Nonlinear dynamics of an archetypal model of ships motions in tandem

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Abstract

Large converted tankers operate as floating platforms for the production of oil in offshore fields whose depth approaches 2000 m. Their dynamics are influenced by wind, currents, and waves, and also by the physical characteristics of mooring lines. In addition to the single-vessel operation mode, a two-ship tandem arrangement is employed during the transfer of cargo (offloading) from the main production vessel to a shuttle vessel that takes the oil to other processing plants. It is essential to design the system in a way that ensures that the dynamical behavior of the vessels during offloading is safe. Mathematical models that represent the motions of moored ships in the sea can be complex, particularly when detailed modeling of hydro- and aerodynamic effects is required. In this work a simplified (archetypal) model is developed and explored that includes wind and current effects, and the elastic interaction between vessels. The model is validated against time series and bifurcation diagram results of a complete, industrial-strength model, and also through comparison with experimental data, showing good agreement. The use of a simplified model produces here a more refined exploration of dynamical features of the system such as its bifurcational structure and basins of attraction. The engineering relevance of these results is also evaluated through the use of basins of attraction.

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1. Introduction

During the past few decades large converted tankers, also called Floating Production Storage and Offloading units (FPSOs), have been deployed as platforms for the exploitation of offshore oil and gas fields. FPSOs have some advantages over other floating platforms. First of all, they often have lower initial costs; a semisubmersible oil platform is more expensive than a converted tanker. Secondly, the storage capacity of a converted tanker is much larger. Other reasons can be also pointed out, such as mobility. However, one significant disadvantage of such system relates to its dynamic behavior. The dynamic response of an FPSO tanker is that of an ordinary ship, therefore its motions due to external forces (such as waves, current and wind) can be considerably larger than those of a semisubmersible platform, particularly when environmental forces act upon the lateral of the vessel. Thus the FPSO has to be kept in place by a mooring system that connects it to the seabed. Two designs have been favored in recent times. The Differentiated Compliant Anchoring System (DICAS) is a spread mooring system which by careful choice of pretensions of mooring lines allows the vessel some freedom to rotate and self-align (partially) with the direction of resultant forces. Because of its limited ability to self-align, the FPSO’s design alignment has to be judiciously chosen according to the predominant direction of incoming current and wind. The other mooring system often used is the Turret type, in which the FPSO has to be equipped with a revolving structure (the turret) that receives both the mooring lines and the risers (ducts that carry the products from the reservoir to the vessel). This is a more expensive and sometimes less robust design that has the advantage of allowing the FPSO total freedom of self-alignment with the weather.

Also significant is the fact that two-ship tandem configurations are employed during the offloading process, when a shuttle tanker is connected to the FPSO through an elastic cable (hawser) in order to take the cargo to onshore processing plants. During this process, more complex, dynamic scenarios are observed [1]. Depending on the external forces, the ships can display undesirable equilibrium points and/or motions that bring the vessels into dangerous proximity or unacceptable relative positioning [2]. This is regarded by operation crews as unsafe situations, and their prevention should ideally be made at the time of the system design. Good knowledge of the rich dynamic behavior exhibited by this system is therefore an essential step in the design process. Such knowledge starts with suitable mathematical models of the whole system, which take into account the hydrodynamic, aerodynamic, and elastic effects. However, a comprehensive investigation of such models is difficult to carry out due to the complexity of the
ensuing equations. Numerical simulation can be used, but because realistic mathematical models are highly vessel-specific and also include a large number of empirical parameters, it is difficult to gain basic knowledge of the system behavior based on those simulations. Also, a more generic analysis can be helpful in guiding more detailed numerical investigations.

Based on such considerations our approach here is that of defining an archetypal model of two vessels in tandem, and to use it to develop some fundamental insight into the complex dynamic involved. The system chosen for this study is an FPSO attached to the seabed by a spread mooring system of the DICAS design (Differentiated Compliant Anchoring System). The analyses performed include the production of bifurcation diagrams of the equilibrium points and their respective basis of attraction.

2. Equations of motion

2.1. Complete model

An FPSO-DICAS system with a shuttle tanker can be seen in Fig. 1 along with the coordinate systems adopted. The equations of motion of a ship expressed in a reference frame attached to its body and under the action of current and wind can be described as follows (see [3,4] for details of the mathematical modeling):

\[
\begin{align*}
(m - m_{11}) \cdot \dot{u} &= (m - m_{22}) \cdot vr - (m \cdot x_g - m_{26}) \cdot r^2 \\
&\quad - (m_{11} - m_{12}) \cdot v_c \cdot r + X, \\
(m - m_{22}) \cdot \dot{v} &= (m_{11} - m) \cdot ur - (m \cdot x_g - m_{26}) \cdot r - (m_{11} - m_{12}) \cdot u_c \cdot r + Y,
\end{align*}
\]
\[ (I_z - m_{66}) \cdot \dot{r} = (m_{22} - m_{11}) \cdot u \cdot v - (m \cdot x_g - m_{26}) \cdot (\dot{v} + r \cdot u) + N, \]

where \( m \) is the ship mass; \( m_{ii}, i = 1, 2, 6 \), are the surge, sway and yaw additional masses, respectively; \( u \) and \( v \) are the surge and sway velocities projected in a reference frame attached to the ship; \( u_c \) and \( v_c \) are the current velocities; \( r \) is the angular (yaw) velocity; \( I_z \) is the moment of inertia in the z-direction; \( X, Y \) and \( N \) are the external forces (current and wind).

The force and moment due to current can be expressed as

\[
X_c(\psi, V) = \frac{1}{2} \rho TL \cdot C_{1c}(\psi) \cdot |V_r|^2, \quad (4)
\]

\[
Y_c(\psi, V) = \frac{1}{2} \rho TL \cdot C_{2c}(\psi) \cdot |V_r|^2, \quad (5)
\]

\[
N_c(\psi, V) = \frac{1}{2} \rho TL^2 \cdot C_{6c}(\psi) \cdot |V_r|^2, \quad (6)
\]

where \( V_r = \sqrt{(u - u_c)^2 + (v - v_c)^2} \).

The coefficients indicated by \( C_{ic}, i = 1, 2, 6 \), are the current force coefficients, and are functions of \( \psi \) (ship heading) as given in Appendix A.

Forces and moment due to wind can be given by

\[
X_w = \frac{1}{2} \cdot C_{Xw}(\psi_{rw}) \cdot \rho_v \cdot V_w^2 \cdot A_T, \quad (7)
\]

\[
Y_w = \frac{1}{2} \cdot C_{Yw}(\psi_{wv}) \cdot \rho_v \cdot V_w^2 \cdot A_L, \quad (8)
\]

\[
N_w = \frac{1}{2} \cdot C_{Nw}(\psi_{rw}) \cdot \rho_v \cdot V_w^2 \cdot A_L \cdot LBP, \quad (9)
\]

\[
\psi_{rw} = \psi_w - \psi.
\]

In the same way, \( C_{Xw}, C_{Yw}, \) and \( C_{Nw} \) represent the wind force coefficients as functions of \( \psi \) as given in Appendix A.

2.2. Archetypal model

Developing a simplified mathematical model of a complex system that retains the main features of its dynamics is not a trivial task. Particularly since the overall dynamic behavior of the original system is unknown, its determination being the primary purpose of the simplified model. There are methods that address the issue of model simplification. Perhaps one the best known methods is the one based on Center Manifold Theory [5], which essentially reduces the dimension of a system by projecting the flow onto the subspace spanned by the center manifold of a given equilibrium point. Naturally, such projection, and
consequently the associated reduction, is locally valid. A global approximation of the original model cannot in general be hereby produced.

The approach employed in present work takes advantage of the fact that some of the major features of the complete model are known through previous studies \[1,2,4,6,7\]. The simplified mathematical model proposed here is arrived at after an energy analysis of the complete model, and the structure of the simplified model is such that it preserves the main dynamical features of the complete model, as described below.

The first aspect of interest is the number and stability properties of equilibrium solutions. Let \( V(x) \) be the total potential energy of the complete system, where \( x \) is the vector of linear and angular displacements. The function \( V(x) \) defines what here is termed a potential well. Clearly, equilibrium solutions \( x^* \) are such that \( \frac{\partial V}{\partial x} |_{x^*} = 0 \). Some basic insight into the nature and distribution of equilibrium points can be gained from the inspection of the overall shape of \( V(x) \).

Taking the potential wells of the complete system described above one can see that the degrees of freedom corresponding to angular motions capture most of the relevant dynamics of the entire system (see Figs. 2 and 3). One can observe that the potential well related to motions in the \( X \) and \( Y \) directions has a shape characterized by just one equilibrium point (a minimum). This is due to the elastic nature of the restoring forces in those directions.

With this feature in mind it is possible to propose a qualitative study of the stability of the system based only on its angular motions.

Another point to be observed is the relationship between the degrees of freedom. It can be seen from the complete model that the wind and current forces

![Fig. 2. Potential well related to angular motions (rad).](image-url)
have the mathematical form of a Van der Pol oscillator, with a displacement variable multiplied by a velocity (see Eqs. (1)–(9) and also Appendix A). This is an important feature of the system, and as such, must be present in any archetypal model to be created.

Taking the potential well defined as follows:

\[ V(\psi)_{arq} = -A \cdot \cos(\psi) \cdot \left(\frac{1}{3} \cdot \sin^2(\psi) + \frac{2}{3}\right) - B \cdot \cos(\psi) \cdot |\cos(\psi)| \]

\[ + C \cdot \sin^2(\psi) + \sigma \cdot \cos(\psi_v - \psi), \]

and considering the inclusion of a Van der Pol oscillator term in the form:

\[ D \cdot \left(-\frac{\sqrt{V_c}}{\sigma} + |\psi|\right) \cdot \dot{\psi}, \]

where \( D \) is a constant value, and \( \sigma \) is the ratio between wind and current velocities \( \left(\sigma = \frac{V_w}{V_c}\right) \), one arrives at the following equation of motion for the simplified model:

\[ \ddot{\psi} + D \cdot \left(-\frac{\sqrt{V_c}}{\sigma} + |\psi|\right) \cdot \dot{\psi} + f(\psi) = 0 \]

with \( f(\psi) = \frac{\partial V(x)_{arq}}{\partial \psi} \).

These are the equations of motion of the archetypal model for one ship. In the tandem configuration Eq. (12) can represent the FPSO-DICAS system. The mooring system prevents large displacements of the main ship in the \( XY \) plane, and therefore one can consider an inertial reference frame attached to the main ship. However, the force between the main ship and the shuttle vessel must also
be represented. To do so one has to add one more mathematical function to the potential well, which represents the interaction between tankers:

\[
 f(\psi_1, \psi_2)_{\text{hawser}} = Kh \cdot \left[ \frac{l_1 \cdot l_2}{2} \cdot \cos(\psi_1) \cdot \sin(\psi_2) - \frac{l_1 \cdot l_2}{2} \cdot \sin(\psi_1) \cdot \cos(\psi_2) \right].
\]  

So the final equation of an archetypal model of an FPSO-DICAS system can be written as follows:

\[
 \ddot{\psi}_1 + D \cdot \left( -\frac{\sqrt{V_c}}{\sigma} + |\psi_1| \right) \cdot \dot{\psi}_1 + f(\psi_1) + f_{\text{hawser}}(\psi_1, \psi_2) = 0.
\]  

3. Comparisons with the complete model

The archetypal model must display behavior similar to the complete model. An important point to be observed considering a nonlinear model is that one cannot just compare time series resulting from a few numerical simulations. Instead, one can use bifurcation diagrams, which encapsulate a greater wealth of information about the system. The strategy here is therefore to compare bifurcation diagrams as well as the response amplitudes obtained from the two models (archetypal and complete). The heading of the shuttle vessel was used to represent the fixed points in the bifurcation diagrams. The control parameter selected was the wind to current speed ratio \( \sigma \).

Figs. 4–7 show the bifurcation diagrams for both models with an angle of incidence of the current of 180°, and two different wind angles: 10° and 0°.
It is possible to identify in both cases (complete and archetypal results) three different regimes in the bifurcation diagram. The first region (marked as I in Figs. 4 and 5) as well as the third one (marked as III in Figs. 4 and 5) are characterized by a small number of equilibrium points. In both diagrams there exists a third region (marked as II in Figs. 4 and 5) in which more complex behavior can be observed, characterized by a larger number of equilibrium points and by the presence of fold bifurcations. The reproduction of this important feature of the complete model is the first crucial test of the simplified model.

Fig. 5. Bifurcation diagram of complete model (wind at 0°, current at 180°).

Fig. 6. Bifurcation diagram of archetypal model (wind at 10°, current at 180°).
A second test can be given by the inspection of time series of the angular displacement of the shuttle vessel. Figs. 8 and 9 compare the time evolution of the heading angle of the shuttle vessel for the complete and archetypal models at two different current speeds. Good agreement was observed for both tests. It is important to note that the numerical parameters of the simplified model were adjusted for the lower current speed and kept constant for the higher speed test, thereby indicating that the structure of the model can cope well with changes in current speed.

Fig. 7. Bifurcation diagram of archetypal model (wind at 0°, current at 180°).

Fig. 8. Comparison between the archetypal and the complete model, $V_c = 0.2$ m/s: Red, archetypal model; Blue, complete model.
4. Experimental validation

The next step in the verification of the simplified model proposed here was a comparison with experimental data. Such a comparison also provided further quantitative basis for the adjustment of parameter values. Experimental data was taken from [8,9]. The experiment consisted of towing a reduced scale model of an FPSO by a rigid bar at controlled speeds. Different speeds of towing simulated the action of different current velocities. Wind was not included in the experimental setup. Typically, after a short transient, the model would undergo periodic oscillations of various amplitudes. Here we have compared the steady-state amplitudes obtained from the simplified model with those recorded in the experiments. Parameters were calibrated for the lower speed (0.2 m/s) and used throughout the tests to examine the ability of the model to adapt to changes in current speeds. Figs. 10 and 11 show experimental data along with numerical results obtained with the simplified model at 0.2 m/s and 0.4 m/s, respectively. The simplified model shows good overall agreement with the experimental data. When current velocity was changed, the model was able to predict well the new period and amplitude of the response, which has an important role in terms of engineering analysis. The higher is the amplitude of the motion, the higher is the likelihood of a collision between the tankers.

5. Basins of attraction

From a practical point of view, one of the most important aspects to be evaluated is the robustness of attracting equilibrium points and limit cycles. Such
robustness results here from the concomitant presence of at least two features. Firstly, the solution must be observed for a wide range of parameter values such as speeds and angles of incidence of current and wind. Secondly, in order to be of practical relevance, the solution must attract to itself a sizable region of

Fig. 10. Experimental validation: $V_c = 0.2$ m/s.

Fig. 11. Experimental validation: $V_c = 0.4$ m/s.
initial conditions (positions and velocities). In other words, it should have a basin of attraction (or catchment region) which is large with respect to other attracting solutions. Failure to exhibit these features will result in a solution that is not likely to be seen in practice, either because environmental conditions will differ slightly from those necessary for its appearance or because other attracting solutions will have captured the behavior.

For the present study we have developed an algorithm to automatically identify and display the basins of attraction of all attractors present for a given environmental condition. The maps created by this algorithm show the kind of attractor identified by a symbol. A limit cycle is identified by a small circle (○) and a point equilibrium by a star (★). The basins of attraction of different attractors are distinguished by different colors and identified by a rectangle. The maps selected for display in this text are all typical in the sense described above, i.e., their general features are common to a large range of environmental situations. They also correspond to conditions examined in the preceding figures. Two such conditions are exemplified here. The first scenario can be inspected in Figs. 12 and 13, which is typical of low wind situations (σ = 0.3). Here the dominance of current forces destabilizes the static equilibrium (see for instance [2]), giving rise to steady-state oscillations. The basin of attraction shown in Fig. 12 effectively covers the whole phase plane, indicating that the periodic attractor is the only attracting solution. The phase diagram shown in Fig. 13 helps to illustrate the behavior of the system. It also shows quite clearly a prevalent feature of this system: yaw motions are heavily damped causing trajectories to converge rapidly onto the stable manifolds of the cyclic attractor. Note that the presence of a single stable periodic solution is consis-
tent with the results shown in Fig. 7 and, also importantly, with the no-wind condition of the experimental setup (see Figs. 10 and 11). The second scenario is depicted in Figs. 14 and 15 where due to stronger wind forces ($r = 1.3$) two static equilibria are now stable (it should be recalled that wind forces tend to stabilize static equilibria aligned with them, see for instance [6]). As expected from the symmetry of this system, the two basins of attraction are largely symmetric, see Fig. 14. Fig. 15 depicts the two attracting equilibria, again displaying very clearly the convergence of trajectories onto their stable manifolds caused by the heavy yaw damping. The passage from one scenario to the other.
is made through a supercritical Hopf bifurcation, which is easily visible in Fig.
7 roughly at ($\sigma = 0.5$).

6. Conclusions

The present work introduced a simplified mathematical model for the study
of the dynamic behavior of two ships in tandem, with particular reference to
the offloading of FPSOs by a shuttle vessel. It was shown that good results
can be obtained restricting the model to just one dynamic variable per vessel:
the angular displacement. The structure of the model was derived from energy
arguments, in which the reproduction of a (generalized) potential function al-
lowed the selection of mathematical terms to be included. These also included
an ad hoc Van der Pol nonlinear damping term.

The model was validated against experimental data obtained from reduced
scale tests. Its predictions were also compared with those of a detailed model
used at the University of São Paulo to simulate ship motions. Results were
compared using bifurcation diagrams covering a range of relevant environmen-
tal parameters, and showed very good qualitative agreement, including the
ability of the simplified model to capture changes in behavior with increasing
current speed. Additional investigation of the model was carried out with the
production of maps of basins of attraction for two of the main scenarios
encountered. The engineering relevance of the results was also assessed and
it was argued that basins of attraction can be a valuable tool in evaluating
the real safety of operations from the point of view of the dynamic behavior
of these systems.
Appendix A

This appendix contains a summary of the main formulae detailing the mathematical models employed throughout this study.

A.1. Current

The forces and moment due to current are given by the following equations [3]:

\[
F_{ci}(\beta, V) = \frac{1}{2} \rho TL p C_{ic}(\beta) |V_c|^{2}, \quad i = 1, 2, 6; \quad p = 1 \text{ for } i = 1, 2;
\]

\[
p = 2 \text{ for } i = 6,
\]

where the hydrodynamic coefficients are given by

\[
C_{1c}(\beta) = \left[ \frac{0.09375}{(\log(Re) - 2)^{2}} \right] \cos(\beta) + \frac{1}{8} \frac{\pi T}{L} (\cos(3\beta) - \cos(\beta)),
\]

\[
C_{2c}(\beta) = \left[ C_Y - \frac{\pi T}{2L} \right] \sin(\beta) |\sin(\beta)| + \frac{\pi T}{2L} \sin^{3}(\beta)
\]

\[
+ \frac{\pi T}{L} \left[ 1 + 0.4 \frac{C_B B}{T} \right] \sin(\beta) |\cos(\beta)|,
\]

\[
C_{6c}(\beta) = -\frac{I_g}{L} \left[ C_Y - \frac{\pi T}{2L} \right] \sin(\beta) |\sin(\beta)| - \frac{\pi T}{L} \sin(\beta) \cos(\beta)
\]

\[
- \left[ 1 + |\cos(\beta)| \right]^{2} \frac{\pi T}{L} \left[ \frac{1}{2} - 2.4 \frac{T}{L} \right] \sin(\beta) |\cos(\beta)|,
\]

where \( B \) and \( T \) are the breadth and draft of the ship respectively; \( C_B \) is the block coefficient; \( C_Y \) is the lateral force coefficient in transversally steady current; \( Re \) is the Reynolds number (based on the length \( L \)); \( I_g \) measures the longitudinal distance between the hull’s centre of mass and the midship section; \( \beta \) is the angle of attack defined as

\[
\beta = \tan^{-1}(v - v_c, u - u_c).
\]

A.2. Damping due to yaw

The damping due to yaw is also calculated based on low aspect ratio wing theory and is given by

\[
X_D = -\frac{1}{4} \rho \cdot \pi \cdot T^{2} \cdot L \cdot v_{r} \cdot r - \frac{1}{16} \rho \cdot \pi \cdot T^{2} \cdot L^{2} \cdot \frac{u_{r}}{|u_{r}|} \cdot r^{2},
\]
The wind forces are determined by conventional drag force formulations, where drag coefficients are obtained from reduced scale model tests:

\[ F_w = \frac{1}{2} \rho \cdot T \cdot L^2 \cdot C_{D,i} \cdot u_r \cdot \dot{r} - 0.035 \cdot \rho \cdot T \cdot L^2 \cdot v_r \cdot \dot{r} - 0.007 \cdot \rho \cdot T \cdot L^3 \cdot |r| \cdot \dot{r}, \]  
(A.7)

\[ N_D = -\frac{1}{2} \rho \cdot T \cdot L^3 \cdot C_{D,6} \cdot |u_r| \cdot \dot{r} - \frac{3}{20} \cdot \rho \cdot T \cdot L^3 \cdot C_\gamma \cdot |v_r| \cdot \dot{r} \]  
(A.8)

\[ u_r = u - u_c, \]  
(A.9)

\[ v_r = v - v_c, \]  
(A.10)

\[ C_{D,2} = \frac{\pi \cdot T}{2 \cdot L} \left( 1 - 4.4 \cdot \frac{B}{L} + 0.16 \cdot \frac{B}{T} \right), \]  
(A.11)

\[ C_{D,6} = \frac{\pi \cdot T}{4 \cdot L} \left( 1 + 0.16 \cdot \frac{B}{T} - 2.2 \cdot \frac{B}{L} \right). \]  
(A.12)

\[ \psi_{rw} = \psi_w - \psi, \]  
(A.14)

where the \( C_{iv} \) are coefficients determined experimentally; \( V_w \) is the wind speed; \( A \) is the corresponding projected area of the vessel and \( \psi_w \) is the direction of the wind.

References