NUMERICAL INVESTIGATIONS ON PARAMETRIC EXCITATION OF A VERTICAL BEAM UNDER PRESCRIBED AXIAL DISPLACEMENTS

Guilherme Rosa Franzini, Alfredo Gay Neto
Polytechnique School at University of São Paulo, Av. Prof. Almeida Prado, 83, 05508-9000, São Paulo, Brazil
emails: gfranzini@usp.br; alfredo.neto@gmail.com

This paper presents and discusses results of parametric excitation. This is done studying the scenario of a vertical beam, subjected to its self-weight. The excitation is done through prescribed axial harmonic displacements at its upper extremity. Two numerical models were employed, namely, an in-house geometric nonlinear finite element code and a reduced-order model. The first is addressed using 3D beam modeling, while the second constructs a 2D solution composed by the first three vibration modes. The reduced-order model considers small deflections and includes nonlinearities arisen from extensibility of the structure. This leads to a nonlinear system of Duffing-like equations, subjected to parametric excitation. The focus of the paper is the presentation of the reduced-order model, as well as comparisons between the mentioned models in a condition in which the first vibration mode is parametrically excited. Results of time histories of displacement in three points along the beam span are presented and showed both qualitative and quantitative adherence between the the numerical approaches herein presented.

1. Introduction

Generally, systems in which one of the parameters is dependent on the time can be subjected to parametric excitation. In the case of time-dependent stiffness, the motion equation is named as Hill’s equation. Particularly, if the stiffness varies harmonically with time, Hill’s equation is renamed as Mathieu’s equation. Depending on the combination of the amplitude and frequency of the parametric excitation, the origin is not a stable solution, such that small perturbations on the initial conditions leads to response significantly different from the origin. A standard approach to check the stability of the trivial solution is by means of the Strutt’s diagram. The books [1] and [2] are examples of the theoretical basis of parametric excitation.

Besides the interest of the academic community on the theme, parametric excitation is also important on several fields of engineering. Particularly to the offshore engineering, the dynamics of slender structures such as risers and tethers of Tension Leg Platforms (TLPs) may be significantly affected by the motions of the floating units. These motions lead to modulation in the geometric stiffness (i.e., the stiffness associated to the axial force) and, depending on the combination of the amplitude and frequency of the motion of the floating units, parametric excitation of risers/tether may occur. Examples of studies on the parametric excitation of slender offshore structures are the papers [3] and [4].
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The objective of this investigation is to present a very first comparison of two different numerical approaches that are being adopted at University of São Paulo to study the parametric excitation of a slender beam. The first approach is based on a geometric non-linear finite element model (FEM). The second one is based on a unidirectional three-mode reduced-order model (ROM). Herein, we will present briefly both approaches and will compared results for a particular level of parametric excitation in a frequency that is favorable to the appearance of Mathieu’s instability.

The paper is structured in others five sections. In Section 2 the properties of the case study are presented and, in Section 3 the finite-element model and the software used for the simulations are detailed. The ROM will be derived from the continuum motion equation of the beam. Section 5 presents the results obtained from the mentioned approaches and Section 5 shows the final remarks and perspectives of further works.

2. Case study

As a case study, we focused on the simulations of a specific beam that has experimentally studied immersed in water, as described in [5]. Experiments of parametric excitation of the same beam carried out in air will be presented in [6]. Hence, the parameters chosen for this investigation are justified, since they will allow directly experimental-numerical correlations. Additionally, an enhanced version of the ROM herein prescribed will allow to investigate the parametric excitation of the submerged beam ([7] and [8]). Tab. 1 presents the relevant (for this investigation) beam geometric and stiffness properties and also the static tension at its upper end.

![Figure 1: Sketch of the vertical beam.](image)

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>External diameter $D$</td>
<td>22.2 mm</td>
</tr>
<tr>
<td>Unstretched length $L_0$</td>
<td>2552 mm</td>
</tr>
<tr>
<td>Stretched length $L$</td>
<td>2602 mm</td>
</tr>
<tr>
<td>Linear mass $m_l$</td>
<td>1.19 kg/m</td>
</tr>
<tr>
<td>Bending stiffness $EI$</td>
<td>0.056 Nm$^2$</td>
</tr>
<tr>
<td>Axial stiffness $EA$</td>
<td>1200 N</td>
</tr>
<tr>
<td>Static tension at the top $\bar{T}_t$</td>
<td>38.36 N</td>
</tr>
</tbody>
</table>

It is important to highlight that the bending stiffness of the beam is very small, so that the geometric stiffness is found to be dominant on the structure. Hence, a cable-like behavior is expected. Strictly speaking, the vibration modes of an non-extensible and vertical cable are defined in terms of Bessel functions owing to the spanwise variation of the tension (see, for example, [9]). The modes of vibration are quite important for the ROM definition and deserve a more detailed discussion that will be carried out in Section 4.

3. The non-linear finite element model

The nonlinear finite element model (FEM) used in this paper application is described in this section. Geometric nonlinear beam elements were used. Each cross section is considered to be a rigid body (Timoshenko beam). Finite rotations in space are considered, such as large displacements. The rotations in space are addressed using Rodrigues’ parameters, which leads to convenient algebraic expressions for rotation tensor. Updated lagrangian frameworks are constructed along the solution of

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1The geometric stiffness is related to the tension.
the nonlinear problem, in order to handle with possibilities of high geometric nonlinearities. All the
details of the formulation can be found in [10], which explains the beam static model.

Applications of this model for offshore risers, involving structural instabilities, are found in [11].
This includes large twisting examples, coupled with bending, in loop formation problems. This shows
that the applicability of the chosen model can include situations of important high-nonlinear behavior.
Thus, the nonlinear FEM can be seen as a benchmark for simplified linearized models, since the
intention is to compare the validity of the linearization and the projection of the solution on a small
number of modes.

Since here we are interested in dynamics, an integration along time has to be adopted. For that,
the Newmark method was chosen. Then, a geometric nonlinear transient analysis could be employed.
Furthermore, the structural damping was considered. For that, a classical Rayleigh damping was
employed. The details about how to implement Newmark method in the here used FEM formulation,
such as the Rayleigh damping model, are found in [12]. All simulations were performed in the
“Generic Interface Readily Accessible for Finite Elements (GIRAFFE)” code. It is a C++ in-house
code from University of São Paulo, Brazil.

The vertical beam was discretized using 100 elements, each one with 3 nodes (quadratic interpo-
lation on displacements and rotations). Then, prior to start the dynamic solution, a pre-static analysis
was done. The static analysis prepares the initial condition for the dynamic simulation. It includes
some loads and imposed displacements in the model, as described in the sequence. First, a displace-
ment field is imposed in the whole structural domain. The objective of it is to include an initial
geometric imperfection along beam span. The chosen imperfection is a harmonic field, with wave
length $\lambda = 2L_0$ and maximum amplitude 1 mm (in the middle of the specimen). This imperfection
induces lateral movement during beam axial excitation. It is essential for simulations once, if not
applied, we would handle with a perfect structure which would never bend. The self-weight was also
included as a static load and the structure suffers deformation due to it, prior to dynamics. As a last
static effect, a pre-tension is applied in the beam. This is done by imposing a 50mm vertical displace-
ment at the top node, leading to the length variation from $L_0$ to $L$. The first natural frequency of this
model was calculated numerically using mass and stiffness matrices, leading to $f_1 = 0.86$ Hz. This
matrices were calculated based on the final configuration of the static model (the initial condition for
dynamics).

A structural damping ratio $\zeta = 1\%$ was assumed. This was used to calibrate the Rayleigh damping
model. The calibration was done through a dynamic simulation. For that, the initial configuration was
considered to be the imperfect structure, already mentioned. It was simulated dynamically, with no
other external loads (only the self-weight was kept during dynamics, as a constant load, such as the
pre-tensioning performed in the pre-static analysis). The mass matrix damping coefficient $\alpha$ was
assumed to be null. The stiffness matrix damping coefficient $\beta$ resulted in $\beta = 0.0037$. The $\beta$
coefficient was calibrated using the logarithm decrement method, imposing the assumed damping
ratio. For that, the lateral movement of the beam central node was monitored along time.

The simulations of interest, to study possible parametric excitation of the beam, were finally
carried out. For that, a top excitation was applied during dynamics. This was done by imposing a top
$z$-direction harmonic displacement, as a support excitation. The simulations total time is 320s, which
showed to be enough to ensure steady-state response. The time-step used was 0.02s.

4. The Reduced-Order Model

The first step adopted, aiming at the construction of the ROM, is to define the equation of lat-
eral motion $u(t, z)$ of the structure. Fig. 1 sketches the vertical beam and the reference system. It
is assumed that the linear mass ($m_i$), the axial and bending stiffness ($EA$ and $EI$ respectively), the
structural damping coefficient ($c$) are independent of the spanwise position. By applying the Hamilton’s Principle along the unstretched configuration, reference [13] presents the following equation of
motion for a hinged-hinged beam.

\[
(1) \quad m_t \frac{\partial^2 u}{\partial t^2} + c \frac{\partial u}{\partial t} + EI \frac{\partial^4 u}{\partial z^4} = - \frac{E A}{2L_0} \frac{\partial^2 u}{\partial z^2} \int_0^{L_0} \left( \frac{\partial u}{\partial z} \right)^2 dz = 0
\]

Herein, the focus is the oscillations in lower modes. Hence, it will be assumed solutions in the form

\[
(2) \quad u(t, z) = \sum_{k=1}^{3} \psi_k(z) A_k(t)
\]

At this point, some aspects deserve more detailed discussions. The first one is related to the last term of Eq. (1) which takes into account the extensibility of the beam. The nonlinear character of this term leads to the coupling between the modes and the higher the number of modes selected for the projection (in this paper, restricted to three), more complex the ROM becomes.

The second aspect is related to the choice of modal functions \( \psi_k(z) \). As already mentioned, the vibration modes of a vertical and non-extensible cable is given by Bessel functions. However, for the sake of simplicity, herein it has been adopted trigonometric functions.

In order to obtain the set of coupled equations of motion, Eq. (2) was introduced in Eq. (1) and then Galerkin’s scheme was employed, i.e., multiply each term of the resulting equation by \( \psi_k(z) \) and integrate along the unstretched length of the beam. Considering the nondimensional quantities \( A_k = A_k/D, \tau = t/\omega_1 \) and \( n = \Omega/\omega_1 \) (ratio between the top motion frequency and the first natural frequency), the following nonlinear system of coupled ordinary differential equations is obtained in the nondimensional form:

\[
(3) \quad \frac{d^2 \hat{A}_1}{d\tau^2} + \alpha_1 \frac{d \hat{A}_1}{d\tau} + (\delta_1 + \epsilon_1 \cos(n\tau)) \hat{A}_1 + \alpha_2 \hat{A}_2 + \alpha_3 \hat{A}_3 + \alpha_4 \hat{A}_2^3 \hat{A}_1 + \alpha_5 \hat{A}_1 \hat{A}_3^2 = 0
\]

\[
(4) \quad \frac{d^2 \hat{A}_2}{d\tau^2} + \beta_1 \frac{d \hat{A}_2}{d\tau} + (\delta_2 + \epsilon_2 \cos(n\tau)) \hat{A}_2 + \beta_2 \hat{A}_3 + \beta_3 \hat{A}_1 + \beta_4 \hat{A}_2 \hat{A}_1^2 + \beta_5 \hat{A}_3^3 + \beta_6 \hat{A}_2 \hat{A}_3^2 = 0
\]

\[
(5) \quad \frac{d^2 \hat{A}_3}{d\tau^2} + \gamma_1 \frac{d \hat{A}_3}{d\tau} + (\delta_3 + \epsilon_3 \cos(n\tau)) \hat{A}_3 + \gamma_2 \hat{A}_2 + \gamma_3 \hat{A}_3 \hat{A}_2^2 + \gamma_4 \hat{A}_3 \hat{A}_1^2 + \gamma_5 \hat{A}_3^3 = 0
\]

where:

\[
(6) \quad \epsilon_k = \frac{EA}{L_0} \left( \frac{k\pi}{L_0} \right)^2 \frac{A_t}{m_1 \omega_1^2}, \quad \delta_k = \frac{EI}{m_1 \omega_1^4} \left( \frac{k\pi}{L_0} \right)^4 - \frac{1}{2} \gamma \frac{L_0}{m_1 \omega_1^2} \left( \frac{k\pi}{L_0} \right)^2 + \frac{1}{2} \frac{\bar{I}_I}{m_1 \omega_1^2}
\]

and the others nondimensional parameters are presented in Tab. 2.

A time-step \( \Delta \tau = 0.1 \) was employed in the simulations, carried out using a 4th–5th order Runge-Kutta scheme available in MATLAB®. The structural damping coefficient was taken as \( c = 0.09 Ns/m^2 \), leading to first-mode structural damping ratio \( \zeta_1 = 0.8\% \). Notice that this value very well agrees with that calibrated in the FEM model. This is necessary to perform a good comparison between the models.

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2 As the bending stiffness is almost negligible, the hinged-hinged assumption is reasonable for the problem presented in Fig. 1.
Table 2: Parameters for the reduced-order model.

<table>
<thead>
<tr>
<th>m</th>
<th>$\alpha_m$</th>
<th>$\beta_m$</th>
<th>$\gamma_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\alpha_1 = \frac{3}{40} \frac{\pi \omega_0}{L_0}$</td>
<td>$\beta_1 = \frac{7}{2} \frac{312 \omega_0}{L_0}$</td>
<td>$\gamma_1 = \frac{\pi \omega_0}{L_0}$</td>
</tr>
<tr>
<td>2</td>
<td>$\alpha_2 = \frac{1}{9} \frac{D^2 E A}{m \omega_0^2} \left(\frac{\pi}{L_0}\right)^4$</td>
<td>$\beta_2 = \frac{3}{2} \frac{312 \omega_0}{L_0}$</td>
<td>$\gamma_2 = \frac{\pi \omega_0}{L_0}$</td>
</tr>
<tr>
<td>3</td>
<td>$\alpha_3 = \frac{1}{4} \frac{D^2 E A}{m \omega_0^2} \left(\frac{\pi}{L_0}\right)^4$</td>
<td>$\beta_3 = \frac{1}{2} \frac{6 \pi \omega_0}{L_0}$</td>
<td>$\gamma_3 = \frac{9}{4} \frac{D^2 E A}{m \omega_0^2} \left(\frac{\pi}{L_0}\right)^4$</td>
</tr>
<tr>
<td>4</td>
<td>$\alpha_4 = \frac{1}{4} \frac{D^2 E A}{m \omega_0^2} \left(\frac{\pi}{L_0}\right)^4$</td>
<td>$\beta_4 = \frac{1}{2} \frac{6 \pi \omega_0}{L_0}$</td>
<td>$\gamma_4 = \frac{9}{4} \frac{D^2 E A}{m \omega_0^2} \left(\frac{\pi}{L_0}\right)^4$</td>
</tr>
<tr>
<td>5</td>
<td>$\alpha_5 = \frac{9}{4} \frac{D^2 E A}{m \omega_0^2} \left(\frac{\pi}{L_0}\right)^4$</td>
<td>$\beta_5 = \frac{1}{4} \frac{2 \pi \omega_0}{L_0}$</td>
<td>$\gamma_5 = \frac{1}{4} \frac{D^2 E A}{m \omega_0^2} \left(\frac{\pi}{L_0}\right)^4$</td>
</tr>
<tr>
<td>6</td>
<td>-</td>
<td>$\beta_6 = \frac{9}{4} \frac{D^2 E A}{m \omega_0^2} \left(\frac{\pi}{L_0}\right)^4$</td>
<td>-</td>
</tr>
</tbody>
</table>

5. Results and Discussions

In this research paper the comparison between ROM and FEM results was carried out. This was performed by discussing the time histories of lateral displacement at three different points of the flexible beam, namely $z/L_0 = 1/4$, $z/L_0 = 1/2$ and $z/L_0 = 3/4$. Due to limitation in the paper length, we will discuss only the results obtained in the principal Mathieu’s instability region ($\Omega/\omega_1 = 2$) and one value of top-motion amplitude $A_t/L_0 = 1\%$.

The FEM analysis directly gives the time histories at the mentioned points. From the modal-amplitude time histories obtained with the ROM, the time histories of displacements can be obtained in a straightforward way simply using Eq. 2 with the corresponding spanwise position coordinates.

Fig. 2 presents the time histories at $z/L_0 = 1/4$, $z/L_0 = 1/2$ and $z/L_0 = 3/4$. As seen in Figs. 2(a), 2(c) and 2(e), the simulations were carried out with a time-span sufficient so that the system can reach a steady-state regime. In Figs. 2(b), 2(d) and 2(f) it is showed zoomed regions of the steady-state regime and the amplitude spectra.

One can notice that there is qualitative agreement between FEM and ROM results for the three spanwise positions considered. Particularly, for the midspan position (see Fig. 2(d)), the steady-state amplitude obtained with the FEM model is slightly lower than that from the ROM. Considering $z/L_0 = 1/4$, the FEM results indicates a slightly larger steady-state amplitude than that obtained with the ROM. However, at $z/L_0 = 3/4$, the steady-state amplitude is significantly larger for the ROM in comparison with the FEM model’s result. All the spectral analysis indicate that the dominant oscillation frequency (i.e., the component in the amplitude spectrum that presents the higher amplitude) is the first natural frequency of the model ($f_4/f_1 \approx 1$).

A conjecture that can be made regarding the differences between the models at $z/L_0 = 1/4$ and $z/L_0 = 3/4$ is related to the choice of sinusoidal functions as modes of vibration and projection functions in the Galerkin’s method. As already mentioned in Section 1, the vibration modes of a non-extendable vertical cable is given by Bessel functions, which present larger displacement below the midspan. The analysis of the modal-amplitude time histories can offer a complementary insight aiming at discussing this conjecture.

Fig. 3 presents the modal-amplitude time histories obtained as solution of the ROM. As expected, in a favorable scenario for the principal Mathieu’s instability of the first vibration mode (as is the case in which $\Omega/\omega_1 = 2$), the modal-amplitude time history corresponding to the first vibration mode $\dot{A}_1(\tau)$ is clearly dominant when comparison with $\ddot{A}_2(\tau)$ and $\ddot{A}_3(\tau)$. Hence, the contribution of the first vibration mode for the oscillations at $z/L_0 = 1/4$ and $z/L_0 = 3/4$ is much larger than the other modes. It must also be emphasized that this conjecture may be deeper discussed after the

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3 The amplitude spectra were calculated only for the steady-state regime.

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Figure 2: Time histories of displacement at different sections. $A_t/L_0 = 1\%$ and $\Omega/\omega_1 = 2$. 
modal analysis of the FEM model and the plot of the modal shapes. Other aspect that deserves a more detailed investigation is related to the phase-shift between $A_1(\tau)$ and $A_2(\tau)$. This phase-shift certainly affects the regeneration of the time histories at some spanwise positions.

As sinusoidal functions were adopted as a simpler choice for vibration modes, the first vibration mode contributes with equal magnitude to the oscillation at $z/L_0 = 1/4$ and $z/L_0 = 3/4$. However, as Bessel functions present larger amplitude below the mid-span than at the upper part of the mode, it is expected that the ROM herein adopted leads to an increase in the oscillation of the upper part and to a decrease in the oscillation of the lower part if compared to a higher-order model, such as FEM one. Although, it is important to highlight that the results obtained with both approaches present a very good agreement and that sinusoidal functions are much simpler to deal than Bessel functions. Then, in a reduced-order model with focus on the parametric excitation problem described in this research paper, this choice is strongly acceptable.

![Figure 3: Modal-amplitude time-histories. ROM results. $A_t/L_0 = 1\%$ and $\Omega/\omega_1 = 2$.](image)

**Final Remarks**

This paper investigated numerically the parametric excitation of a slender and vertical beam subjected to prescribed harmonic and axial top-motion. The top-motion frequency $\Omega$ was twice the first natural frequency of the beam $\omega_1$ and its amplitude corresponded to $1\%$ of the unstretched length of the beam. Two numerical approaches were adopted, namely a geometric nonlinear finite element model and a three-mode reduced-order model based on sinusoidal functions for the vibration modes and projection functions.

In this very first comparison, we focus on the discussion of the differences obtained in the time histories corresponding to three distinct points of the beam, namely $z/L_0 = 1/4$, $z/L_0 = 1/2$ and $z/L_0 = 3/4$. Qualitative and quantitative similarities were pointed out. At the midspan, it was observed a marked adherence between the two numerical approaches. Differences were observed in the other spanwise positions.

A conjecture to explain these differences was based on the fact that, strictly speaking, the sinusoidal functions do not correspond to the vibration modes of a vertical beam. As Bessel functions are correspond to the vibration modes of a non-extensible vertical beam, the amplitudes above the midspan are larger than those at the upper part. Since the first vibration mode dominates the response when $\Omega = 2\omega_1$, the regeneration of the local time histories of displacement based on the modal-amplitude time histories obtained with the ROM based on sinusoidal functions leads to local amplitudes larger than FEM results in the upper part of the beam.

Further works include ROM/FEM comparisons with others parametric excitation amplitudes and frequencies, as well as comparisons with experimental data. Additionally, comparisons with the same beam immersed in water are planned. This last condition is typical in offshore engineering scenario.
and present experimental data available in the literature, allowing numerical-experimental correlations and discussions.

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REFERENCES