Notes on "Notes on Conditional Previsions"

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Abstract

These notes comment on Williams' fundamental essay *Notes on Conditional Previsions*, written as a research report in 1975 and published in the present issue. Basic aspects of that work are discussed, including historical background and relevance to the foundations of probability; examples are supplied to help understanding.

Key words: Previsions, Imprecise Previsions, Envelope Theorem, Natural Extension, Coherence, Generalized Bayes Rule.

1 Precise previsions and precise conditional previsions

P. M. Williams wrote his *Notes on Conditional Previsions* [22] in 1975, one year after the English translation [9] of de Finetti's book *Teoria delle probabilità* was published. We offer some guidance on the main contributions of Williams' essay, as a revised version by the author appears in the present issue (we use the numbering in the new version of the paper).

To better understand Williams' essay, it is useful to recall some of the innovative ideas in de Finetti's work:

(a) A precise evaluation of a *random quantity* X (i.e., a *bounded* random variable, or a *gamble* [21]) can be made by selecting a real number $P(X)$ called

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the *prevision* of X. To elicit $P(X)$ it is not necessary to preliminarily assess a probability distribution on the possible values of X, a distribution function for X or something else. However, $P(X)$ must be a *coherent* evaluation. A coherent $P(X)$ can be interpreted in an idealized betting scheme as an individual's fair price for either buying or selling X, that is the price for which she would accept any random gain $G = c(X - P(X))$, where c is an arbitrary real value chosen by a competitor (when c is positive, the individual buys X; when c is negative, she sells it, or buys $-X$). Obviously the bet would not be acceptable if $\sup G < 0$ for some c and a given $P(X)$, because then the individual would lose money whatever happens. Thus coherence for $P(X)$ requires that $\sup G \geq 0$.

(b) Previsions can be announced on *any* set D of random quantities (that is, D need not be a linear space, nor a cone, nor any structured set). Coherence requires that any bet on any finite number of random quantities in D is such that the supremum of the corresponding overall gain is non-negative. That is to say, P is coherent on D if and only if, for all $n \in \mathbf{N}$, for $X_1, \ldots, X_n \in D$, for real $c_1, \ldots, c_n$, defining $G = \sum_{i=1}^{n} c_i(X_i - P(X_i))$, we have $\sup G \geq 0$.

(c) An important feature in this approach is that it must be possible to extend a coherent prevision on D to any superset D' of D, in such a way that the extension is coherent on D'. Thus an individual can always extend the set of random quantities under evaluation, without necessarily modifying her earlier judgements.

The role of a prevision is similar to that of an expectation. In fact, recall from (a) that assessing a prevision $P(X)$ for X does not require assessing also a distribution function $F(X)$ for X; however, when $F(X)$ and hence the expectation $E(X)$ is known, then the prevision $P(X)$ is uniquely determined and $P(X) = E(X)$ (the correspondence is less immediate with unbounded random quantities [6]). Not surprisingly then, (coherent) previsions are linear functionals. Note that de Finetti (and Williams) uses the same symbol for both an event and its indicator function—the reader must operate himself the distinction in each case. Thus $P(A)$ denotes the prevision of the indicator function of event A, and it is also taken as the definition of the probability of event A. Furthermore, Williams's usage of the symbol $X|E$ should be better interpreted as the random quantity X conditional on the event E.

Let us illustrate some features of de Finetti's approach to probability with a simple example (a didactic discussion of de Finetti's theory in the finite case can be found in reference [14]).

Example 1 *You are in a pub where two people are about to throw darts. For each player, there are two outcomes:* 1 *if player hits the bull's eye;* -1 *otherwise. Call* X_1 *and* X_2 *the outcomes for the first and the second player respectively; both are random quantities with values in* $\mathcal{A} = \{-1, 1\}$*. You can bet on the outcomes: a transaction on* X_i *with the* j*-th person in the pub*

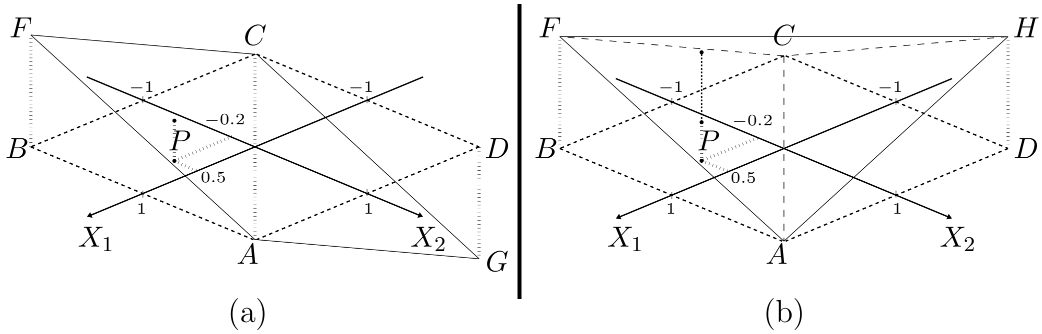

Fig. 1. Random quantities and sets of coherent previsions in Example 1.

requires you to give $c_{ij}P(X_i)$ *dollars and to receive* $c_{ij}X_i$ *dollars—here* c_{ij} *is chosen by the* j*-th person and* $P(X_i)$ *is your prevision (fair price) for* X_i*. Letting* $c_i := \sum_j c_{ij}$*, your net gain is* $G = c_1(X_1 - P(X_1)) + c_2(X_2 - P(X_2))$*. Coherence requires you to assess previsions such that for no pair* (c_1, c_2)*,* G *is negative for all the values that* X_1 *and* X_2 *may assume in* $\mathcal{A}$*. For example,* $P(X_2) = 2$ *is not a coherent assessment, as* $(c_1, c_2) = (0, 1)$ *leads to a sure loss:* $G = X_2 - 2$ *is negative whatever the outcome* X_2*.*

Suppose you assess the prevision P *on* $D = \{X_1, X_2\}$ *such that* $P(X_1) = 0.5$ *and* $P(X_2) = -0.2$*. De Finetti shows that assessment* $(P(X_1), P(X_2))$ *is coherent when it lies in the convex hull of* $\mathcal{B} = \mathcal{A} \times \mathcal{A}$ *(this is the square with vertices A, B, C and D in Fig. 1.a). So your prevision* $(0.5, -0.2)$*, indicated by point* P *in the figure, is coherent.*

Consider now transactions on other random quantities, such as $X_3 = X_1 - X_2$*. The coherent values of* (X_1, X_2, X_3) *are* $\{(x_1, x_2, x_3) : (x_1, x_2) \in \mathcal{B}, x_3 = x_1 - x_2\}$*. The possible values of* $(P(X_1), P(X_2), P(X_3))$ *consist of the convex hull of this set, i.e. the polytope with vertices A, F, C and G in Fig. 1.a, where the non-displayed vertical axis represents* X_3 *(hence* F *is point* $(1, -1, 2)$ *and* G *is point* $(-1, 1, -2)$*). When* $(P(X_1), P(X_2)) = (0.5, -0.2)$*, the coherent value of* $P(X_3)$ *is precisely 0.7 (this is an example of what de Finetti called the* fundamental theorem of prevision*). A simpler argument uses the linearity of previsions:* $P(X_3) = P(X_1 - X_2) = P(X_1) - P(X_2) = 0.7$*. Now consider a* nonlinear *relationship, where* $X_4 = |X_1 - X_2|$*. The space of possibilities for* (X_1, X_2, X_4) *is* $\{(x_1, x_2, x_4) : (x_1, x_2) \in \mathcal{B}, x_4 = |x_1 - x_2|\}$*. Its convex hull is the tetrahedron with vertices A, C, F and H in Fig. 1.b, where the non-displayed vertical axis is now* X_4*, and the coordinates of the new point* H *are* $(-1, 1, 2)$*. The intersection of such a set with the vertical line through* P *is a segment, whose extremes are* $(0.5, -0.2, 0.7)$ *and* $(0.5, -0.2, 1.7)$*, representing the set of all previsions that are coherent with your assessment* P*. The (coherent) prevision* $P(X_4)$ *is* imprecise*, as we are only able to constrain it to the interval* $[0.7, 1.7]$ *rather than to a single point.*

Now suppose you are allowed to buy and sell bets after *the first player has thrown his dart. Therefore, you are interested in your previsions conditional on the observation of the first outcome. Let us assume that the first player* *hits the bull's eye, i.e., let us consider the event $E := (X_1 = 1)$. Denote by $P(X|E)$ your fair price for a bet, on a random quantity X, that is called off unless $E = 1$. Such a fair price is called the* conditional prevision *for X given E. When X is an event, the conditional prevision is called* conditional probability*, and it is called* probability *when further E is the sure event.* □

De Finetti discussed extensively the above items (a)–(c), and gave a fully positive answer to (c) proving an extension theorem in [7] (see also [8], pp. 78–79). The proof employed the Axiom of Choice and was a model for later extension theorems (among them, for coherent rates of exchange [2], and for previsions for unbounded random quantities [6]). De Finetti was also concerned in [9] with conditional previsions, but considered only some special cases and did not tackle the extension problem (c). It is important to understand that even in the special case of conditional probabilities de Finetti was not merely interested in the Kolmogorovian set-up where conditional probability is *defined* from unconditional (σ–additive) probability. De Finetti focused on functions $P(\cdot|\cdot)$ such that $P(\cdot|A)$ is a (finitely additive) probability measure for any event A, $P(A|A) = 1$ for any A, and $P(A \cap B) = P(A|B)P(B)$ for any A and B. Such measures had been discussed already by Keynes [12] and appeared (later than de Finetti's first proposals) in various works in different research areas [5,10]. Many AI techniques connected to ordinal uncertainty, default reasoning and counterfactual reasoning can be linked to such measures [1,3,13]. We should note that these "full" conditional measures do allow one to define conditional probability $P(A|B)$ even when $P(B)$ is equal to zero (in fact, the reader will note that Williams discusses the behavior of zero probabilities on more occasions in his paper). The important point here is that conditional previsions did not receive a "coherence-based" formal derivation in de Finetti's own work. Williams comments extensively on the difficulties of de Finetti's approach in his current Section 5.

These unsolved questions are the starting point for the first of Williams' paper objectives, that is, to give a coherence condition for conditional previsions in a sufficiently general case (this is his condition (A) in Sec. 3.4) and to prove an extension result for them (Sec. 4.3). There are also other remarkable results on conditional previsions in his essay, like their characterization in Proposition 6. Williams' solution predates several other justifications of "full" conditional measures: coherence-based derivations were to appear only in 1985 (independently by Holzer [11] and by Regazzini [18]), and related derivations based on preferences appeared later [4,15].

Williams' other main objective was that of formalizing a different coherence notion, always in a conditional environment, to generalize conditional previsions to imprecise conditional previsions. Furthermore, he solved fundamental problems in the theory of imprecise previsions, as we will see in the next section. However, research about imprecise evaluations was not very widespread in the seventies. This probably explains why Williams presented several results on imprecise previsions nearly in passing, as instrumental for his first objective, and why, for instance, most of the final Sec. 5 is devoted to precise rather than imprecise previsions.

2 Williams' imprecise previsions

Studies on imprecise probabilities (a general term which includes many uncertainty models, and imprecise previsions as well) have greatly developed since 1975. A basic reference is Walley's book [21], and we assume here some knowledge of the main notions developed there (in fact, Walley's work was influenced by Williams', as Walley acknowledges in the book's preface). Note that Williams tends to argue in terms of upper previsions, while Walley favours lower previsions. This is no real problem, since the conjugacy equality $P_*(X|E) = -P^*(-X|E)$ between lower (P_*) and upper (P^*) conditional previsions may be assumed.

A noteworthy characteristic of Williams' essay is that several important concepts are defined in terms of cones of *acceptable*, or *desirable* in the currently prevailing later terminology [21], random quantities. So Williams does not start directly from conditions on "betting" (these are obtained from cones, for example in Proposition 1). Nor Williams starts from partially ordered preferences, as many authors do—note that a partial ordering on preferences satisfying some simple conditions leads to a cone of random quantities; several relationships between cones and preferences are discussed by Walley [21, Chapter 3].

Perhaps a simple example will clarify the intuitions behind the cones discussed by Williams. Consider the set $\mathcal{X}$ of all random quantities X with two values, $X(\omega_1)$ and $X(\omega_2)$. The notation here suggests on purpose that there is an underlying possibility space with two elements, ω_1 and ω_2. Now each element of $\mathcal{X}$ is a point in the plane drawn in Fig. 2; some relevant cones can be seen in the figure. The cone $\mathcal{C}$ contains $\mathcal{X}^{\geq}$, and its intersection with $\mathcal{X}^{\leq}$ is the origin. This corresponds to conditions (C1) and (C2) in Williams' essay. We can think of b_1 (and likewise b_2) as made up of "marginally" acceptable random quantities: if $X \in b_1$, increasing (decreasing) $X(\omega_1)$ or $X(\omega_2)$ by any positive amount would make the resulting random quantity acceptable (not acceptable). We can also consider $-\mathcal{C}$, the cone spanned by $-b_1$ and $-b_2$; no random quantity in this "negative" cone can be acceptable; in fact, for $X \in -\mathcal{C}$, we know that $-X$ is acceptable. And the region between $\mathcal{C}$ and $-\mathcal{C}$ contains random quantities that are neither acceptable nor unacceptable. (As a technical aside, the *polar cone* of $-\mathcal{C}$ intersects the interval from (0,1) to (1,0); this intersection defines a set of probability measures with vertices p_1

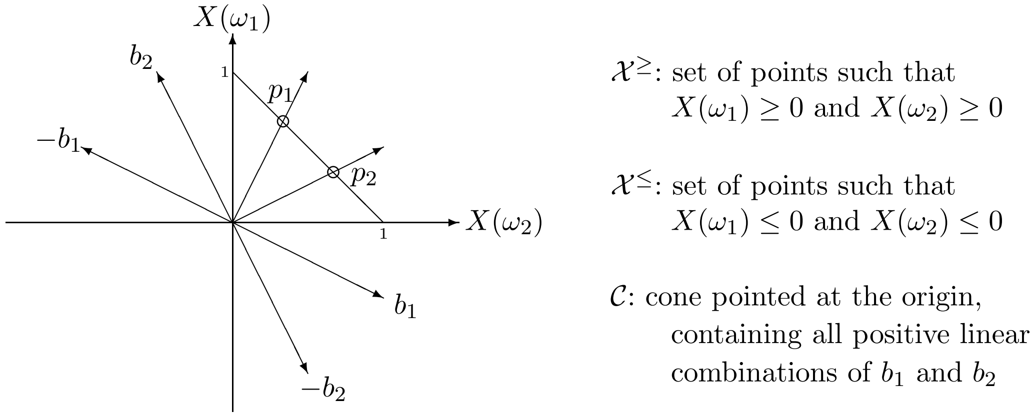

Fig. 2. Cones of random quantities (points p_1 and p_2 indicated by small circles).

and p_2.)

Let us now see in more detail the basic results achieved by Williams. The coherence condition for conditional upper previsions proposed by Williams is condition (A*) in Sec. 3.1. This condition will be referred to as *W-coherence*. Unlike de Finetti, he discussed first some simple desirability conditions ((C1') and (C2') in Sec. 3.1), and then proved that condition (A*) holds if and only if the above desirability conditions hold (Sec. 3). This approach leads to an extension theorem for upper previsions (Theorem 1, Sec. 4) *without* making use of the Axiom of Choice, thus departing from de Finetti's classical proof. It is very interesting to note that the upper prevision P^* appearing in Theorem 1 is a notable case of *natural extension* (using terminology later introduced by Walley). Therefore Williams operated a change of perspective in the proof of extension problems, not proving that a generic extension exists, but finding the least-committal one, i.e., the natural extension. This technique and the concept of natural extension were greatly emphasized later by Walley [21].

Another very important result is Theorem 2, in Sec. 4.2, which is an *envelope theorem* in current terminology. It states that an upper conditional prevision is W-coherent if and only if it is the upper envelope of some family of (precise) coherent conditional previsions. The "only if" clause is very useful in theoretical derivations, while the "if" clause also guarantees coherence of a practical way of assessing *indirectly* imprecise previsions, as envelopes of precise previsions.

Thus W-coherent upper previsions have a natural extension and are characterized by an envelope theorem. These features differentiate W-coherence from other concepts of coherence for conditional random quantities. For example, consider a form of coherence proposed by Walley [21, Sec. 7.1.4 (b)]. An imprecise conditional prevision which satisfies this kind of coherence is also W-coherent, but when replacing W-coherence with it the "only if" clause in the envelope theorem does not always hold, nor is a natural extension of its kind guaranteed to exist. Because of these advantages, W-coherence has a prominent role in the still open problem of understanding which notion of coherence is more appropriate in a conditional environment.

There are other results on W-coherent upper (and lower) previsions in Williams' essay. Often, they are introduced because they are needed for the proof of Theorem 2, but several of them are important on themselves. In particular, a special case of property (A4*) was later investigated under the name of *Generalized Bayes Rule* [21]. Furthermore, it is not difficult to prove, using the material in the paper, that the following characterization of W-coherent conditional upper previsions holds, which is analogue to that of Proposition 6, Sec. 3.4 concerning precise previsions: if P^* is a conditional upper prevision on $\mathcal{X}$, then W-coherence (i.e., condition (A*)) is equivalent to the set of four conditions (A1*)–(A4*).

Let us now consider a new example that aims at clarifying some of the concepts discussed in the present section.

Example 2 *Reconsider the set-up of Example 1, assuming that you have little or no information at all about the players. You ask three people, who are locally regarded as betting experts, for this kind of information. Each of them indeed provides you with a (precise) evaluation.*

Defining $p(x_1, x_2) := P((X_1 = x_1) \wedge (X_2 = x_2))$, one can completely describe (X_1, X_2) when a probability $p := (p(-1,-1), p(-1,1), p(1,-1), p(1,1))$ is assessed. The previsions given by experts 1, 2, 3 are based on their respective probabilities: $p_1 = (0.2, 0.1, 0.2, 0.5)$, $p_2 = (0.1, 0.2, 0.3, 0.4)$, $p_3 = (0.3, 0.1, 0.3, 0.3)$.

In fact, remember that expectations uniquely determine previsions. Therefore p_k uniquely determines a coherent (precise) prevision on the linear space $\mathcal{X}_\Omega$ of all random quantities $X : \mathcal{B} \to \mathbb{R}$ by setting $P_k(X) := \sum_{(x_1,x_2)\in\mathcal{B}} X(x_1, x_2) \cdot p_k(x_1, x_2)$. We obtain, for example, $P_1(X_1) = -1 \cdot p_1(-1,-1) - 1 \cdot p_1(-1,1) + 1 \cdot p_1(1,-1) + 1 \cdot p_1(1,1) = 0.4$, and, analogously, $P_1(X_2) = 0.2$, $P_2(X_1) = 0.4$, $P_2(X_2) = 0.2$, $P_3(X_1) = 0.2$, $P_3(X_2) = -0.2$.

At this point you have three coherent unconditional previsions on $\mathcal{X}_\Omega$. Without any further assumption, Williams' envelope theorem (Theorem 2, in Sec. 4.2) allows you to build your coherent upper prevision on $\mathcal{X}_\Omega$ out of them. This is achieved by simply setting $P^*(X) := \max_{k\in\{1,2,3\}} P_k(X)$ for all X. Your coherent lower prevision for X is then $P_*(X) = \min_{k\in\{1,2,3\}} P_k(X)$, from the conjugacy equality $P_*(X) = -P^*(-X)$. Consider, for instance, $X_3 = X_1 - X_2$ as in Example 1. By the linearity of coherent previsions, we have $P_1(X_3) = P_1(X_1) - P_1(X_2) = 0.2$, and, analogously, $P_2(X_3) = 0.2$, and $P_3(X_3) = 0.4$, from which $P_*(X_3) = 0.2$, and $P^*(X_3) = 0.4$.

It is useful to point out that there is a clear betting interpretation of coherent *lower and upper previsions. $P_*(X)$ is the supremum price μ' for which you accept buying any bet with net gain $c'(X - \mu')$, and $P^*(X)$ is the infimum price μ'' for which you accept selling any bet with net gain $c''(\mu'' - X)$, where c' and c'' are arbitrary* non-negative *real values chosen by competitors. For any price between $P_*(X)$ and $P^*(X)$ you do not express a preference between buying or selling a bet: i.e., you are undecided. (It is actually an essential feature of imprecise previsions that they allow you to express indecision.)*

Now consider the event $E = (X_1 = 1)$, as in Example 1. You are interested in evaluating your upper conditional prevision $P^*(X_2|E)$. To this extent, you can use Bayes rule to update each expert's beliefs given E, obtaining the following three probabilities: $P_1(X_2 = -1|E) = 0.2/0.7$, $P_2(X_2 = -1|E) = 0.3/0.7$, $P_3(X_2 = -1|E) = 0.3/0.6$, and, consequently, the three coherent conditional previsions $P_1(X_2|E) \simeq 0.57$, $P_2(X_2|E) \simeq 0.86$, and $P_3(X_2|E) = 1$ (note that by property (A2) in Williams' paper, you also know the experts' previsions on $aX_2|E$, for any real a). Using Williams' envelope theorem to take their upper envelope you then obtain the coherent upper prevision: $P^*(X_2|E) = 1$.*

Lastly, return to the beginning of the example and suppose the story takes another direction: you realize that you can bet on $D' = \{X_1, X_2, X_2|E\}$, and the experts only supply you with their own precise previsions on D', without letting you know any further assessment of theirs (since previsions for random quantities not in D' are unnecessary for your betting purposes). Can you still apply Williams' envelope theorem? Strictly speaking, the answer is no, because Williams' coherence is not even defined on sets like D', it requires some structure properties. For instance $X_1 \in D', X_2 \in D'$ should imply $X_1 + X_2 \in D'$. Of course, there is (more than) one way–out. The matter will be discussed in the next section.

3 Beyond Williams' coherence?

When considering W-coherence (condition (A*)), one might wonder: does it correspond to bullet (b) in our Sec. 1? In other words, does it apply to arbitrary sets of conditional random quantities? Strictly speaking, the answer is no, because the quantities $X|E$ in (A*) are such that the events E are (non-impossible and) essentially arbitrary (their corresponding indicator functions must belong to an arbitrarily large linear space), but for each E, $P^*(X|E)$ is assigned for any X in a linear space $\mathcal{X}_E$ (Sec. 3.1). In Example 2 for instance, D' should be "fatter," including in particular also $\mathcal{X}_\Omega$. However a W-coherence definition without any constraint on each X and E can be introduced, and was employed, for instance, in [20]. It is sufficient for its consistency to note that this version of W-coherence allows for coherent extensions on any superset, and hence also on supersets satisfying the conditions of (A*) (this can be proved following the scheme of de Finetti's extension theorem, or also adapting results and proofs concerning the convex natural extension in [16], Sec. 5). Therefore W-coherence can be slightly generalized to a nimbler, structure-free version, which in the unconditional case reduces to the coherence condition in [21], Sec. 2.5.4 (a).

An interesting question is: are there consistency notions more general than W-coherence, that substantially preserve essential properties of W-coherence, like the envelope theorem or the Generalized Bayes Rule? This problem was investigated in [16,17], where (centered) convex conditional previsions were introduced. It turns out that (Williams') Generalized Bayes Rule holds for this kind of conditional previsions too, while envelope theorems can be stated in some special cases, but in the general case no simple analogue of Williams' envelope theorem could so far be found. Although convex previsions turn out to be useful for various kinds of problems, including applications to risk measurement, W-coherence seems to be rather close to the most general consistency notion for which characterizations via envelope theorems are easily applicable.

Another generalization concerns the possibility of enlarging W-coherence to conditional random quantities that are *not necessarily* bounded. This area has been investigated only very recently; some early results can be found in [19].

Finally, although Williams' contribution offers solutions to many questions about conditional imprecise previsions, there are several other open problems. To the best of our knowledge, the one raised in the 1975 report and now forming Sec. 4.2.1 is still among them.

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