $\mathcal{A}$, its *evenly convex hull* $\text{eco}\mathcal{A}$ is the intersection of all evenly convex sets containing $\mathcal{A}$; so $\text{eco}\mathcal{A}$ is the intersection of all open halfspaces that contain $\mathcal{A}$. Note that $\text{co}\mathcal{A} \subseteq \text{eco}\mathcal{A}$, where $\text{co}\mathcal{A}$ is the convex hull of $\mathcal{A}$. Fig. 4 depicts sets with identical evenly convex hull.

There are many characterizations of evenly convex sets [12,17,19]. In particular, we will use the following result in the proof of Theorem 7: a convex set $\mathcal{A}$ is evenly convex iff for every $X_0 \in \text{cl}\mathcal{A} \backslash \mathcal{A}$, and every $\{X_i\}_{i \geq 1} \subset \mathcal{A}$, and every $\{\lambda_i\}_{i \geq 1}$ such that $\lambda_i > 0$, we have $X_0 - \lim_i \lambda_i (X_i - X_0) \notin \mathcal{A}$ whenever the limit exists [12, Corollary 6].

Appendix B introduces a separation property for evenly convex cones that is used later, but that is of independent interest.

3.1. Evenly convex sets of desirable gambles

We introduce the following condition:

Even continuity If $X_i \succ 0$ for every i, and $Y \succ 0$ is false, then $\lim_i (\lambda_i Y - X_i) \succ 0$ is false for any sequence of $\lambda_i > 0$ such that the limit exists.

Even though the condition is somewhat long, it is quite reasonable: one cannot take an undesirable gamble Y and make it desirable, not even in the limit, by multiplying it by a positive number and subtracting from it a desirable gamble.[7]

To make later results more concise, we introduce the following definition:

Definition 6. A preference ordering $\succ$ is *coherent* when it satisfies monotonicity, cancellation, and even continuity.

We then obtain:

Theorem 7. *If a preference ordering $\succ$ is coherent, then there is an evenly convex cone $\mathcal{D}$ of gambles, not containing the origin but containing the interior of the positive orthant, such that $X \succ Y$ iff $X - Y \in \mathcal{D}$.*

Proof. Take the set of desirable gambles produced by Proposition 2.

For a fixed $Y \in \text{cl}\mathcal{D} \backslash \mathcal{D}$ (hence $Y \notin \mathcal{D}$) and $X_i \in \mathcal{D}$ for every i, and $\lambda_i > 0$, compute $\lambda'_i = 1 + \lambda_i$ and $X'_i = \lambda_i X_i$. Clearly $\lambda'_i > 0$ and $X'_i \in \mathcal{D}$. By even continuity, $\lim_i (\lambda'_i Y - X'_i) \notin \mathcal{D}$; hence $\lim_i ((1 + \lambda_i) Y - \lambda_i X_i) \notin \mathcal{D}$, and then $Y - \lim_i \lambda_i (X_i - Y) \notin \mathcal{D}$. Thus $\mathcal{D}$ is evenly convex [12, Corollary 6]. □

Note that coherence implies Aumann's continuity condition:

[7] One might consider a weaker condition (as suggested by a reviewer): If $X_i \succ 0$ and not $Y \succ 0$, then not $\lim_i (Y - X_i) \succ 0$. But this is implied by SSK-continuity: if $X_i \succ 0$, then if $Y \succ \lim_i X_i$ then $Y \succ 0$ by SSK-continuity, implying that if $X_i \succ 0$, then if not $Y \succ 0$ then not $Y \succ \lim_i X_i$.

Proposition 8. *Suppose a preference ordering $\succ$ is coherent. If $\alpha X + (1-\alpha)Y \succ Z$ for all $\alpha > 0$, then either $Y \succ Z$ or $Y \nsim Z$.*

Proof. If $X_i \succ 0$ for every i, then the fact that $\neg(0 \succ 0)$ and even continuity imply $\neg(-X \succ 0)$ for $X = \lim_i X_i$. Now, if $\alpha X + (1-\alpha)Y \succ Z$, then take $\alpha_i = 1/2^i$ and $X_i = \alpha_i(X - Z) + (1-\alpha_i)(Y - Z)$; hence $X_i \succ 0$, implying that $\neg(Z - Y \succ 0)$, so either $Y \succ Z$ or $Y \nsim Z$. □

3.2. Evenly convex credal sets

Evenly convex sets of desirable gambles can be nicely represented by evenly convex sets of probability measures, as described by the next theorem. In the next proof and later we use the nonempty cone

$$\mathcal{C} = \{p : X \cdot p > 0, \forall X \in \mathcal{D}\}. \tag{1}$$

Theorem 9. *If a preference ordering $\succ$ is coherent, then there is a unique maximal evenly convex credal set $\mathcal{K}$ such that $X \succ Y$ iff for all $\mathbb{P} \in \mathcal{K}$ we have $\mathbb{E}_{\mathbb{P}}[X] > \mathbb{E}_{\mathbb{P}}[Y]$.*

Proof. Part 1) For any $X \notin \mathcal{D}$, there is p such that $X \cdot p \leq 0$ and $Y \cdot p > 0$ for all $Y \in \mathcal{D}$ by Theorem 24. So $\mathcal{C}$ is nonempty, and in fact it is a cone (if p' and p'' satisfy the constraints, then so does $p' + p''$ and $\lambda p'$ for $\lambda > 0$). Hence if $X \notin \mathcal{D}$ then $\exists p \in \mathcal{C} : X \cdot p \leq 0$; equivalently, if $\forall p \in \mathcal{C} : X \cdot p > 0$, then $X \in \mathcal{D}$.

Part 2) By construction, if $X \in \mathcal{D}$ then $X \cdot p > 0$ for all $p \in \mathcal{C}$; using this and Part 1, $X \in \mathcal{D} \Leftrightarrow \forall p \in \mathcal{C} : X \cdot p > 0$.

Part 3) We now show that $\mathcal{C}$ is equivalent to a set of probability measures $\mathcal{K}$. Denote by $\mathbf{1}$ a vector of ones, and $\mathbf{1}_i$ a vector whose ith element is 1 and all other elements are zero. By monotonicity, $\mathbf{1} \cdot p > 0$ for all $p \in \mathcal{C}$, so $\sum_i p_i > 0$. Also, for every $p \in \mathcal{C}$: $(\mathbf{1}_i + \epsilon) \cdot p > 0$ for every $\epsilon > 0$; hence $p_i + \epsilon \sum_j p_j > 0$ for every ϵ, implying that $p_i \geq 0$ (if $p_i < 0$ then for $\epsilon < -p_i / \sum_j p_j$ we have $p_i + \epsilon \sum_j p_j < 0$, a contradiction). Hence we can normalize each p in $\mathcal{C}$, thus obtaining a set of probability measures $\mathcal{K}$ that is a representation for $\mathcal{D}$: $X \in \mathcal{D} \Leftrightarrow \forall \mathbb{P} \in \mathcal{K} : \mathbb{E}_{\mathbb{P}}[X] > 0$.

Part 4) Note that $\mathcal{K}$ is the intersection of $\mathcal{C}$ and the unitary simplex $\sum_i p_i = 1$: If p belongs to this intersection, it is normalized element of $\mathcal{C}$, so $p \in \mathcal{K}$; and if $p \in \mathcal{K}$, then $p \in \mathcal{C}$ and also it is normalized so it belongs to the unitary simplex.

Part 5) The cone $\mathcal{C}$ is defined as the intersection of open halfspaces, hence by definition it is evenly convex. And $\mathcal{K}$ is the intersection of those open halfspaces and the unitary simplex (itself the intersection of open halfspaces), hence $\mathcal{K}$ is evenly convex.

Part 6) Note that if two sets of probability measures $\mathcal{K}_1$ and $\mathcal{K}_2$ represent a preference ordering, then $\mathcal{K}_1 \cup \mathcal{K}_2$ also represents the same preference ordering (if $X \succ 0$, then $\forall \mathbb{P} \in \mathcal{K}_1 : \mathbb{E}_{\mathbb{P}}[X] > 0$ and $\forall \mathbb{P} \in \mathcal{K}_2 : \mathbb{E}_{\mathbb{P}}[X] > 0$, implying that $\forall \mathbb{P} \in \mathcal{K}_1 \cup \mathcal{K}_2 : \mathbb{E}_{\mathbb{P}}[X] > 0$; conversely, if $\forall \mathbb{P} \in \mathcal{K}_1 \cup \mathcal{K}_2 : \mathbb{E}_{\mathbb{P}}[X] > 0$, then $\forall \mathbb{P} \in \mathcal{K}_1 : \mathbb{E}_{\mathbb{P}}[X] > 0$, hence $X \succ 0$). Thus there is a unique maximal credal set that represents $\succ$; to show that $\mathcal{K}$ is this credal set, suppose there is $\mathcal{K}'$ that represents $\succ$, and $\mathbb{P}' \in \mathcal{K}'$ but $\mathbb{P}' \notin \mathcal{K}$. If $\mathbb{P}' \notin \mathcal{K}$, then by the definition of $\mathcal{K}$ we must have some $X \in \mathcal{D}$ such that $\mathbb{E}_{\mathbb{P}'}[X] \leq 0$. However, because $\mathcal{K}'$ represents $\succ$, for any $X \in \mathcal{D}$ we must have $\mathbb{E}_{\mathbb{P}}[X] > 0$ for all $\mathbb{P} \in \mathcal{K}'$; that is, $\mathbb{E}_{\mathbb{P}'}[X] > 0$. Hence we get a contradiction, implying that no representing credal set can contain probability measures outside of $\mathcal{K}$. □

In fact *many* sets of probability measures may encode the same ordering. For instance, if a representing $\mathcal{K}$ is a closed set, then the set of its extreme points $\mathsf{ext}\mathcal{K}$ is an equivalent representation for $\succ$; that is, $X \succ Y \Leftrightarrow \forall \mathbb{P} \in \mathsf{ext}\mathcal{K} : \mathbb{E}_{\mathbb{P}}[X] > \mathbb{E}_{\mathbb{P}}[Y]$. We can actually state a more interesting fact:

Theorem 10. *Suppose $\succ$ is a coherent preference ordering, and the credal set $\mathcal{K}$ has been built as in the proof of Theorem 9. A set of probability measures $\mathcal{K}'$ represents $\succ$ iff $\mathsf{eco}\mathcal{K}' = \mathcal{K}$.*

Proof. We need only to consider preferences with respect to the zero gamble.

Take a credal set $\mathcal{K}'$ such that $\mathsf{eco}\mathcal{K}' = \mathcal{K}$. Clearly if $X \succ 0$ then $\forall \mathbb{P} \in \mathcal{K} : \mathbb{E}_{\mathbb{P}}[X] > 0$ then $\forall \mathbb{P} \in \mathcal{K}' : \mathbb{E}_{\mathbb{P}}[X] > 0$ as $\mathcal{K}' \subseteq \mathsf{eco}\mathcal{K}'$. Now suppose $\forall \mathbb{P} \in \mathcal{K}' : \mathbb{E}_{\mathbb{P}}[X] > 0$. Consider that $\mathsf{eco}\mathcal{K}'$ is the set of all p such that $Y \cdot p > 0$ for all Y such that for all $q \in \mathcal{K}'$ we have $Y \cdot q > 0$. As X satisfies the last set of inequalities, then $X \cdot p > 0$ for all $p \in \mathsf{eco}\mathcal{K}'$, hence $\mathbb{E}_{\mathbb{P}}[X] > 0$ for all $\mathbb{P} \in \mathcal{K}$, and then $X \succ 0$. Hence $\mathcal{K}'$ represents $\succ$.

Now suppose $\mathcal{K}'$ represents $\succ$. Then its elements must satisfy the constraints $X \cdot p > 0$ for all $X \in \mathcal{D}$. Suppose $\mathcal{K}'$ also satisfies a nontrivial constraint $Y \cdot p > \alpha$ for some Y and α; that is, there is p' that satisfies all other constraints but such that $Y \cdot p' \leq \alpha$. Because every p is a probability measure, $(Y - \alpha) \cdot p > 0$ is an equivalent constraint. Hence $(Y - \alpha) \cdot p > 0$ for all $p \in \mathcal{K}'$; because $\mathcal{K}'$ represents $\succ$, $Y - \alpha$ is a desirable gamble. However there is $p' \in \mathcal{K}$ such that $(Y - \alpha) \cdot p' \leq 0$, implying $Y - \alpha \notin \mathcal{D}$, a contradiction. So there is no additional nontrivial strict linear inequality that distinguishes $\mathcal{K}'$ and $\mathcal{K}$, and consequently they share the same evenly convex hull. □

This theorem shows that if two evenly convex sets are different, then they represent distinct preference orderings. Fig. 4 shows several different credal sets that have the same evenly convex hull, and hence represent the same coherent preference ordering.

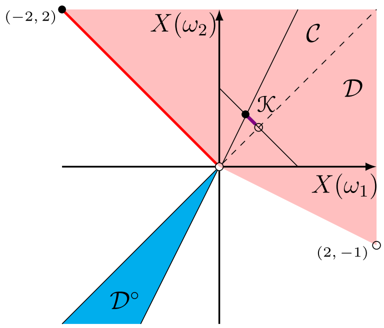

Fig. 5. An evenly convex cone $\mathcal{D}$, its polar cone $\mathcal{D}^\circ$, the cone $\mathcal{C}$ and the credal set $\mathcal{K}$ corresponding to $\mathcal{D}$. The dual cone $\mathcal{D}^\star$ is equal to the closure of the cone $\mathcal{C}$ as $\mathcal{D}^\star$ is the mirror image of $\mathcal{D}^\circ$. The boundary of $\mathcal{D}$ consists of two half-lines extending from the origin. One of these half-lines is in $\mathcal{D}$ except for the origin (this part of the boundary of $\mathcal{D}$ is shown in red), and the corresponding face of cl$\mathcal{C}$ is not in $\mathcal{C}$; this face of cl$\mathcal{C}$ appears as a dashed line. The other half-line on the boundary of $\mathcal{D}$ is not in $\mathcal{D}$, and the corresponding face of $\mathcal{C}$ is indeed in $\mathcal{C}$; this face cl$\mathcal{C}$ appears as a thick line. The credal set $\mathcal{K}$ is the interior of the line segment between $(1/2, 1/2)$ and $(1/3, 2/3)$ plus the latter point.

3.3. A bit of duality

Additional insight can be obtained by investigating the duality between cl$\mathcal{D}$ and cl$\mathcal{C}$, as we do in this section.

To start, recall some needed definitions. Given a convex set $\mathcal{A}$, its *polar set* is $\mathcal{A}^\circ = \{p : \forall X \in \mathcal{A} : X \cdot p \leq 1\}$ [5]; if $\mathcal{A}$ is a convex cone its polar set is equal to its *polar cone*, defined as $\{p : \forall X \in \mathcal{A} : X \cdot p \leq 0\}$ (because any inequality with right hand side larger than zero is redundant). The *dual cone* is simply the mirror image of the polar cone: $\mathcal{A}^\star = -\mathcal{A}^\circ$. Also, $\mathcal{A}^{\circ\circ} = \{X : \forall p \in \mathcal{A}^\circ : X \cdot p \leq 0\} = \{X : \forall -p \in -\mathcal{A}^\circ : X \cdot p \leq 0\} = \{X : \forall p \in \mathcal{A}^\star : X \cdot p \geq 0\} = \mathcal{A}^{\star\star}$.

If we have a set of desirable gambles $\mathcal{D}$, then the definition of the cone $\mathcal{C}$ is such that cl$\mathcal{C}$ is exactly the dual of $\mathcal{D}$, and then cl$\mathcal{D}$ is the dual of cl$\mathcal{C}$. Now, the larger a cone, the smaller its dual cone. So, when the set of desirable gambles is maximal (an open halfspace, indicating minimum uncertainty concerning preferences), the corresponding credal set is minimal (a single probability measure, indicating minimum uncertainty); also, the smaller the set of desirable gambles, the larger the corresponding credal set. If a set of desirable gambles is the union of several cones, then the corresponding credal set is obtained by intersecting the various credal sets corresponding to those cones. And even the faces of cl$\mathcal{D}$ and cl$\mathcal{C}$ are related: a face of cl$\mathcal{D}$ that is in $\mathcal{D}$ corresponds to a face of cl$\mathcal{C}$ that is not in $\mathcal{C}$, and the reverse is true under some conditions. Fig. 5 depicts these relationships in a simple two-dimensional case.

As $\mathcal{C}$ is nonempty, cl$\mathcal{C} = \{p : \forall X \in \mathcal{D} : X \cdot p \geq 0\}$; hence cl$\mathcal{C}$ is by definition the dual cone of $\mathcal{D}$ [4, Section 2.6]. Then $(\text{cl}\mathcal{C})^\star$ is just the closure of $\mathcal{D}$, as cl$\mathcal{D} = \mathcal{D}^{\star\star}$ [5, Theorem 6.2]. Also, if a cone $\mathcal{F} \subset \mathcal{D}$ (say a proper face of $\mathcal{D}$), then $\mathcal{D}^\star \subset \mathcal{F}^\star$, and if we have several cones $\{\mathcal{D}_i\}_i$, then $(\cup_i \mathcal{D}_i)^\star = \cap \mathcal{D}_i^\star$ [20, Theorem 23.3].

It is also possible to establish the announced connection between the faces of cl$\mathcal{D}$ and cl$\mathcal{C}$, as follows. The following definition is necessary: for any face $\mathcal{F}$ of a closed convex cone $\mathcal{A}$, define its dual face $\mathcal{F}^\triangle = \mathcal{A}^\star \cap \mathcal{F}^\perp$ [31, Section 2.13], where the superscript $\perp$ denotes orthogonal complement (that is, $\mathcal{B}^\perp = \{p : \forall X \in \mathcal{B} : X \cdot p = 0\}$). If for two faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{A}$ we have that $\mathcal{F}_1$ is a face of $\mathcal{F}_2$, then $\mathcal{F}_2^\triangle$ is face of $\mathcal{F}_1^\triangle$ [32, Proposition 2.4]. In fact, if all faces of $\mathcal{A}$ are exposed, then the mapping between faces of $\mathcal{A}$ and its dual is one-to-one and onto, in such a way that $\mathcal{F}_1$ is a face of $\mathcal{F}_2$ iff $\mathcal{F}_2^\triangle$ is face of $\mathcal{F}_1^\triangle$ [32, Corollary 2.6]. In particular if cl$\mathcal{D}$ is the convex hull of some *finite* set of gambles, then all its faces are exposed and the mapping is indeed one-to-one and onto the faces of cl$\mathcal{C}$ [31, Theorem 2.13.2]. Of course, this applies similarly to faces of cl$\mathcal{C}$ and its dual.

Note that if a face of cl$\mathcal{D}$ does intersect $\mathcal{D}$, its dual face does not intersect $\mathcal{C}$:

Theorem 11. *If $\mathcal{F}$ is a face of* cl$\mathcal{D}$*, and $\mathcal{F} \cap \mathcal{D} \neq \emptyset$, then $\mathcal{F}^\triangle \cap \mathcal{C} = \emptyset$.*

Proof. As $\mathcal{F} \cap \mathcal{D} \neq \emptyset$, pick $X \in \mathcal{F} \cap \mathcal{D}$. For any $p \in \mathcal{F}^\perp$ we must have $X \cdot p = 0$, so p cannot be in $\mathcal{C}$; hence $\mathcal{F}^\perp \cap \mathcal{C} = \emptyset$ and consequently $\mathcal{F}^\triangle \cap \mathcal{C} = \mathcal{D}^\star \cap \mathcal{F}^\perp \cap \mathcal{C} = \emptyset$. $\square$

The converse can be shown for any face that is finitely generated[8]:

Theorem 12. *If $\mathcal{F}$ is a face of* cl$\mathcal{D}$ *that is the convex hull of a finite set of gambles, and $\mathcal{F} \cap \mathcal{D} = \emptyset$, then $\mathcal{F}^\triangle \cap \mathcal{C} \neq \emptyset$.*

Proof. We have that $\mathcal{F}$ is the conic hull of a finite set of gambles $\{X_1, \ldots, X_n\}$. Suppose that no element of $\mathcal{F}^\perp = \{p : \forall X \in \mathcal{F} : X \cdot p = 0\}$ belongs to $\mathcal{C}$. Then for each $p \in \mathcal{C}$ there is at least a $X \in \mathcal{F}$ such that $X \cdot p > 0$. Write X as $\sum_i \alpha_i X_i$ (where all

[8] Whether or not Theorem 12 holds for general faces is an open question.

$\alpha_i \geq 0$) to obtain that $\sum_i \alpha_i X_i \cdot p > 0$; if we have $X_i \cdot p \geq 0$ for all X_i, then it must be that $X_i \cdot p > 0$ for at least one X_i. (To conclude that $X_i \cdot p \geq 0$ for all X_i, reason as follows. As any $Y \in \mathcal{F}$ is on the boundary of $\mathcal{D}$, for all such Y we have, for all $p \in \mathcal{C}$ and all $\epsilon > 0$, that $(Y + \epsilon) \cdot p > 0$. So for all $Y \in \mathcal{F}$ and all $p \in \mathcal{C}$ we must have $Y \cdot p \geq 0$ to satisfy $Y \cdot p > -\epsilon \sum_i p_i$ for all $\epsilon > 0$.) Consequently the convex combination $Z = \sum_{i=1}^{n} X_i/n$ must satisfy $Z \cdot p > 0$ for all $p \in \mathcal{C}$, and then $Z \in \mathcal{D}$. But Z must belong to $\mathcal{F}$, so Z cannot be in $\mathcal{D}$ by assumption. Hence there must be an element of $\mathcal{F}^{\perp}$ in $\mathcal{C}$; this proves the theorem as $\mathcal{F}^{\perp} \cap \mathcal{C} = \mathcal{F}^{\perp} \cap \mathcal{C} \cap \mathrm{cl}\mathcal{C} = \mathcal{F}^{\perp} \cap \mathcal{C} \cap \mathcal{D}^{\star} = \mathcal{F}^{\triangle} \cap \mathcal{C}$. $\square$

3.4. Back to SSK-continuity

Note that SSK-continuity is satisfied by coherent preference orderings:

Proposition 13. *If $\succ$ is a coherent preference ordering, then SSK-continuity holds.*

Proof. Take $\{X_i\} \to X$ and $\{Y_i\} \to Y$ such that $X_i \succ Y_i$. Take the representing credal set $\mathcal{K}$; any probability measure $\mathbb{P} \in \mathcal{K}$ satisfies $\mathbb{E}_{\mathbb{P}}[X_i - Y_i] > 0$, so $\lim_i \mathbb{E}_{\mathbb{P}}[X_i - Y_i] \geq 0$; then $\mathbb{E}_{\mathbb{P}}[\lim_i X_i] \geq \mathbb{E}_{\mathbb{P}}[\lim_i Y_i]$ as the state space is finite, hence $\mathbb{E}_{\mathbb{P}}[X] \geq \mathbb{E}_{\mathbb{P}}[Y]$. If additionally $Y \succ Z$, then $\mathbb{E}_{\mathbb{P}}[Y] > \mathbb{E}_{\mathbb{P}}[Z]$ for every $\mathbb{P} \in \mathcal{K}$, so $\mathbb{E}_{\mathbb{P}}[X] > \mathbb{E}_{\mathbb{P}}[Z]$ for every $\mathbb{P} \in \mathcal{K}$, and then $X \succ Z$ as desired. $\square$

The natural question is whether SSK-continuity implies even continuity. It does not; but to appreciate the matter, it is interesting to note that SSK-continuity implies even continuity in an important case. Start by considering a consequence of SSK-continuity that is quite reasonable as a property of preferences:

Proposition 14. *Suppose $\succ$ is a preference ordering satisfying SSK-continuity. If $\alpha W + (1 - \alpha)X \succ Y \succ 0$ for $\alpha \in (0, 1]$, then $X \succ 0$.*

Proof. Take $\alpha_i = 1/2^i$, $X_i = \alpha_i W + (1 - \alpha_i)X$ and $Y_i = Y$. As $X_i \succ Y_i$, $\{X_i\} \to X$, $\{Y_i\} \to Y$, and $Y \succ 0$, SSK-continuity implies $X \succ 0$ as desired. $\square$

This result leads to:

Proposition 15. *Suppose $\succ$ is a preference ordering satisfying monotonicity, cancellation, and SSK-continuity, with representing set of desirable gambles $\mathcal{D}$. If $X \in \mathcal{D}$ and $Y \in \mathrm{cl}\mathcal{D}$, then $\alpha X + (1 - \alpha)Y \in \mathcal{D}$ for $\alpha \in (0, 1)$.*

Proof. Take $X \in \mathcal{D}$, $Y \in \mathrm{cl}\mathcal{D}$, $\alpha \in (0, 1)$, and $Z = \alpha X + (1 - \alpha)Y$. For some $\delta > 0$ we have $Y + \delta \in \mathrm{relint}\mathcal{D}$ by monotonicity; hence $\beta(Y + \delta) + (1 - \beta)Y \in \mathcal{D}$ for $\beta \in (0, 1]$ [27, Theorem 6.1]. Note that $Y = \gamma Z - \alpha\gamma X$ where $\gamma = (1 - \alpha)^{-1}$; thus $\beta(\gamma Z - \alpha\gamma X + \delta) + (1 - \beta)(\gamma Z - \alpha\gamma X) \succ 0$. Hence $\beta(\gamma Z + \delta) + (1 - \beta)(\gamma Z) \succ \alpha\gamma X$ for $\beta \in (0, 1]$. By assumption $X \succ 0$, so $\alpha\gamma X \succ 0$; by Proposition 14, we obtain $\gamma Z \succ 0$, hence $Z \in \mathcal{D}$ as desired. $\square$

A cone $\mathcal{A}$ whose closure is the intersection of finitely many closed halfspaces is evenly convex iff it satisfies: if $X \in \mathcal{A}$ and $Y \in \mathrm{cl}\mathcal{A}$, then the segment between X and Y is in $\mathcal{A}$ [14, Section 3.5]. Hence we obtain:

Theorem 16. *Suppose $\succ$ is a preference ordering satisfying monotonicity, cancellation, and SSK-continuity, with representing set of desirable gambles $\mathcal{D}$. If the closure of $\mathcal{D}$ is the intersection of finitely many closed halfspaces, then $\mathcal{D}$ is evenly convex.*

However, in general SSK-continuity does not imply even convexity of sets of desirable gambles. To prove the latter sentence, we build an example as follows. First, we take a coherent preference ordering $\succ'$ and its representing set of desirable gambles $\mathcal{D}'$, so that $\mathcal{D}'$ contains a *non-exposed* but extreme ray R_0 that goes through a gamble X_0 (that is, $R_0 = \{\lambda X_0 : \lambda > 0\}$), such that the auxiliary set of desirable gambles $\mathcal{D}'' = \mathcal{D}' \backslash R_0$ has the same evenly convex hull as $\mathcal{D}'$. Then we define a preference ordering $\succ''$ as $X \succ'' Y$ iff $X - Y \in \mathcal{D}''$. We finally show that this preference ordering satisfies SSK-continuity even though $\mathcal{D}''$ is not evenly convex.

Example 17. Suppose $\Omega = \{\omega_1, \omega_2, \omega_3\}$; a gamble X can be encoded as a triple of numbers (x_1, x_2, x_3), meaning $(X(\omega_1), X(\omega_2), X(\omega_3))$. Consider $\mathcal{B}$ as the union of the *open* circle with center $(1/4, 1/4, 1/2)$ and radius $\sqrt{3/2}$ drawn on the simplex consisting of $x_1 + x_2 + x_3 = 1$, and the *closed* polygon with four vertices $(3/4, 3/4, -1/2)$, $(-1/4, -1/4, 3/2)$, $(-2, 3/2, 3/2)$, $(-1, 5/2, -1/2)$. Take $X_0 = (-1/4, -1/4, 3/2)$, a non-exposed extreme point of $\mathcal{B}$. Fig. 6 depicts the set $\mathcal{B}$. Take the cone $\mathcal{D}''$ as the set of all rays emanating from the origin and going through points of $\mathcal{B}$ except X_0.

We now show that, even though $\mathcal{D}''$ is not an evenly convex set, $\succ''$ built as described satisfies SSK-continuity.

As the cone $\mathcal{D}'$ is evenly convex, we can build its representing credal set $\mathcal{K}'$. By construction $X \succ'' 0$ implies that for all $\mathbb{P} \in \mathcal{K}'$ we have $\mathbb{E}_{\mathbb{P}}[X] > 0$; also by construction $X \succ'' 0$ implies $X \notin R_0$. Also, if for all $\mathbb{P} \in \mathcal{K}'$ we have $\mathbb{E}_{\mathbb{P}}[X] > 0$ *and* $X \notin R_0$, then $X \succ'' 0$. That is, we have the representation: $X \succ'' 0 \Leftrightarrow (X \notin R_0) \wedge \left(\forall \mathbb{P} \in \mathcal{K}' : \mathbb{E}_{\mathbb{P}}[X] > 0\right)$.

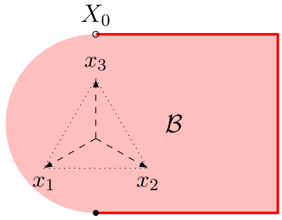

Fig. 6. The set $\mathcal{B}$ in Example 17, viewed from point $(1, 1, 1)$. The axes appear as dashed lines. The set $\mathcal{B}$ includes the simplex where $x_1 + x_2 + x_3 = 1$ and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$; this simplex is depicted using dotted lines. Thus the set $\mathcal{B}$ is not contained in the positive orthant; the smallest cone from the origin that contains $\mathcal{B}$ also contains the positive orthant.

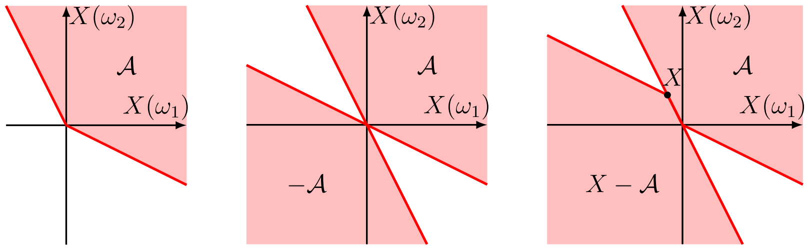

Fig. 7. A closed convex cone $\mathcal{A}$ (left), the cones $\mathcal{A}$ and $-\mathcal{A}$ (middle), and the cones $\mathcal{A}$ and $X - \mathcal{A}$ for X in an extreme ray of $\mathcal{A}$ (right).

Note that $\succ''$ satisfies cancellation and monotonicity because $\mathcal{D}''$ is a cone containing the interior of the positive orthant. By Proposition 5, we need to show that $\{X_i\} \to X$, $\{Y_i\} \to Y$, $X_i \succ'' Y_i$, $Y \succ'' 0$ imply $X \succ'' 0$. If $Y \in R_0$, then $Y \succ'' 0$ is false and there is nothing to prove; hence assume that $Y \notin R_0$. We distinguish two cases: $X \notin R_0$ and $X \in R_0$.

Take $X \notin R_0$. To prove that $X \succ'' 0$, note that $\mathbb{E}_{\mathbb{P}}[X_i - Y_i] > 0$ for every $\mathbb{P} \in \mathcal{K}'$, so $\lim_i \mathbb{E}_{\mathbb{P}}[X_i - Y_i] \geq 0$ and therefore $\mathbb{E}_{\mathbb{P}}[X] \geq \mathbb{E}_{\mathbb{P}}[Y]$ for $\mathbb{P} \in \mathcal{K}'$. Thus $\mathbb{E}_{\mathbb{P}}[X] \geq \mathbb{E}_{\mathbb{P}}[Y] > \mathbb{E}_{\mathbb{P}}[0] = 0$ and then $\mathbb{E}_{\mathbb{P}}[X] > 0$ for every $\mathbb{P} \in \mathcal{K}'$, implying $X \succ'' 0$ as desired.

Now take $X \in R_0$; note that X is in an extreme ray of $\mathrm{cl}\mathcal{D}'$. In the next paragraph we show that if $\{X_i\} \to X$, $\{Y_i\} \to Y$, $X_i \succ'' Y_i$, then $Y \succ'' 0$ must be false. Hence it is irrelevant to consider $X \in R_0$ as the premise of SSK-continuity is never satisfied in this case, and the argument is finished.

To conclude we show that, if $\mathcal{A}$ is a convex cone, $\{X_i\} \to X$, $\{Y_i\} \to Y$, $X_i - Y_i \in \mathcal{A}$, and X belongs to an extreme ray of $\mathrm{cl}\mathcal{A}$ but $X \notin \mathcal{A}$, then $Y \notin \mathcal{A}$. We have that $Y \in \mathrm{cl}\mathcal{A}$ and, as $X_i - Y_i \in \mathcal{A}$ for every i, $X - Y \in \mathrm{cl}\mathcal{A}$ (the closure is the set of limiting points). So we have both Y and $X - Y$ in $\mathrm{cl}A$. If $Y \neq \lambda X$, then $X/2$ is the convex combination $Y/2 + (X - Y)/2$ of two points not in the ray containing X, a contradiction with the assumption that X is in an extreme ray of $\mathrm{cl}\mathcal{A}$. So $Y = \lambda X$ for some λ, and then $Y \notin \mathcal{A}$. (This result is illustrated by Fig. 7: Y must belong to the closure of $\mathcal{A}$ and to the closure of $X - \mathcal{A}$, so it belongs to the line from the origin through X.) $\square$

4. Almost preferences and natural extension

In practice we may have a set of assessments indicating preferences, and we may be interested in the consequences of these assessments. It seems "natural" to study the smallest preference ordering that agrees with given assessments, in accordance with a least commitment strategy. In this section we explore this idea to adapt Walley's natural extensions [33] to the context of evenly convex credal sets.

Given an arbitrary set of preference assessments, their *natural extension* is the smallest preference ordering that is obtained from the original assessments by applying the coherence conditions. We contemplate two kinds of assessments here. First, the assessment "X is preferred to Y" is interpreted as $X \succ Y$. Second, the assessment "X is almost-preferred to Y" is interpreted as $X + \epsilon \succ Y$ for every $\epsilon > 0$. Almost-preference is the basis of Walley's theory of lower previsions [33, Section 3.7]; in short, an almost-preference of X over Y indicates that X and Y are not necessarily exchangeable in a given transaction, but any positive change to X makes it preferable to Y.

Suppose then that we have a set of strict preferences $\mathbf{S} = \{X_j \succ Y_j\}_{j \in J}$ and a set of almost-preferences $\mathbf{A} = \{X_k + \epsilon \succ Y_k, \epsilon > 0\}_{k \in K}$, for arbitrary sets of indexes J and K. When there is a smallest coherent preference ordering $\succ$ that satisfies both $\mathbf{S}$ and $\mathbf{A}$, we refer to $\succ$ as the *natural extension* of $\mathbf{S}$ and $\mathbf{A}$. And we call the credal set representing $\succ$ the *credal natural extension* of $\mathbf{S}$ and $\mathbf{A}$.

Theorem 18. *Given strict preferences $\mathbf{S} = \{X_j \succ Y_j\}_{j \in J}$ and almost-preferences $\mathbf{A} = \{X_k + \epsilon \succ Y_k, \epsilon > 0\}_{k \in K}$, consider rays $s_j = \{\lambda(X_j - Y_j)\}_{\lambda > 0}$, for each $j \in J$, and $a_{k,\epsilon} = \{\lambda(X_k - Y_k + \epsilon)\}_{\lambda > 0}$, for each $k \in K$ and $\epsilon > 0$. Denote by $\mathcal{D}_{\mathbf{S},\mathbf{A}}$ the evenly convex hull of points in rays s_j and $a_{k,\epsilon}$ and points in the interior of the positive orthant. Then there exists a natural extension $\succ$ of $\mathbf{S}$ and $\mathbf{A}$ iff the cone $\mathcal{D}_{\mathbf{S},\mathbf{A}}$ does not contain the origin. And if $\mathcal{D}_{\mathbf{S},\mathbf{A}}$ does not contain the origin, then the preference ordering induced by*

$\mathcal{D}_{\mathbf{S},\mathbf{A}}$ is the natural extension of **S** *and* **A**. *Moreover, in this case the credal natural extension of* **S** *and* **A** *is the evenly convex set* $\{p : \sum p_i = 1, p_i \geq 0 \text{ for all } i, (X_j - Y_j) \cdot p > 0 \text{ for all } j \in J, (X_k - Y_k) \cdot p \geq 0 \text{ for all } k \in K\}$.

Proof. Suppose for a moment that the natural extension $\succ$ exists, and that we wish to build a set of desirable gambles $\mathcal{D}$ such that $X \succ Y$ iff $X - Y \in \mathcal{D}$. So consider a cone $\mathcal{D}'$ consisting of all rays s_j and $a_{k,\epsilon}$ and the interior of the positive orthant (the latter must be in $\mathcal{D}$ due to monotonicity). Now to satisfy transitivity, cancellation and even continuity, $\mathcal{D}$ must be an evenly convex cone, as argued in the proof of Proposition 2 and Theorem 7. Hence to produce the smallest such cone, we must take the evenly convex hull of $\mathcal{D}'$. Now if $\mathrm{eco}\mathcal{D}'$ contains the origin, there is no natural extension (otherwise we would have $X \succ X$); and if $\mathrm{eco}\mathcal{D}'$ does not contain the origin, then all requirements are satisfied by taking $\mathcal{D}_{\mathbf{S},\mathbf{A}} = \mathrm{eco}\mathcal{D}'$ as described in the theorem.

Suppose the origin is not in $\mathcal{D}_{\mathbf{S},\mathbf{A}}$, so there is a natural extension, and build the cone $\mathcal{C} = \{p : X \cdot p > 0, \forall X \in \mathcal{D}_{\mathbf{S},\mathbf{A}}\}$. Now consider another cone:

$$\mathcal{C}' = \{p : (\mathbf{1}_i + \epsilon) \cdot p > 0, (X_j - Y_j) \cdot p > 0, (X_k - Y_k + \epsilon) \cdot p > 0\},$$

where each $\mathbf{1}_i$ is a vector whose ith element is 1 and all other elements are zero, j ranges over preferences in **S**, and k ranges over preferences in **A**, and ϵ ranges over the positive reals. If $p \in \mathcal{C}$, then $p \in \mathcal{C}'$ as every constraint in $\mathcal{C}'$ also appears in $\mathcal{C}$. In the next paragraph we show that if $p \in \mathcal{C}'$ then $p \in \mathcal{C}$; consequently, $\mathcal{C} = \mathcal{C}'$. Each restriction $(X + \epsilon) \cdot p > 0$ for every $\epsilon > 0$ is equivalent to $X \cdot p \geq 0$, so $\mathcal{C} = \mathcal{C}' = \{p : p_i \geq 0, (X_j - Y_j) \cdot p > 0, (X_k - Y_k) \cdot p \geq 0\}$. Now the credal set representing the natural extension $\succ$ is the intersection of $\mathcal{C}$ and the unitary simplex; hence it is the intersection of $\{p : p_i \geq 0, (X_j - Y_j) \cdot p > 0, (X_k - Y_k) \cdot p \geq 0\}$ and $\{p : \sum_i p_i = 1, p_i \geq 0\}$. The intersection we seek is $\{p : \sum_i p_i = 1, p_i \geq 0, (X_j - Y_j) \cdot p > 0, (X_k - Y_k) \cdot p \geq 0\}$, and the proof is finished.

It remains to show that $p \in \mathcal{C}'$ implies $p \in \mathcal{C}$. Fix some q that belongs to $\mathcal{C}'$; that is, $X \cdot q > 0$ for every $X \in \mathcal{D}'$. Now, there may be $Y \in \mathrm{eco}\mathcal{D}' \backslash \mathcal{D}'$. So, suppose we have $Y \in \mathrm{eco}\mathcal{D}' \backslash \mathcal{D}'$ but $Y \cdot q \leq 0$. But then we would be able to add the constraint $X \cdot q > 0$ (here q is fixed and X runs through all gambles) to the evenly convex hull of $\mathcal{D}'$, and this constraint would eliminate Y from $\mathrm{eco}\mathcal{D}'$, contradicting the assumption that $Y \in \mathrm{eco}\mathcal{D}'$. Hence q satisfies the property that $X \cdot q > 0$ for every $X \in \mathrm{eco}\mathcal{D}'$, implying that $q \in \mathcal{C}$ as desired. □

5. Conditioning

In Kolmogorov's probability theory, the conditional probability of event A given event B, denoted $\mathbb{P}(A|B)$, is simply the ratio $\mathbb{P}(A \cap B)/\mathbb{P}(B)$ whenever $\mathbb{P}(B) > 0$; conditional probability is left undefined when $\mathbb{P}(B) = 0$.

When preferences are encoded by credal sets, the definition of conditional probability requires additional care. First of all, suppose $\mathbb{P}(B) > 0$ for each probability measure $\mathbb{P}$ in the credal set $\mathcal{K}$. Then we can define the conditional credal set given B, denoted by $\mathcal{K}_B$, to be the set of all conditional probability measures $\mathbb{P}(\cdot|B)$ obtained by applying Kolmogorov's ratio definition to probability measures in $\mathcal{K}$. A second possibility is that $\mathbb{P}(B) = 0$ for all probability measures in $\mathcal{K}$; inspired by Kolmogorov's definition, we may then leave undefined the conditional credal set given B. And there is a third possibility: we may have some measures in $\mathcal{K}$ such that $\mathbb{P}(B) = 0$, and yet other measures in $\mathcal{K}$ such that $\mathbb{P}(B) > 0$. One strategy could be to leave $\mathcal{K}_B$ undefined in this case [16]. However, this event B may happen with some positive probability, thus the possibility that B is observed cannot be ignored; what can be said about preferences in case B is indeed observed?[9]

To avoid this difficulty, one may adopt a theory of probability that allows conditioning on events of zero probability, as advocated by de Finetti [13], Renyi [26], and Popper [23]. In fact there has been work on sets of de Finetti-style probabilities [7] or sets of full conditional probabilities in Dubin's sense [9]. Another strategy is pursued by Walley, who also applies conditioning regardless of the conditioning event, but moves away from probabilities as the basic representation (Walley's theory is discussed later). Such approaches are quite distinct from the one we have followed in this paper, as we have focused here on sets of "usual" probability measures.

If one is to stay with sets of probability measures, the most popular way to define conditioning for credal sets is *regular conditioning*, where $\mathcal{K}_B$ is computed by discarding measures assigning probability zero to B:

$$\mathcal{K}_B = \{\mathbb{P}(\cdot|B) : \mathbb{P} \in \mathcal{K}, \mathbb{P}(B) > 0\}; \tag{2}$$

if the latter set is empty, then $\mathcal{K}_B$ is left undefined. This concept is closely related to Walley's *regular extensions*, to be discussed later in this section.

We have the following result. Here, and in the remainder of this paper, we use $\mathbb{I}_B$ to refer to the indicator function of B.

Theorem 19. *If $\mathcal{K}$ is an evenly convex credal set and $\mathcal{D}$ is the set of desirable gambles induced by $\mathcal{K}$ (that is, $X \in \mathcal{D}$ iff $\forall \mathbb{P} \in \mathcal{K}$: $\mathbb{E}_{\mathbb{P}}[X] > 0$), and B is an event such that $\mathbb{P}(B) > 0$ for some $\mathbb{P} \in \mathcal{K}$, then*

---

[9] A more extreme situation is this. Suppose we have a credal set such that each state $\omega \in \Omega$ is assigned probability zero for some measure in $\mathcal{K}$. Leaving $\mathcal{K}_\omega$ undefined for each ω would be unreasonable: whatever state ω we might observe, nothing would be said about preferences given ω.

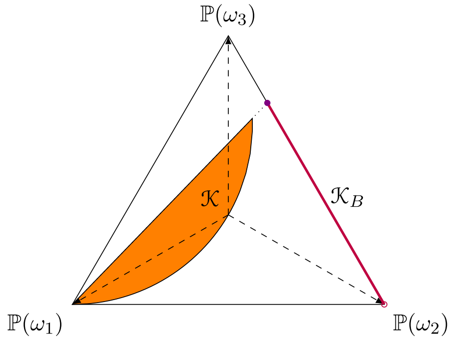

Fig. 8. The unitary simplex, the credal set $\mathcal{K}$ described in Example 20, and the conditional credal set $\mathcal{K}_B$, where $B = \{\omega_2, \omega_3\}$, viewed from point $(1, 1, 1)$. The credal set $\mathcal{K}$ is closed and touches the probability measure that assigns $\mathbb{P}(\omega_1) = 1$; this measure is discarded when regular conditioning is applied given B. The conditional credal set $\mathcal{K}_B$ is an interval from $(0, 1, 0)$ to $(0, 1/4, 3/4)$, containing the latter point but not the former one; it does not contain the conditional probability measure that assigns $\mathbb{P}(\omega_2|B) = 1$.

$$\mathcal{K}_B = \{\mathbb{I}_B p : \forall X \in \mathcal{D} : X \cdot p > 0 \textit{ and } \mathbb{I}_B \cdot p = 1\},$$

an evenly convex credal set.

Proof. As $\mathcal{K}$ is evenly convex, it is the intersection of a cone $\mathcal{C} = \{p : X \cdot p > 0, \forall X \in \mathcal{D}\}$ and the unitary simplex. To produce $\mathcal{K}_B$, we first focus on the points $q = \mathbb{I}_B p$, for each $p \in \mathcal{C}$ such that $\mathbb{I}_B \cdot p > 0$. That is, we start with the set

$$\mathcal{K}'_B = \{q : q = \mathbb{I}_B p \text{ and } \forall X \in \mathcal{D} : X \cdot p > 0 \text{ and } \mathbb{I}_B \cdot p > 0\}.$$

This set is a cone: if $q_1 = \mathbb{I}_B p_1$ and $q_2 = \mathbb{I}_B p_2$ satisfy the defining constraints, $q_1 + q_2$ is equal to $\mathbb{I}_B(p_1 + p_2)$ and λq_1 is equal to $\mathbb{I}_B \lambda p_1$ for all $\lambda > 0$, and clearly we have $X \cdot (p_1 + p_2) > 0$ and $\lambda p_1 > 0$ for all $X \in \mathcal{D}$ and $\mathbb{I}_B \cdot (p_1 + p_2) > 0$ and $\mathbb{I}_B \lambda p_1 > 0$; hence both $q_1 + q_2$ and λq_1 for $\lambda > 0$ belong to the set.

So, take the intersection of $\mathcal{K}'_B$ with the unitary simplex to "normalize" each point in the cone (this is equivalent to dividing q by $\mathbb{I}_B \cdot q$ when conditioning). By collecting these constraints, we obtain

$$\mathcal{K}_B = \{q : q = \mathbb{I}_B p \text{ and } \forall X \in \mathcal{D} : X \cdot p > 0 \text{ and } \mathbb{I}_B \cdot q = 1 \text{ and } \mathbb{I}_B \cdot p > 0\},$$

a set defined solely by linear constraints (thus evenly convex). As the constraint $\mathbb{I}_B \cdot q > 0$ is redundant given $\mathbb{I}_B \cdot q = 1$, we can simplify the description of $\mathcal{K}_B$ to obtain the expression in the theorem — an expression that is akin to the Charnes–Cooper transformation employed in linear fractional programming [6]. □

To understand why a theory of evenly convex sets is needed when regular conditioning is adopted, note that $\mathcal{K}_B$ may have open faces *even if* $\mathcal{K}$ *is closed*. This fact has been noticed before, and one can find a relatively simple example in Ref. [11, Example 1]. Here is an even simpler example:

Example 20. Suppose $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and consider the credal $\mathcal{K}$ set containing all probability measures that satisfy

$$3\mathbb{P}(\omega_2) \geq \mathbb{P}(\omega_3) \quad \text{and} \quad (\mathbb{P}(\omega_1) - 1)^2 + \mathbb{P}(\omega_2)^2 + (\mathbb{P}(\omega_3) - 1)^2 \leq 1.$$

The resulting credal set is depicted in Fig. 8. Denote by B the event $\{\omega_2, \omega_3\}$. Using regular conditioning, $\mathcal{K}_B$ is an evenly convex set consisting of all conditional probability measures such that $\mathbb{P}(\omega_2|B) \in [1/4, 1)$. This conditional credal set is also depicted in Fig. 8. □

Expression 2 defines regular conditioning solely in terms of probabilities; clearly it would be very useful to extract regular conditioning directly from preferences. Alas finding such a characterization with an intuitive content seems to be a difficult problem that is left open here.

Walley also considers the connection between regular extension and preferences [33, Appendix F]. His approach deserves some discussion as it combines representations based on lower previsions with regular extensions.[10] Recall that a *lower prevision* $\underline{P}$ is, under mild assumptions, a superadditive functional (that is, $\underline{P}(X+Y) \geq \underline{P}(X) + \underline{P}(Y)$) such that $\underline{P}(X) \geq \inf_\omega X(\omega)$ and $\underline{P}(\lambda X) = \lambda \underline{P}(X)$ for $\lambda > 0$. A lower prevision can be represented by a closed convex credal set $\mathcal{K}$ as $\underline{P}(X) = \inf_{\mathbb{P} \in \mathcal{K}} \mathbb{E}_{\mathbb{P}}[X]$. A lower prevision $\underline{P}$ can always be represented by some convex cone of gambles $\mathcal{G}$ in the sense that, for any X, $\underline{P}(X) = \sup(\lambda : X - \lambda \in \mathcal{G})$. And the *upper prevision* of X is $\overline{P}(X) = -\underline{P}(-X)$. For an event B with indicator function $\mathbb{I}_B$,

[10] The remainder of this section perhaps requires some understanding of Walley's theory of imprecise probabilities [33].

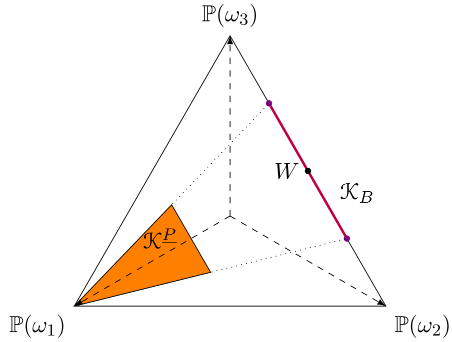

Fig. 9. The credal set $\mathcal{K}^{\underline{P}}$ in Example 21, viewed from point $(1, 1, 1)$. The conditional credal set $\mathcal{K}_B$ is a closed interval in the plane defined by $\mathbb{P}(\omega_1) = 0$. The gamble W introduced in that example is also shown.

the *lower probability* of B, denoted by $\underline{P}(B)$, is defined to be $\underline{P}(\mathbb{I}_B)$; the *upper probability* of B, denoted by $\overline{P}(B)$, is $\overline{P}(\mathbb{I}_B)$. Given a number of assessments $\underline{P}(X_i) = \alpha_i$, Walley's natural extension $\underline{E}$ is the smallest lower prevision that agrees with the assessments. For any given X, $\underline{E}(X)$ can be calculated by building the (closed convex) credal set $\mathcal{K}^{\underline{P}}$ that consists of all probability measures satisfying $\mathbb{E}_{\mathbb{P}}[X_i] \geq \alpha_i$, and then returning $\underline{E}(X) = \inf_{\mathbb{P}\in\mathcal{K}^{\underline{P}}} \mathbb{E}_{\mathbb{P}}[X]$ for any X.

The idea of Walley's regular extensions [33, Appendix F] is to first extract a set of gambles from $\underline{P}$, to be interpreted as the set of gambles that are "actually" desirable:

$$\mathcal{R}_2 = \{X : (\underline{P}(X) \geq 0 \text{ and } \overline{P}(X) > 0) \text{ or } (X \geq 0)\};$$

however this set is not appropriate here because it contains the origin. A solution is to use Couso and Moral's approach to regular extensions, as they adapt Walley's definition to a setting where the origin is not desirable [8, Expression (10)]. In short, Couso and Moral extract from a given $\underline{P}$ the following set of gambles:

$$\mathcal{G}_{\underline{P}} = \{X : (\underline{P}(X) \geq 0 \text{ and } \overline{P}(X) > 0) \text{ or } (X \geq 0 \text{ and } X \neq 0)\}.$$

The regular extension is then defined as a lower prevision induced by $\mathcal{G}_{\underline{P}}$:

$$\underline{P}(X|B) = \sup\left(\lambda : B(X - \lambda) \in \mathcal{G}_{\underline{P}}\right).$$

With this scheme, as desired, we have that $\underline{P}(X|B) = \inf_{\mathbb{P}\in\mathcal{K}^{\underline{P}}:\mathbb{P}(B)>0} \mathbb{E}_{\mathbb{P}}[X|B]$ when $\overline{P}(B) > 0$; besides, when $\overline{P}(B) = 0$, then $\underline{P}(X|B) = \inf\{X(\omega) : \omega \in B\}$.[11]

This approach is elegant because it concentrates all conditional information into a single set of gambles $\mathcal{G}_{\underline{P}}$. However, the interpretation of this set of gambles in terms of probabilities may be difficult, as suggested by the next example.

Example 21. Suppose $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and consider assessments $\underline{P}(\omega_1) = 1/2$ and $\underline{P}(3\mathbb{I}_{\omega_2} - \mathbb{I}_{\omega_3}) = 0$ and $\underline{P}(3\mathbb{I}_{\omega_3} - \mathbb{I}_{\omega_2}) = 0$. Then

$$\mathcal{K}^{\underline{P}} = \{\mathbb{P} : \mathbb{P}(\omega_1) \geq 1/2 \quad \text{and} \quad 3\mathbb{P}(\omega_3) \geq \mathbb{P}(\omega_2) \geq \mathbb{P}(\omega_3)/3\}$$

is a closed and convex set, as depicted in Fig. 9.

The constraints clearly allow $\mathbb{P}(\omega_1) = 1$, in which case the probability of $B = \{\omega_2, \omega_3\}$ is zero. Indeed, we have $\underline{E}(\mathbb{I}_B) = 0$. Now, is it the case that B can have probability zero? In Walley's approach, it seems the answer is no. To understand this, consider a gamble W such that $W(\omega_1) = 0$ and $W(\omega_2) = W(\omega_3) = 1/2$. This gamble belongs to $\mathcal{G}_{\underline{P}}$ as $W \geq 0$ and $W \neq 0$. If $\mathcal{G}_{\underline{P}}$ is to encode the "actually" desirable gambles, then it must be that $\sum_i \mathbb{P}(\omega_i) W(\omega_i) > 0$; that is, $\mathbb{P}(\omega_2)/2 + \mathbb{P}(\omega_3)/2 > 0$, or equivalently $\mathbb{P}(B) > 0$. By attributing desirability to elements of $\mathcal{G}_{\underline{P}}$, a "hidden" constraint is imposed, a constraint that requires $\mathbb{P}(B) > 0$.

That is, a conditioning event with zero *lower* probability cannot be interpreted, in Walley's approach, as an event that can actually have zero probability, even when the input assessments allow this zero probability to be assigned. Although "hidden" constraints removing zero probabilities do not affect lower/upper previsions, they may affect a representation of $\mathcal{G}_{\underline{P}}$ based on credal sets. There is a difference between the closed convex credal set that represents the input lower previsions

[11] Here is a short proof. If $\overline{P}(B) = 0$ then $\underline{P}(B(X-\lambda)) = \overline{P}(B(X-\lambda)) = 0$ and we have $\underline{P}(X|B) = \sup(\lambda : B(X-\lambda) \geq 0 \text{ and } B(X-\lambda) \neq 0)$; hence $\underline{P}(X|B) = \inf(X(\omega) : \omega \in B)$. And if $\overline{P}(B) > 0$, then the conditions $B(X-\lambda) \geq 0$ and $B(X-\lambda) \neq 0$ become irrelevant as $\underline{P}(X|B) \geq \inf(X(\omega) : \omega \in B)$ with or without them; we thus have $\underline{P}(X|B) = \sup(\lambda : \inf_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}[BX] - \mathbb{E}_{\mathbb{P}}[B\lambda]) \geq 0 \text{ and } \sup_{\mathbb{P}}(\mathbb{E}_{\mathbb{P}}[BX] - \mathbb{E}_{\mathbb{P}}[B\lambda]) > 0)$. Consequently, $\underline{P}(X|B) = \sup(\lambda : \inf_{\mathbb{P}:\mathbb{P}(B)>0} \mathbb{E}_{\mathbb{P}}[X|B] \geq \lambda \text{ and } \sup_{\mathbb{P}:\mathbb{P}(B)>0} \mathbb{E}_{\mathbb{P}}[X|B] > \lambda)$; that is, $\underline{P}(X|B) = \min(\inf_{\mathbb{P}:\mathbb{P}(B)>0} \mathbb{E}_{\mathbb{P}}[X|B], \sup_{\mathbb{P}:\mathbb{P}(B)>0} \mathbb{E}_{\mathbb{P}}[X|B]) = \inf_{\mathbb{P}:\mathbb{P}(B)>0} \mathbb{E}_{\mathbb{P}}[X|B]$.

and the evenly convex credal set that should be used to represent the "actually" desirable gambles, but Walley's approach is to focus on the former representation: the difference is washed over by taking infima and suprema.

As another point, take a gamble Z such that $Z(\omega_1) = 0$, $Z(\omega_2) = -1$, $Z(\omega_3) = 3$. This gamble also belongs to $\mathcal{G}_{\underline{P}}$ as $\underline{\mathbb{P}}(Z) = 0$ and $\overline{\mathbb{P}}(Z) = 1$. If we were to encode the desirability of Z by imposing $\mathbb{E}_{\mathbb{P}}[Z] > 0$ for all $\mathbb{P}$ in the representing credal set, it should be the case that $3\mathbb{P}(\omega_3) - \mathbb{P}(\omega_2) > 0$. But this constraint removes all probability measures on the lower boundary of the set $\mathcal{K}^{\underline{P}}$, hence removing every probability measure such that $\mathbb{P}(\omega_2|B) = 3/4$. Because lower/upper previsions are not affected by boundary behavior, we obtain $\overline{P}(\omega_2|B) = 3/4$ even though we cannot have $\mathbb{P}(\omega_2|B) = 3/4$. However, regular conditioning applied to $\mathcal{K}^{\underline{P}}$ produces a *closed* conditional credal set $\mathcal{K}_B$ also depicted in Fig. 9: each extreme point is actually attained using probability measures within $\mathcal{K}^{\underline{P}}$. To repeat, in Walley's approach there is a difference between the representation of lower previsions and the representation of "actually" desirable gambles when regular conditioning is used. The difficulty, of course, is that lower previsions are always represented by closed convex credal sets in Walley's theory, while regular conditioning requires evenly convex ones. □

6. Conclusion

We have presented a few axioms on preference orderings that, together, imply a representation through evenly convex credal sets. This representation lets one handle assessments of strict inequality for probabilities, and go beyond what can be done with closed convex credal sets. The main idea is to adopt a novel Archimedean condition (even continuity) that implies even convexity. A similar representation can be obtained using SSK-continuity in many, but not all, cases.

We have also discussed natural extensions and, more importantly, the concept of regular conditioning, a concept that is inextricably linked to evenly convex credal sets.

Future work should look at more *general* sets of probabilities, mimicking results by Seidenfeld, Schervish, and Kadane [30].

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Appendix A. Obtaining a unique probability measure

Aumann's continuity suffices to obtain a unique probability measure when absence of preference is an equivalence relation. This result is in essence well-known, but it seems that a detailed proof is not available in the literature.

Proposition 22. *If a preference ordering $\succ$ satisfies monotonicity, cancellation, and Aumann's continuity, and absence of preference $\not\sim$ is an equivalence, then its representing set of desirable gambles $\mathcal{D}$ is an open halfspace, and its representing credal set $\mathcal{K}$ is a singleton.*

We start with a useful property [15, Theorem 2.1]: when $\not\sim$ is an equivalence, then

$$X \not\sim Y \wedge Y \succ Z \Rightarrow X \succ Z. \tag{A.1}$$

To prove this, assume $X \not\sim Y$ and $Y \succ Z$, and note: if $X \not\sim Z$, by transitivity of $\not\sim$ we have $Y \not\sim Z$ violating $Y \succ Z$; and if $Z \succ X$, by transitivity of $\succ$ we have $Y \succ X$ violating $X \not\sim Y$.

Proof. Part 1) Take X on the boundary of $\mathcal{D}$; to create a contradiction, assume $X \in \mathcal{D}$. Consider that for any $\delta > 0$, both $X - \delta$ and $X - \delta/2$ are not in $\mathcal{D}$ (if they were, then X would not be on the boundary). So, either $X - \delta \not\sim 0$ or $0 \succ X - \delta$, and likewise $X - \delta/2 \not\sim 0$ or $0 \succ X - \delta/2$. In fact, $0 \succ X - \delta$ for every $\delta > 0$ [to show this, suppose that $X - \delta \not\sim 0$ for some $\delta > 0$; if $X - \delta/2 \not\sim 0$ as well, then by transitivity $X - \delta/2 \not\sim X - \delta$, violating monotonicity; if instead $0 \succ X - \delta/2$, then $X - \delta \succ X - \delta/2$ (using Expression (A.1)), again violating monotonicity]. But this implies that $-X + \delta \succ 0$ for every $\delta > 0$, so $\alpha(-X + \delta) + (1 - \alpha)(-X) \succ 0$ for all $\alpha > 0$. By Aumann's continuity, we have either $-X \succ 0$ or $-X \not\sim 0$, contradicting the assumption $X \succ 0$. So a gamble on the boundary of $\mathcal{D}$ is not in $\mathcal{D}$, hence $\mathcal{D}$ is open.

Part 2) Because $\mathcal{D}$ is open, it can be represented by a credal set $\mathcal{K}$. Suppose there are distinct $\mathbb{P}_1$ and $\mathbb{P}_2$ and X such that $\mathbb{E}_{\mathbb{P}_1}[X] < \mathbb{E}_{\mathbb{P}_2}[X]$; then one can find μ and $\delta > 0$ such that $\mathbb{E}_{\mathbb{P}_1}[X] < \mu < \mu + \delta < \mathbb{E}_{\mathbb{P}_2}[X]$. Now $\mathbb{E}_{\mathbb{P}_1}[X - \mu] < 0$ implies $\neg(X - \mu \succ 0)$, and $\mathbb{E}_{\mathbb{P}_2}[X - \mu] > 0$ implies $\neg(0 \succ X - \mu)$; hence $X - \mu \not\sim 0$. But similarly $X - \mu - \delta \not\sim 0$, and by transitivity of $\not\sim$ we have $0 \not\sim -\delta$, violating monotonicity. Hence $\mathbb{P}_1 = \mathbb{P}_2$. Now note that if the credal set is a singleton containing $\mathbb{P}$, then the set of desirable gambles is the open halfspace $\mathbb{E}_{\mathbb{P}}[f] > 0$. □

Appendix B. A separation property for evenly convex cones

If $\mathcal{A}$ is evenly convex, then if $X \in \mathcal{A}$ and $Y \in \mathrm{cl}\mathcal{A}$ we have $\alpha X + (1 - \alpha)Y \in \mathcal{A}$ for $\alpha \in (0, 1)$ [14, Section 3.5]. Consequently:

Lemma 23. *Suppose $\mathcal{A}$ is evenly convex and $0 \notin \mathcal{A}$. If X and $-X$ belong to $\mathrm{cl}\mathcal{A}$, then neither is in $\mathcal{A}$.*

Proof. If $X \in \mathcal{A}$, then $-X \in \mathrm{cl}\mathcal{A}$ implies $X/2 + (-X)/2 = 0 \in \mathcal{A}$, a contradiction; hence $X \notin \mathcal{A}$. By similar reasoning, $-X \notin \mathcal{A}$. □

We then obtain the following separation property:

Theorem 24. *Suppose $\mathcal{A}$ is an evenly convex cone such that $0 \notin \mathcal{A}$. If $X \notin \mathcal{A}$, then there is p such that $X \cdot p \leq 0$ and $Y \cdot p > 0$ for all $Y \in \mathcal{A}$.*

Proof. Part 1) Suppose $X \in \mathrm{cl}\mathcal{A}$, but $X \notin \mathcal{A}$. Because $\mathcal{A}$ is evenly convex, there is p and β such that $X \cdot p = \beta$ and $Y \cdot p > \beta$ for all $Y \in \mathcal{A}$ [17, Proposition 3.1(ii)]. If $\beta > 0$, then for any Y in a neighborhood of 0 we have $\epsilon Y \cdot p < \beta$ for some $\epsilon > 0$; this is a contradiction because some such Y is in $\mathcal{A}$, and for this Y we must have $\epsilon Y \cdot p > \beta$. Hence $\beta \leq 0$. We now show that actually $\beta = 0$.

For $Y \in \mathcal{A}$, $Y \cdot p > \beta = X \cdot p$, hence $(Y - X) \cdot p > 0$. Because X is on the boundary of $\mathcal{A}$, there is a gamble Y in a neighborhood of X that belongs to $\mathcal{A}$; define $Z = Y - X$, and note that the segment from Y to X (excluding X) is in $\mathcal{A}$ [14, Section 3.5]. That is, there is Z such that $Z \cdot p > 0$ and $(X + \epsilon Z) \cdot p > \beta$ for $\epsilon > 0$ in a neighborhood of 0. Now for any $\lambda > 0$ we have $\lambda(X + \epsilon Z) \in \mathcal{A}$. That is, $\lambda(X + \epsilon Z) \cdot p > \beta$, so $X \cdot p > \beta/\lambda - \epsilon Z \cdot p$. Again use $X \cdot p = \beta$, to obtain $\beta > \beta/\lambda - \epsilon Z \cdot p$. Consequently, we have both $\beta \leq 0$ and $\beta > -\epsilon Z \cdot p/(1 - 1/\lambda)$; take say $\lambda = 2$ to obtain the constraint $\beta > -\epsilon(2Z \cdot p)$. These conditions can only be satisfied for $\epsilon > 0$ if $\beta = 0$.

Part 2) Now suppose instead that $X \notin \mathrm{cl}\mathcal{A}$. Consider the cone $\mathcal{B} = \{\lambda X : \lambda \geq 0\}$. Using an appropriate separation result [18, Theorem 2.5], we know that there is p such that $Y \cdot p > 0$ for $Y \in \mathrm{cl}\mathcal{A} \backslash (\mathrm{cl}\mathcal{A} \cap -\mathrm{cl}\mathcal{A})$, $Y' \cdot p = 0$ for $Y' \in (\mathrm{cl}\mathcal{A} \cap -\mathrm{cl}\mathcal{A}) \cup (\mathcal{B} \cap -\mathcal{B})$, $Y'' \cdot p \leq 0$ for $Y'' \in \mathcal{B} \backslash (\mathcal{B} \cap -\mathcal{B})$. Clearly $\mathcal{B} \cap -\mathcal{B}$ contains just the zero gamble. Now note that $\mathrm{cl}\mathcal{A} \cap -\mathrm{cl}\mathcal{A}$ does not intersect $\mathcal{A}$ (if $Y \in \mathrm{cl}\mathcal{A} \cap -\mathrm{cl}\mathcal{A}$, then $Y \in \mathrm{cl}\mathcal{A}$ and $-Y \in \mathrm{cl}\mathcal{A}$, so both are not in $\mathcal{A}$ by Lemma 23). Hence there is p such that $X \cdot p \leq 0$ and $Y \cdot p > 0$ for $Y \in \mathcal{A}$. □

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