HEURISTICS BASED ON LAGRANGIAN RELAXATION FOR THE VEHICLE ROUTING PROBLEM WITH TIME WINDOWS

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ABSTRACT

In this paper, two different heuristics for the vehicle routing problem with time windows (VRPTW) are presented: the Sequential Insertion Heuristic and the Cluster and Sequential Insertion Heuristic. Both are based on the Lagrangian relaxation of the constraints which require each customer to be serviced exactly once. They were evaluated based on the classical six test problem sets described in the literature. The paper also describes a practical application of the Cluster and Sequential Insertion Heuristic to a real-life application involving a wholesale goods distribution in the metropolitan area of São Paulo, Brazil, which yielded good results with significant savings in vehicle-km and total cost.

1. Introduction

The vehicle routing and scheduling problem with time windows (VRPTW) consists of designing a set of minimum cost routes originating and terminating at a central depot for a fleet of vehicles with known capacity. The routes must service a set of customers with known demands. Each customer must be serviced exactly once during its allowable delivery time or time window. The fleet mix may be heterogeneous. There is an upper limit on the maximal time length of the routes.

In the VRPTW, time windows are rigid. A vehicle is not permitted to arrive at a customer after the latest time to initiate service. However, if a vehicle arrives too early at a customer, it is permitted to wait until the earliest time to begin service.

Lenstra and Rinnooy Kan (1981) demonstrated that vehicle routing problems are NP-hard. Other authors have shown, by restriction, that the VRPTW is also NP-hard (Desrosiers et al., 1995, Kolen et al., 1987, Solomon, 1987 and Solomon and Desrosiers, 1988). Thus the search for efficient heuristic models for its solving.

Time constrained routing and scheduling problems are found in a variety of industrial and service sector applications, ranging from logistics to transportation systems. It has been an area of intense research due to its wide applicability. Most routing problems in urban areas have time windows constraints. They involve not only traditional industries such as wholesale goods distribution but also home delivery of appliances, scheduling

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services (e.g. dispatching repair men), school bus routing, pick-up and delivery in overnight mail services, etc.

2. The Vehicle Routing Problem With Time Windows

Early works on the VRPTW included both exact and heuristic approaches. Despite the progress on exact methods, the NP-completeness of the VRPTW requires heuristic solution approaches for most real-life problems. Solomon (1987) proposed and tested several heuristics for the VRPTW.

Good surveys of vehicle routing problems time windows may be found in Desrochers et al. (1988), Desrosiers et al. (1995) and Solomon and Desrosiers (1988).

The VRPTW can be stated as follows:

Let $N$ be the number of customers or nodes to be visited. For each customer $i \in \{1, 2, \ldots, N\}$ there is a task to be performed, which may be a pick-up or a delivery of goods or passengers, depending on the problem. Simultaneous pick-ups and deliveries are not allowed.

Each customer $i \in \{1, 2, \ldots, N\}$ has a non-negative service time $s_i$; a time window $[a_i, b_i]$, within which the customer permits the start of service; a known demand $q_i$ to be picked up or delivered, depending on the problem. Nodes 0 and $N+1$ represent respectively the origin-depot and the destination-depot of all vehicles.

There is a fleet of $NV$ vehicles that are available to be routed and scheduled. Each vehicle $v \in \{1, 2, \ldots, NV\}$ has an available capacity $K^v$; a total fixed daily cost denoted $C_v^f$; a cost per unit of distance travelled $C_v^d$; and a cost per hour $C_v^h$.

The maximum route duration is $H$. In most cases, this duration is related to the maximal length of a shift for the driver and the crew.

There is a travel time $t_{ij}$ associated with servicing customer $j$ immediately after customer $i$, as well as a distance $d_{ij}$ to be travelled between the two nodes. Note that symmetry of travel times and distances is not assumed.

The problem formulation involves the following decision variables:

$$x_{ij}^v = \begin{cases} $$
1, if node \( j \) is serviced immediately after node \( i \) by vehicle \( v \); 
0, otherwise.

- \( T_j \) = the start time of service at node \( i \), \( i \in \{1, 2, \ldots, N\} \)

The problem of finding the minimal cost set of routes satisfying the VRPTW constraints can be formulated as follows:

\[
\text{[VRPTW]} \quad \text{[min]} \sum_{v=1}^{NV} \sum_{i=0}^{N} \sum_{j=1}^{N+1} C_{d,j} x_{ij}^v + \sum_{v=1}^{NV} \sum_{i=1}^{N} \left(T_i + s_j + t_{i,N+1}\right) C_{h,x_i,N+1}^v + \sum_{v=1}^{NV} \sum_{j=1}^{N} C_{f,x_j}^v 
\]

subject to:

\[
\sum_{j=1}^{N+1} x_{ij}^v = 1 \quad i = 0, 1, 2, \ldots, N; \ i \neq j 
\]

\[
\sum_{j=1}^{N+1} x_{0j}^v = 1 \quad v = 1, 2, 3, \ldots, NV 
\]

\[
\sum_{i=0}^{N} x_{ij}^v - \sum_{i=1}^{N} x_{ji}^v = 0 \quad j = 1, 2, 3, \ldots, N; \ v = 1, 2, 3, \ldots, NV 
\]

\[
\sum_{i=0}^{N} x_{j,N+1}^v = 1 \quad v = 1, 2, 3, \ldots, NV 
\]

\[
a_i \leq T_i \leq b_i \quad i = 1, 2, 3, \ldots, N \quad (6) 
\]

\[
T_i + s_j + t_{ij} - T_j \leq (1 - x_{ij}^v)M \quad i = 1, 2, 3, \ldots, N \quad ; \ j = 1, 2, 3, \ldots, N; \ j \neq i 
\]

\[
(7) \\
\sum_{j=1}^{N} y_{ij}^v \leq K^v \quad v = 1, 2, 3, \ldots, NV 
\]

\[
(T_j + s_j + t_{j,N+1}) x_{j,N+1}^v = (T_i - t_{0j}) x_{ji}^v \leq H^v \quad i = 1, 2, 3, \ldots, N \quad ; \ j = 1, 2, 3, \ldots, N 
\]

\[
(9) \\
x_{ij}^v \in \{0,1\} \quad i = 0, 1, 2, 3, \ldots, N+1; \ j = 0, 1, 2, 3, \ldots, N+1; \ v = 1, 2, 3, \ldots, NV \quad (10) 
\]

The objective function (1) represents the total cost. Constraints (2) impose that each customer \( i \in \{1, 2, \ldots, N\} \) be assigned exactly once to a vehicle route. Constraints (3) to (5) describe the flow on the path that vehicle \( v \) will use. If the vehicle is not allocated to a route, it follows the path that links the origin-depot (node 0) to the destination-depot (node \( N+1 \)). Constraints (6) establish the time windows within which nodes must be visited, while constraints (7) describe the compatibility requirements between the flow variables that form the routes and the time variables that give the schedules. Constraints
(8) assure feasibility of the load on each vehicle, while constraints (9) guarantee feasibility of the time duration on each route. Binary conditions on the flow variables are given in (10).

### 3. Solution strategy based on lagrangian relaxation

The Lagrangian relaxation approach is based upon the observation that many difficult integer programming problems can be modelled as a relatively easy problem complicated by a set of side constraints (Fisher, 1985). Relaxing the VRPTW constraints (2) on visiting each customer exactly once results in a Lagrangian subproblem which retains a network structure: it is a shortest path problem with time windows with the special feature that \( NV \) units of flow must follow the shortest path from the origin-depot \( (i = 0) \) to the destination-depot \( (i = N +1) \). The relaxed problem is still hard to solve for problems with more than one vehicle. For instances with only one vehicle \( (NV = 1) \) the subproblem results on a shortest path problem with time windows (denoted SPPTW) with some additional constraints: route duration and vehicle capacity. These constraints do not affect SPPTW’s network structure and properties.

In this context, the subproblem in the latter case can be solved through a very efficient procedure, called the Generalized Permanent Labelling Algorithm (GPLA), proposed by Desrochers and Soumis (1988). Cunha and Swait (2000) proposed additional state-dominance criteria for the GPLA on dense graphs, as naturally arises in routing problems, which markedly improve its performance and has been used in the proposed heuristics. More details on the use of GPLA for the Lagrangian relaxation of the VRPTW may be found in Cunha (1997).

### 4. Heuristics for the VRPTW

Two different heuristics have been proposed for the VRPTW: the Sequential Insertion Heuristic and the Cluster and Sequential Insertion Heuristic. Both are based on the Lagrangian relaxation strategy described above.

They both allow a vehicle to be allocated to a second route, before a new vehicle is picked, if the vehicle returns to the depot early. The early return of a vehicle to the depot usually occurs when the physical capacity of a vehicle is reached due to few customers, whose service time is below the vehicle time duration constraint.
The following two sections present a brief overview of each of these two solution strategies. For more details refer to Cunha (1997).

4.1. The Sequential Insertion Heuristic

In the Sequential Insertion Heuristic, vehicles are considered for routing and scheduling one at a time, in a sequential order. Thus, this heuristic can deal only with homogeneous fleets of vehicles. Customers are sequentially allocated to a vehicle until time or capacity constraints are violated. All customers not yet serviced are candidates to be served by the current vehicle. The GPLA algorithm is applied to the graph defined by the unserviced customers. The costs on the arcs of the graph are affected by Lagrange multipliers, which are updated by subgradient optimization method (Fisher, 1985). The multipliers aim to prevent one customer to be serviced more than once and to assure the inclusion of the remaining customers to the route.

Details on adjusting the Lagrangian multipliers can be found in Cunha (1997).

The subgradient method for a vehicle is terminated either (i) when an arbitrary iteration limit is reached or (ii) when all customers are inserted in a route of the current vehicle. In the former, as some customers may remain unserviced, the best route for the vehicle is found among those generated at each subgradient iteration. A new vehicle is then selected and the subgradient method is repeated for this vehicle, considering only the remaining customers.

This procedure is repeated until all vehicles available are scheduled or all customers have been serviced. Thus, this heuristic can be used for vehicle routing either considering a pre-defined fleet or for determine the minimum cost fleet size and mix and the respective efficient routes.

4.2. The Cluster and Sequential Insertion Heuristic

In the Cluster and Sequential Insertion Heuristic the problem is decomposed into two subproblems. First, non-serviced customers are clustered using a heuristic method based on the minimum cost spanning tree with additional feasibility tests, which take into consideration capacity and journey length constraints. The cost is based on a measure of the spatial and “temporal” (regarding time windows) distance between customers. Then subgradient method for the Lagrangian VRPTW is applied to the set of clustered customers. These customers form a group of candidates to be inserted on the
route of the current vehicle. This cluster-first, route-second strategy proceeds until all customers have been serviced or all vehicles have been used.

The subgradient method is also used for solving each vehicle Lagrangian problem in the same way as described in the Sequential Insertion Heuristic. The route feasibility constraints (e.g., time windows, vehicle capacity and time duration) are not checked in the clustering stage of this heuristic. The clustering stage aims to select a group of potential candidate customers to be in the same route within the set of unserviced customers. This group is usually larger than the effective number of customers to be routed due to the VRPTW constraints. The computation on the best route to be built, for a given vehicle, is done in the routing stage through the Lagrangian relaxation strategy described in the Sequential Insertion Heuristic.

The decomposition of the VRPTW brings about some advantages over the Sequential Insertion Heuristic: first, the grouping of potential candidate customers through the clustering procedure allows the reduction of the computational effort by the subgradient method, since only a subset of the customers not serviced is considered in the Lagrangian relaxation heuristic. The grouping procedure also allows the use of a vehicle choice procedure when the fleet is not homogeneous. This choice was based on the fixed cost per unit of vehicle capacity criterion. It can be easily changed to accommodate the user’s preference or priority on the use of vehicles types and sizes.

5. Computational Results

Both heuristics were programmed and compiled using Borland Pascal version 7 and run on a PC Pentium 100Mhz microcomputer. The heuristics were tested against the six data sets provided by Solomon (1987). The Cluster and Sequential Insertion Heuristic was also applied to a real urban distribution problem in São Paulo Metropolitan Area, Brazil.

Table 1 compares both heuristics proposed with Solomon’s Insertion I1 and I2 as reported in his original paper. For comparison purposes, solution quality is based first on the number of routes, then on the total distance travelled by the vehicles.

For most problem sets, the Cluster and Sequential Insertion Heuristic outperformed the Sequential Insertion Heuristic. This heuristic showed to be efficient and allowed to solve
short scheduling horizon problems in less than 30 seconds. These times can be
dramatically reduced if a faster (e.g. Pentium-III) microcomputer is used.

**TABLE 1 - Comparative Computational Results**

<table>
<thead>
<tr>
<th>Set</th>
<th>Heuristic</th>
<th>Number of routes</th>
<th>Total Distance Travelled</th>
<th>Computation Time (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sequential Insertion</td>
<td>10.0</td>
<td>935.5</td>
<td>11.3</td>
</tr>
<tr>
<td>C1</td>
<td>Cluster and Sequential Insertion</td>
<td>10.0</td>
<td>835.0</td>
<td>4.2</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I1^3</td>
<td>10.0</td>
<td>951.9</td>
<td>25.3</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I2^4</td>
<td>10.1</td>
<td>1049.8</td>
<td>25.3</td>
</tr>
<tr>
<td></td>
<td>Sequential Insertion</td>
<td>14.3</td>
<td>1674.1</td>
<td>145.4</td>
</tr>
<tr>
<td>R1</td>
<td>Cluster and Sequential Insertion</td>
<td>14.4</td>
<td>1546.9</td>
<td>30.8</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I1^3</td>
<td>13.6</td>
<td>1436.7</td>
<td>24.7</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I2^4</td>
<td>14.5</td>
<td>1638.7</td>
<td>25.5</td>
</tr>
<tr>
<td></td>
<td>Sequential Insertion</td>
<td>14.3</td>
<td>1830.6</td>
<td>129.1</td>
</tr>
<tr>
<td>RC1</td>
<td>Cluster and Sequential Insertion</td>
<td>13.9</td>
<td>1653.5</td>
<td>21.4</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I1^3</td>
<td>13.5</td>
<td>1596.5</td>
<td>23.8</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I2^4</td>
<td>14.2</td>
<td>1874.4</td>
<td>24.4</td>
</tr>
<tr>
<td></td>
<td>Sequential Insertion</td>
<td>3.3</td>
<td>781.3</td>
<td>803.6</td>
</tr>
<tr>
<td>C2</td>
<td>Cluster and Sequential Insertion</td>
<td>3.1</td>
<td>685.7</td>
<td>100.8</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I1^3</td>
<td>3.1</td>
<td>692.7</td>
<td>43.0</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I2^4</td>
<td>3.4</td>
<td>921.5</td>
<td>44.5</td>
</tr>
<tr>
<td></td>
<td>Sequential Insertion</td>
<td>3.2</td>
<td>1401.5</td>
<td>1417.4</td>
</tr>
<tr>
<td>R2</td>
<td>Cluster and Sequential Insertion</td>
<td>3.1</td>
<td>1365.3</td>
<td>1151.9</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I1^3</td>
<td>3.3</td>
<td>1402.4</td>
<td>62.6</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I2^4</td>
<td>3.3</td>
<td>1470.7</td>
<td>71.1</td>
</tr>
<tr>
<td></td>
<td>Sequential Insertion</td>
<td>3.8</td>
<td>1584.2</td>
<td>702.8</td>
</tr>
<tr>
<td>RC2</td>
<td>Cluster and Sequential Insertion</td>
<td>3.8</td>
<td>1777.1</td>
<td>505.2</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I1^3</td>
<td>3.9</td>
<td>1682.1</td>
<td>51.7</td>
</tr>
<tr>
<td></td>
<td>Solomon’s Insertion I2^4</td>
<td>4.1</td>
<td>1797.6</td>
<td>54.0</td>
</tr>
</tbody>
</table>

Notes:
1 refers to the average number of routes on all problem instances in each set
2 Average times for each set. The proposed heuristics were run on a 100 Mhz Pentium
   computer; Solomon’s algorithms CPU times reported were processed on a DEC-10
   computer.
3 Solomon’s Insertion Heuristic I1 resulted the best overall performance (Solomon, 1987).
4 Solomon’s Insertion Heuristic I1 resulted the best overall performance (Solomon, 1987).

The Cluster and Sequential Insertion Heuristic outperformed Solomon’s Insertion I2 in all
problem sets. For long scheduling horizon problems C2, R2 and RC2, the Cluster and
Sequential Insertion Heuristic slightly outperformed Solomon’s insertion I1. For problem
set C1, the average distance travelled has been reduced. For problem sets R1 and
RC1, Solomon’s insertion I1 reached the best results.

Though some metaheuristics proposed more recently, most based on tabu search, have
yielded impressive results, as pointed out by Laporte (2000) these algorithms
sometimes lack robustness, since they are governed by several user controlled
parameters, which depend on the problem being solved, and deeply affect the quality of
the solutions produced. Besides, they are quite complex to develop and to implement;
on the other hand, both proposed heuristics are reasonably simple, with codes on the order of 5,000 lines each.

6. Case Study

The Cluster and Sequential Insertion Heuristic was applied to a real-life distribution problem in São Paulo metropolitan area. This problem refers to a third-party logistics provider that consolidates and distributes non-perishable food and consumer goods from industries to retail in general, including supermarkets, small stores, etc., all located in the metropolitan area of the city of São Paulo.

The basic problem consists of routing and scheduling the vehicles on a daily basis. The actual demands are not known prior to the arrival of the TL trucks from the industries to the depot.

On an average day there are about 130 delivery destinations. About 60% of the customers have tight time windows (up to 4 hours). Service time may vary due to congestion problems at customers’ receiving docks. In average, service time was about 90 minutes, with small variation depending on the amount to be unloaded. The fleet is composed of 55 trucks, ranging from 1.0 to 12.7 ton of capacity.

The daily routes were constructed manually by dispatchers, based on geographical and vehicle capacity criteria. Since information on the precise location and time window of each customer was not available to the dispatcher (not to mention the scarce available time for actually forming the routes, of about one hour), assignment and combination of customers in the same route were restricted, resulting in loaded vehicles below their capacity.

In order to perform the application of the Cluster and Sequential Insertion Heuristic to this problem, one crucial task was to obtain the proper necessary data, including geographic customer locations, time windows constraints and fleet availability. In developing countries, like Brazil, locating and geo-coding customers and addresses may be quite troublesome, since address matching is rather imprecise and, not rarely, virtually impossible to be automated. Detailed map databases at street level, which include information on legal and forbidden turns and movements, simply do not exist. In this case, a simplified network was built, based on the coordinates of each customer and calibrated to represent real traffic conditions.
The summary of the results obtained, as well as the manual routes are presented in Table 2. The results clearly indicate that the proposed heuristic markedly improved the manual solution of the dispatchers. The total distance was reduced by 23% and the total cost even more, by 32%, due to better use of the vehicles capacities. An increase of about 30% was achieved on the average load factor of the vehicles.

**TABLE 2 - Comparison of the results obtained for the case study**

<table>
<thead>
<tr>
<th></th>
<th>Routes constructed manually by the firm’s dispatchers</th>
<th>Cluster and Sequential Insertion Heuristic</th>
<th>% change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of customers serviced</td>
<td>136</td>
<td>136</td>
<td>-</td>
</tr>
<tr>
<td>Total load (kg)</td>
<td>152 307</td>
<td>152 307</td>
<td>-</td>
</tr>
<tr>
<td>Number of routes</td>
<td>55</td>
<td>35</td>
<td>36%</td>
</tr>
<tr>
<td>Average load factor (over capacity)</td>
<td>53%</td>
<td>83%</td>
<td>30%</td>
</tr>
<tr>
<td>Total distance run in the routes (km)</td>
<td>3 068.34</td>
<td>2 360.70</td>
<td>23%</td>
</tr>
<tr>
<td>Total time spent on the routes (min)</td>
<td>NA¹</td>
<td>20 164</td>
<td>-</td>
</tr>
<tr>
<td>Total travel time (min)</td>
<td>NA¹</td>
<td>7 076</td>
<td>-</td>
</tr>
<tr>
<td>Total service time (min)</td>
<td>NA¹</td>
<td>13 088</td>
<td>-</td>
</tr>
<tr>
<td>Average commercial speed (km/h)</td>
<td>NA¹</td>
<td>7.02</td>
<td>-</td>
</tr>
<tr>
<td>Average load per route (kg)</td>
<td>2 720</td>
<td>4 353</td>
<td>60%</td>
</tr>
<tr>
<td>Total cost of the schedule (R$)</td>
<td>3 722.81</td>
<td>2 538.23</td>
<td>32%</td>
</tr>
</tbody>
</table>

7. Conclusions

Two heuristics for the VRPTW have been presented. Both are based on the Lagrangian relaxation of the constraints that require each customer to be serviced exactly once. Test problems proposed by Solomon (1987) were used to test the proposed heuristics. The computational results have shown that, for most problem sets, the Cluster and Sequential Insertion Heuristic outperformed Solomon’s Sequential Insertion Heuristic. The Cluster and Sequential Insertion Heuristic was successfully applied to a real-life urban distribution problem. The case study has shown the potential of the heuristic to solve real-life routing problems with time windows. The application of the heuristic resulted in significant savings in fleet reduction and distribution costs compared to the original manual solution produced by the dispatchers.

The application of mathematical models to vehicle routing requires large effort in input data preparation, what naturally inhibits testing and even utilising these models. The case study has evidenced this problem. A relevant and time-consuming effort was devoted to setting and calibrating a street road network to truly represent average real
traffic conditions. However, the benefits from the model application have largely overcome the data preparation effort.

**Bibliography**


